Data-Driven Decision-Making
In Enterprise Applications

Linear Programming II

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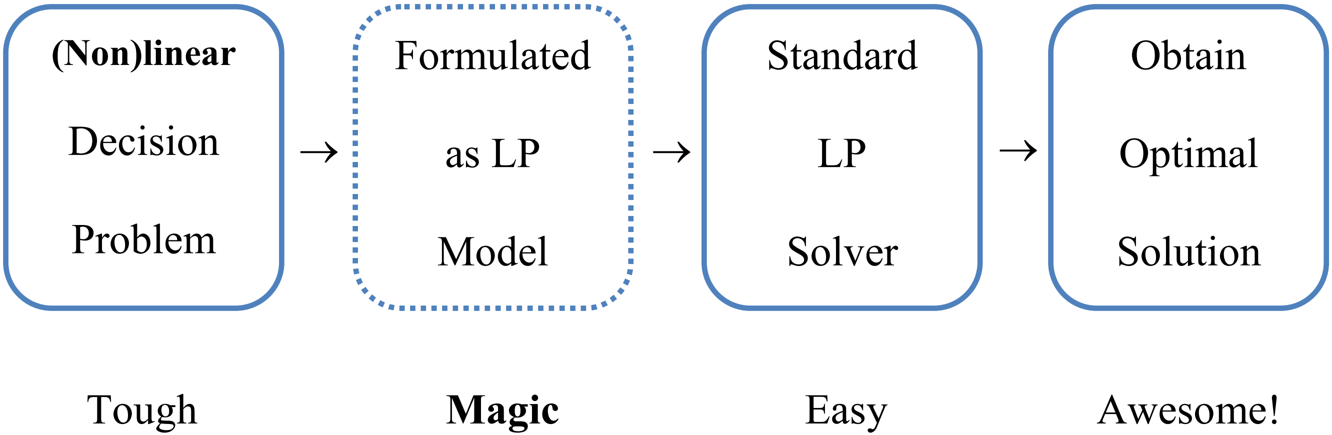
Decision-Making Using Linear Programming



Linear Programming

- Questions regarding last week?
- Implemented/Solved Examples (using AMPL)?
- Today: Tricks to Circumvent Non-Linearities
- Penalty Approaches & Continuous Relaxations
- Description of the HPI Master Project Assignment Problem

Nonlinear Programming Models

- Often non-linear expressions are needed within a model
- (–) Linear solvers cannot be used anymore
- (–) NL solvers often cannot guarantee optimality
- (+) So-called “mild” nonlinearities can be expressed linearly
- (+) This is very valuable as we can exploit LP solvers and their optimality
- The price of such transformations is acceptable:

  More variables and constraints

I Linearization of “and” in the Constraints

Objective: $\min_{x_1, x_2 \in \{0,1\}} 2 \cdot x_1 + x_2$

Constraints NL: . . .

$x_1 = 1 \; and \; x_2 = 1$ (e.g. needed as joint condition)

Objective: $\min_{x_1, x_2 \in \{0,1\}} 2 \cdot x_1 + x_2$

Constraints LIN: . . .

$x_1 + x_2 = 2$

II Linearization of "or" in the Constraints

Objective: $\min_{x_1, x_2 \in \{0,1\}, x_3 \in [0,M]} 2 \cdot x_1 + x_2 + x_3$

Constraints NLa: $x_1 = 1 \; or \; x_2 = 1$ (e.g. needed as joint condition)

Constraints NLb: $x_1 = 1 \; or \; x_2 = 0$

Constraints NLc: $x_3 = 0 \; or \; x_3 \geq 3$

Objective: $\min_{x_1, x_2 \in \{0,1\}, x_3 \in [0,M], z \in \{0,1\}} 2 \cdot x_1 + x_2 + x_3$

Constraints LINa: $x_1 + x_2 \geq 1$

Constraints LINb: $x_1 + (1 - x_2) \geq 1$

Constraints LINc: $x_3 \leq M \cdot z$, $x_3 \geq 3 \cdot z$

III Linearization of "max" in the Objective

Objective NL: $\min_{x_1,\ldots,x_N \in \mathbb{R}} \left\{ \max_{i=1,\ldots,N} x_i \right\}$

Constraints: . . .

