Fair Allocation with Initial Utilities

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Traditional Setting

Agents A: 👤 🎓 🕵 (1, 2, 3)

Indivisible Resources R: 🏠 🚗 💻 💎 🍴

Agent $i \in A$ has **utility function** u_i

👤 : $u_1(\{🏠\}) = 1000,\ u_1(\{🍴\}) = 5, \dots$ 🎓 : $u_2(\{🏠\}) = 500, \dots$ 🕵 : $\dots$

Utility function is **additive**: $u_1(\{🏠, 🍴\}) = u_1(\{🏠\}) + u_1(\{🍴\})$.

Goal: "Fair" **allocation** $\mathcal{X}$ of resources to agents

$X_1 = \{🏠\}, \quad X_2 = \{🚗\}, \quad X_3 = \{💻, 💎, 🍴\}$.

Our Extension: Initial Utilities

- Traditional setting ignores potential **initial disparities** before resources are allocated!
- Capture "starting positions" via **initial utility** $b_i \in \mathbb{R}$ for every agent $i \in A$:

  👤 : $b_1 = 0$, 🎓 : $b_2 = 50$, 🕵 : $b_3 = 10$

  ▶ **Goal:** Develop fairness notions attuning to *equality of outcome*.
- Related work on extending given allocation of *frozen* resources to envy-free allocation [DEIG25, VIV25].

Adapting Envy-Freeness: A First Attempt

An allocation $\mathcal{X}$ is **envy-free (up to one resource)**, if for every pair of agents $i, j \in A$, we have $X_j = \emptyset$ or (there exists a resource $r \in X_j$ such that):

Envy-Freeness (EF-init)

$$\underbrace{b_i + u_i(X_i)}_{i\text{'s initial utility} + i\text{'s utility for own resources}} \geq \underbrace{b_j}_{j\text{'s initial utility}} + \underbrace{u_i(X_j)}_{i\text{'s utility for } j\text{'s resources}}$$

Envy-Freeness up to one resource (EF1-init)

$$\underbrace{b_i + u_i(X_i)}_{i\text{'s initial utility} + i\text{'s utility for own resources}} \geq \underbrace{b_j}_{j\text{'s initial utility}} + \underbrace{u_i(X_j \setminus \{r\})}_{i\text{'s utility for } j\text{'s resources after removing } r}$$

Results:

- Contrast to standard setting:
  - ▸ **Deciding existence** of complete **EF1-init** allocation is **NP-hard**.
  - ▸ Allocation **may not exist** even on simple instance **with identical resources**.
- Positive algorithmic results for identical resources and few agents.

Example where complete EF1-init allocation does not exist:

Agents 👤 : $b_1 = 10$, 🎓 : $b_2 = 1$, identical resources $4 \times$ 🚗 and utilities 👤 : $u_1(\{🚗\}) = 10$, 🎓 : $u_2(\{🚗\}) = 3$.

Then $|X_2| \geq 3$, but then agent 1 envies agent 2.

A Satisfiable Envy Notion

An allocation $\mathcal{X}$ is **minimum-EF1-init** (min-EF1-init), if for every pair of agents $i, j \in A$, we have $X_j = \emptyset$ or:

- If $b_i \leq b_j$, then i does not envy j under EF1-init.
- If $b_i > b_j$, then there exists a resource $r \in X_j$ and a subset $X^* \subseteq X_j$ with $\underbrace{\sum_{r' \in X^*} \min_{j' \in A\colon b_{j'} < b_i} u_{j'}(\{r'\}) < b_i - b_j}_{X^* \text{ accounts for initial difference between } i \text{ and } j}$, such that $\underbrace{u_i(X_i) \geq u_i(X_j \setminus (X^* \cup \{r\}))}_{\text{remaining allocation fair to } i}$.

▶ Guaranteed to exist & can be computed by polynomial-time adaptation of round-robin.

▶ Implies EF1-init when utility for a resource is diminishing in initial utility.

[VIV25] V. Prakash H. V., A. Igarashi, and R. Vaish, "Fair and Efficient Completion of Indivisible Goods", AAAI '25.

[DEIG25] A. Deligkas, E. Eiben, R. Ganian, T.-L. Goldsmith, and S. D. Ioannidis, "The Complexity of Extending Fair Allocations of Indivisible Goods", AAAI '25.