Homework 3 – “Heuristic Optimization”

This homework is due **Wednesday, May 13, 12:45**. This homework is to be submitted at the mailbox titled “Heuristic Optimization” at the reception desk of the main building.

We use the following definitions on $O$-notation: Let $g : \mathbb{N} \to \mathbb{R}^+$ be a mapping from the natural numbers to the positive real numbers. We let

$$O(g) = \{ f : \mathbb{N} \to \mathbb{R}^+ | \exists c > 0 \exists n_0 \in \mathbb{N} \forall n \geq n_0 : f(n) \leq c \cdot g(n) \}.$$ 

Furthermore, we let

$$\Omega(g) = \{ f : \mathbb{N} \to \mathbb{R}^+ | \exists c > 0 \exists n_0 \in \mathbb{N} \forall n \geq n_0 : g(n) \leq c \cdot f(n) \}.$$ 

Finally, we let $\Theta(g) = O(g) \cap \Omega(g)$. For any set of functions $A$ and any function $f$ we write $f = A$ instead of $f \in A$.

For any set of functions $A$ we can define operations on this set as follows.

$\forall r : A + r = \{ f + r | f \in A \}$;

$\forall r : r \cdot A = \{ rf | f \in A \}$;

$2^A = \{2^f | f \in A \}.$

Note that when $A$ is a set of functions with positive range, then $-A$ is not, but $2^{-A}$ is. If there is some $n_0$ such that something holds for all $n$ larger than $n_0$, we also say that this something holds “for $n$ large enough”.

**Exercise 4** Show the following claims.

(a) Let $f : \mathbb{N} \to \mathbb{R}^+$. Suppose that there is a $c \in (0, 1)$ (the open interval from 0 to 1) such that, for all $n$ large enough, $f(n + 1) \leq cf(n)$. Then $f = 2^{-\Omega(n)}$.

(b) There is an $f$ with $f = 2^{-\Omega(n)}$ such that there is no $c \in (0, 1)$ such that, for all $n$ large enough, $f(n + 1) \leq cf(n)$.

(c) Let $f, g : \mathbb{N} \to \mathbb{R}^+$. If, for all $n$ large enough, $f(n) \leq g(n)$, then $2^f = O(2^g)$.

(d) Let $f, g : \mathbb{N} \to \mathbb{R}^+$. If $2^f = O(2^g)$, then $f = O(g)$.

(e) There are $f, g : \mathbb{N} \to \mathbb{R}^+$ such that $f = O(g)$ but **not** $2^f = O(2^g)$. 