Objective LIN: $\min_{x_1,\ldots,x_N \in \mathbb{R}, z \in \mathbb{R}} z$

Constraints: . . .

new $z \geq x_i$ for all $i=1,\ldots,N$

IV Linearization of "min" in the Objective

Objective NL: $$\max_{x_1,\dots,x_N \in \mathbb{R}} \left\{ \min_{i=1,\dots,N} x_i \right\}$$

Constraints: . . .

Objective LIN: $$\max_{x_1,\dots,x_N \in \mathbb{R}, z \in \mathbb{R}} z$$

Constraints: . . .

new $z \le x_i$ for all $i=1,\dots,N$

V Linearization of "min" in the Constraints

Objective: $\min_{x_1, x_2 \in [0, M]} 2 \cdot x_1 + x_2$

Constraints NL: $4 \le \min(x_1, x_2) \le 7$

Objective: $\min_{x_1, x_2 \in [0, M], z_1, z_2 \in \{0,1\}} 2 \cdot x_1 + x_2$

Constraint LIN: $4 \le x_i$ for all i=1,2

new $M \cdot z_i \ge x_i - 7$ for all i=1,2

new $z_1 + z_2 \le 1$

VI Linearization of “abs” in the Objective

Objective NL: $\min_{x_1, x_2 \in \mathbb{R}} 2 \cdot x_1 + abs(3 - x_2)$

Constraints: . . .

Objective LIN: $\min_{x_1, x_2 \in \mathbb{R}, z \in \mathbb{R}} 2 \cdot x_1 + z$

Constraints: . . .

new $x_2 - 3 \leq z$

new $3 - x_2 \leq z$

VII Linearization of “abs” in the Constraints

Objective: $\min_{x_1, x_2 \in \mathbb{R}} 2 \cdot x_1 + x_2$

Constraints NL: $abs(3 - x_2) \leq x_1$

Objective LIN: $\min_{x_1, x_2 \in \mathbb{R}, z \in \mathbb{R}} 2 \cdot x_1 + x_2$

Constraints: $z \leq x_1$

new $x_2 - 3 \leq z$

new $3 - x_2 \leq z$

VIII Linearization of "if-then-else"

Objective NL: $$\min_{x_1,x_2\in\{0,1,2,\dots,M\}} 2\cdot x_1 + \left(\mathit{if}\ x_2 \le 5.5\ \mathit{then}\ a\ \mathit{else}\ b\right)$$

Constraints: . . .

Objective LIN: $$\min_{x_1,x_2\in\{0,1,2,\dots,M\},z\in\{0,1\}} 2\cdot x_1 + b\cdot z + a\cdot(1-z)$$

Constraints: . . .

new $$x_2 - 5.5 \le M\cdot z$$

new $$5.5 - x_2 \le M\cdot(1-z)$$

IX Linearization of a Product of Binary Variables

Objective: $$\min_{x_1, x_2 \in \{0,1\}} 2 \cdot x_1 + x_2$$

Constraints NL: including the term: $x_1 \cdot x_2$

Objective: $$\min_{x_1, x_2 \in \{0,1\}, z \in \{0,1\}} 2 \cdot x_1 + x_2$$

Constraints LIN: include the term z instead, where

$$z \leq x_i, \qquad \text{for } i=1,2$$

$$z \geq x_1 + x_2 - 1$$

X Linearization of a Binary x Continuous Variable

Objective: $$\min_{x_1 \in \{0,1\}, x_2 \in [0,M]} 2 \cdot x_1 + x_2$$

Constraints NL: including the term: $x_1 \cdot x_2$

Objective: $$\min_{x_1 \in \{0,1\}, x_2 \in [0,M], z \in [0,M]} 2 \cdot x_1 + x_2$$

Constraints LIN: include the term z instead, where

$z \leq M \cdot x_1$, for i=1,2

$z \leq x_2$

$z \geq x_2 - (1 - x_1) \cdot M$

Solution Tuning

Recall Example IV: Project Assignment Problem

$x_{i,j} \in \{0,1\}$ whether project i, i=1,...,N, is assigned to worker j, j=1,...,N

LP:
$$\max_{x_{i,j} \in \{0,1\}^{N \times N}} \sum_{i=1,\dots,N,\ j=1,\dots,N} w_{i,j} \cdot x_{i,j}$$

s.t.
$$\sum_{i=1,\dots,N} x_{i,j} = 1 \quad \text{for all } j=1,\dots,N \quad \text{(each worker gets 1 project)}$$
$$\sum_{j=1,\dots,N} x_{i,j} = 1 \quad \text{for all } i=1,\dots,N \quad \text{(each project is assigned)}$$

