Heuristic Optimization

Lecture 4

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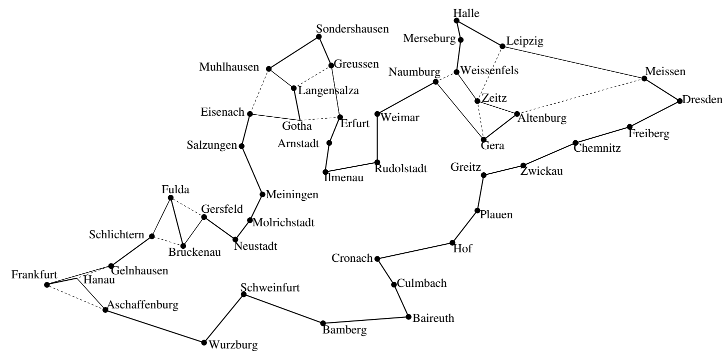
A historical problem with obscure roots

1832 – Der Handlungſreiſende – wie er ſein ſoll und waſ er zu tun hat, um Aufträge zu erhalten und eineſ glücklichen Erfolgſ in ſeinen Geſchäften gewiß zu ſein – Von einem alten Commiſ-Voyageur



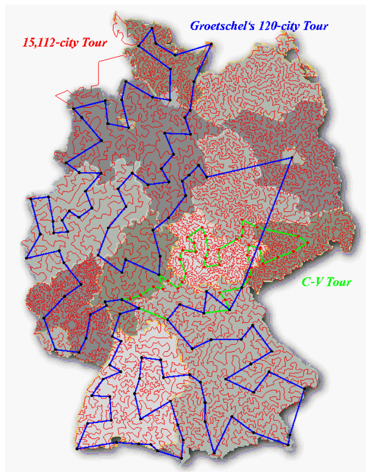
Alexander Schrijver, "On the history of combinatorial optimization (till 1960)", Handbook of Discrete Optimization, (K. Aardal, G.L. Nemhauser, R. Weismantel, eds.), Elsevier, Amsterdam, 2005, pp. 1–68.

A historical problem with obscure roots

Karl Menger in 1930: at *mathematisches Kolloquium* in Vienna – shortest path through a set of points in space

"[T]his problem is solvable by finitely many trials. Rules which would push the number of trials below the number of permutations are not known. The rule that one first should go from the starting point to the closest point, then to the point closest to this, etc., in general does not yield the shortest route."

A historical problem with obscure roots



1832 tour

Martin Groetschel, *Mathematical Systems in Economics* (1977)

15 112 city instance

TSP

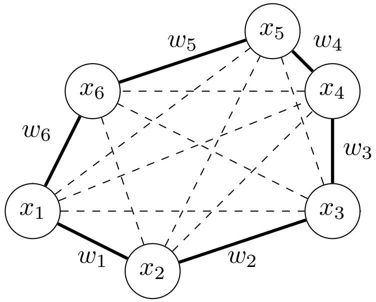
Let $G = (V, E)$ be an undirected complete graph with positive edge weights $w : E \to \mathbb{R}^+$.

- **tour** – a cycle that visits every vertex $v \in V$ exactly once
- **cost** – if x is a tour defined by a sequence of vertices, the cost of x is:

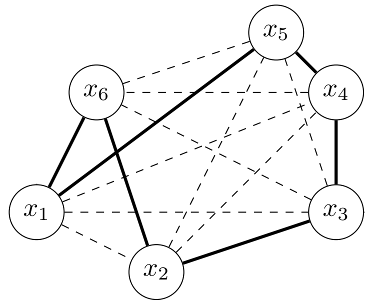
$$f(x) = \sum_{i=1}^{n-1} w(x_i, x_{i+1}) + w(x_1, x_n)$$

$x = (x_1, x_2, x_3, x_4, x_5, x_6)$

$f(x) = w_1 + w_2 + w_3 + w_4 + w_5 + w_6$



Nearest-neighbor heuristic



Can make a **bad local decision** early

Could lead to an arbitrarily bad tour

Approximating the TSP

TSP is NP-hard: no polynomial time algorithm is known to solve it exactly

Do we have hope to find an approximate solution?

Definition

A ρ-approximation algorithm is an algorithm that guarantees a solution x with $f(x) \le \rho\mathrm{OPT}$ where OPT is the value of the optimal solution.

Approximating the TSP: edge weights matter!