- Will the allocation always be fair?
- How "outliers" can be avoided?
- Approaches: (i) utility functions, (ii) max min, (iii) multi-objective

Approach (i): Fair Project Assignment (Non-linear)

$x_{i,j} \in \{0,1\}$ whether project i, i=1,...,N, is assigned to worker j, j=1,...,N

NLP:
$$\max_{x_{i,j} \in \{0,1\}^{N \times N}} \sum_{j=1,\ldots,N} u\left(\sum_{i=1,\ldots,N} w_{i,j} \cdot x_{i,j} \right)$$

using, e.g., $u(z) := \ln(z)$, $u(z) := z^{0.6}$, $u(z) := -e^{-0.1 \cdot z}$

s.t.

$\sum_{i=1,\ldots,N} x_{i,j} = 1$ for all j=1,...,N (each worker gets 1 project)

$\sum_{j=1,\ldots,N} x_{i,j} = 1$ for all i=1,...,N (each project is assigned)

- Idea: Avoiding low scores is better than including high scores
- Disadvantage (i): Non-linear solver is needed

Approach (ii): Fair Project Assignment (Linear!)

$x_{i,j} \in \{0,1\}$ whether project i, i=1,...,N, is assigned to worker j, j=1,...,N

LP: $\max\limits_{x_{i,j} \in \{0,1\}^{N \times N}, z \in \mathbb{R}} z$ s.t. $z \leq \sum\limits_{i=1,\ldots,N} w_{i,j} \cdot x_{i,j}$ for all j=1,...,N

$\sum\limits_{i=1,\ldots,N} x_{i,j} = 1$ for all j=1,...,N (each worker gets 1 project)

$\sum\limits_{j=1,\ldots,N} x_{i,j} = 1$ for all i=1,...,N (each project is assigned)

- Idea: Optimize the lowest willingness (cf. worst case criteria)
- Disadvantage (ii): Total willingness score can be low

Approach (iii): Fair Project Assignment (Linear!)

$x_{i,j} \in \{0,1\}$ whether project i, i=1,...,N, is assigned to worker j, j=1,...,N

LP: $$\max_{x_{i,j} \in \{0,1\}^{N \times N}, z \in \mathbb{R}} \sum_{i=1,\ldots,N,\ j=1,\ldots,N} w_{i,j} \cdot x_{i,j} + \alpha \cdot z,$$ with parameter $\alpha \geq 0$

s.t. $$z \leq \sum_{i=1,\ldots,N} w_{i,j} \cdot x_{i,j} \quad \forall j$$

$$\sum_{i=1,\ldots,N} x_{i,j} = 1$$ for all j=1,...,N (each worker gets 1 project)

$$\sum_{j=1,\ldots,N} x_{i,j} = 1$$ for all i=1,...,N (each project is assigned)

- Idea: Combine both objectives as a weighted sum
- Disadvantage (iii): Suitable weighting factor α has to be determined

Penalty Approaches & Efficient Frontiers

Penalty Formulations for Pareto-Optimal Relaxations

Objective: $\max\limits_{x_1,\dots,x_N \in \{0,1\}} \sum\limits_{i=1,\dots,N} u_i \cdot x_i$ Knapsack example

Constraints: $\sum\limits_{i=1,\dots,N} s_i \cdot x_i \leq C$ (One) Hard Constraint

Penalty-Objective: $\max\limits_{x_1,\dots,x_N \in \{0,1\}} \sum\limits_{i=1,\dots,N} u_i \cdot x_i - \alpha \cdot \sum\limits_{i=1,\dots,N} s_i \cdot x_i$ (Soft Constraint)

Constraints: none

Results: Pareto-optimal combinations of “Utility” and “Space”

Pareto-Optimal Relaxations (int. vs. cont. solutions)

(i) Optimal integer solution (blue): $\min\limits_{\vec{x}\in\{0,1\}^N} F(\vec{x}) \;\; s.t. \;\; M(\vec{x}) \leq A \;\Rightarrow\; \vec{x}^*(A) \;\; optimal$

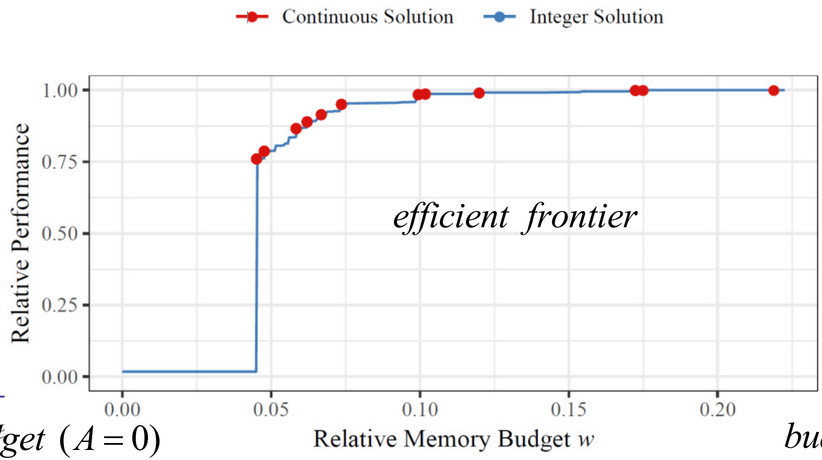
(ii) Continuous relaxation: $\rightarrow \min\limits_{\vec{x}\in[0,1]^N} F(\vec{x}) \;\; s.t. \;\; M(\vec{x}) \leq A \;\Rightarrow\; \vec{x}^*(A) \in \{0,1\}^N \,?$

(iii) Penalty formulation (red): $\min\limits_{\vec{x}\in[0,1]^N} F(\vec{x}) + \alpha \cdot M(\vec{x}) \;\Rightarrow\; \vec{x}^*(\alpha) \in \{0,1\}^N \;\; and$

$\uparrow$ $\qquad Pareto-optimal!$

Performance:

Runtime saved

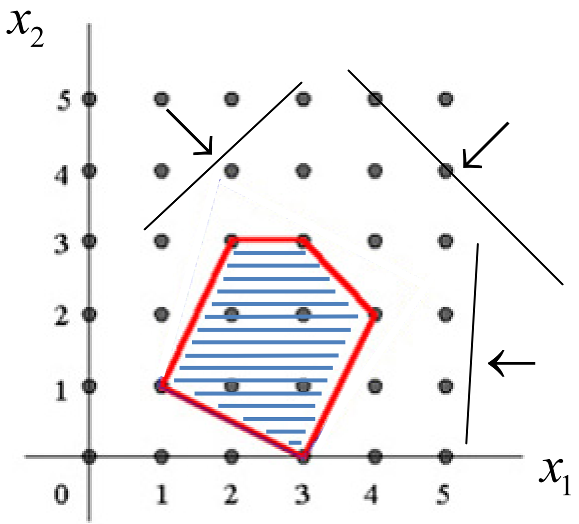


efficient frontier

no budget $(A=0)$ *budget* A

When do Integer & Continuous Solutions Coincide?

maximize $a \cdot x_1 + b \cdot x_2$ s.t. . . . with $x_1, x_2 \in \mathbb{R}$ vs. $x_1, x_2 \in \mathbb{N}$



- **Answer**: The corners of the polygon have to be "integers"!

HPI Master Project Assignment Problem

Description HPI Master Project Assignment Problem

- Each worker gets 1 project
- 0, 3, 4, 5, or 6 students per project
- Each student has a “first/second/third choice” project
- Can a student exclude one/two projects? -> no
- Average number of students/projects?

- Maximize the number of first choices?
- Minimize the number of unfulfilled dreams?
- Weighted sum?

Next Week

Homework: Try to apply/implement the Linearizations I-X!

Formulate the HPI Master Project Assignment Problem

Outlook:

- Nonlinear Programming and Suitable Solvers
- Linear Regression
- Logistic Regression
- Probabilities & Random Variables

Overview

2	April 25	Linear Programming I
3	April 29	Linear Programming II
4	**May 2**	**Linear/Logistic Regression + Homework (2 weeks time)**
5	May ?	Exercise Implementations (postponed)
6	May 16	Dynamic Programming I
7	May 20	Dynamic Programming II
8	May 23	Response Strategies / Game Theory
9	May 27	Project Assignments
10	June 3	Robust Optimization
11	June 13	Workshop / Group Meetings
12	June 20	Presentations (First Results)
13/14	June 24/27	Workshop / Group Meetings
15/16	July 1/4	Workshop / Group Meetings
17	July 11	Presentations (Final Results), Feedback, Documentation (Aug 31)