Arbitrary weights

- $w \colon E \to \mathbb{R}^+$ arbitrary
- NP-hard to find a ρ-approximation for any fixed ρ

Metric

- w satisfies the triangle inequality: $w(a, b) + w(b, c) \ge w(a, c)$.
- Constant-factor approximation available in polytime

Euclidean

- Each point $v \in V$ associated with coordinates in some $\mathbb{R}^d$
- $w(u, v)$ is Euclidean distance between u and v
- Special case of metric

Constant factor approximation for Metric TSP

Algorithm 1: ApproxTSPTour

input : A graph $G = (V, E)$
output: A tour x
Construct the minimum spanning tree T of G;
Let $x = (x_1, x_2, \ldots, x_n)$ be the nodes in a pre-order traversal of T;

Constant factor approximation for Metric TSP

Theorem.
The ApproxTSPTour algorithm is a 2-approximation algorithm for metric TSP.

Proof.

- Let $x^\star$ be the optimal tour. Let $T^\star$ be the tree obtained by removing one edge from $x^\star$,
- Denote as MST the weight of the minimal spanning tree for G. Then $MST \leq w(T^*) < f(x^\star)$
- If x is created by ApproxTSPTour, then $f(x) \leq 2MST$ (shortcut argument)
- We conclude $f(x) \leq 2MST \leq 2f(x^\star)$
- Time bound

The Christofides algorithm

Another approach

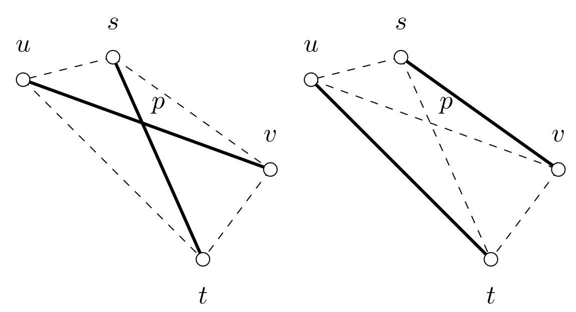
- build the MST T
- find a perfect matching M on the vertices of odd degree in T
- construct Eulerian tour on $T \cup M$ (repeat vertices removed)

Christofides' algorithm: a 3/2-approximation

2-opt

Euclidean TSP: Remove edges that cross, and replace with ones that do not

Must improve the tour

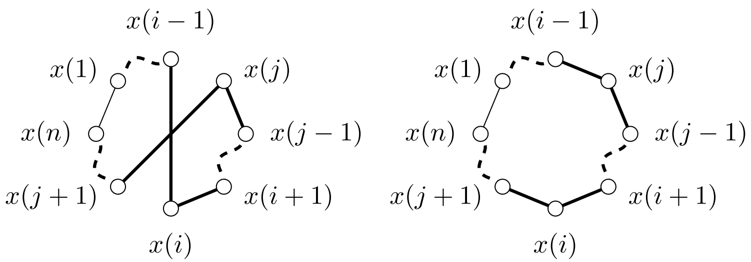


2-opt

Invert a subsequence of the tour from i to j.

$$(x_1, \ldots, x_{i-1}, \underbrace{x_i, x_{i+1}, \ldots x_{j-1}, x_j}_{\text{inversion}}, x_{j+1}, \ldots x_n)$$

$$(x_1, \ldots, x_{i-1}, \overbrace{x_j, x_{j-1}, \ldots x_{i+1}, x_i}, x_{j+1}, \ldots x_n)$$



Local search

Idea

- 2-opt operator generates a “neighborhood” of tours
- iteratively choose the best 2-opt neighbor until no longer possible

Generalizations

- k-opt
  - remove k mutually disjoint edges
  - reassemble remaining fragments into a legal tour
- Lin-Kernighan
  - switch between 2-opt and 3-opt

No performance guarantees currently exist

Can take exponential time to find a local optimum (even on Euclidean instances)[1]

[1] Matthias Englert, Heiko Röglin, and Berthold Vöcking Worst Case and Probabilistic Analysis of the 2-Opt Algorithm for the TSP In Proc. of the 18th SODA (New Orleans, USA), pp. 1295-1304, 2007.

Other methods

Held-Karp algorithm (1962)

- Fix a home vertex h. For any $S \subseteq V \setminus \{h\}$,
- $D(S, v) :=$ minimum tour starting at h, ending at v and using only cities in S
- $D(S, v) := \min_{u \in S \setminus \{v\}} D(S \setminus \{v\}, u) + w(u, v)$
- Solution to find $\min_{u \in V \setminus \{h\}} D(V \setminus \{h\}, u) + w(u, h)$
- Dynamic programming for D in time $O(n^2 2^n)$

Mixed-integer linear programming

- State TSP as an integer program
- Solve linear relaxation, add further constraints

Tools

CONCORDE: `http://www.math.uwaterloo.ca/tsp/concorde.html`