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Counting Homomorphisms over Fields of Prime Order

Abstract

Homomorphisms are a fundamental concept in mathematics expressing the similarity of structures. They provide a framework that captures many of the central problems of computer science with close ties to various other fields of science. Thus, many studies over the last four decades have been devoted to the algorithmic complexity of homomorphism problems. Despite their generality, it has been found that non-uniform homomorphism problems, where the target structure is fixed, frequently feature complexity dichotomies. Exploring the limits of these dichotomies represents the common goal of this line of research.

We investigate the problem of counting homomorphisms to a fixed structure over a finite field of prime order and its algorithmic complexity. Our emphasis is on graph homomorphisms and the resulting problem $\#_p\textsc{Hom}(H)$ for a graph H and a prime p. The main research question is how counting over a finite field of prime order affects the complexity.

In the first part of this thesis, we tackle the research question in its generality and develop a framework for studying the complexity of counting problems based on category theory. In the absence of problem-specific details, results in the language of category theory provide a clear picture of the properties needed and highlight common ground between different branches of science. The proposed problem $\#_p\textsc{Mor}^{\mathbf{C}}(B)$ of counting the number of morphisms to a fixed object B of $\mathbf{C}$ is abstract in nature and encompasses important problems like *constraint satisfaction problems*, which serve as a leading example for all our results. We find explanations and generalizations for a plethora of results in counting complexity. Our main technical result is that specific matrices of morphism counts are non-singular. The strength of this result lies in its algebraic nature. First, our proofs rely on carefully constructed systems of linear equations, which we know to be uniquely solvable. Second, by exchanging the field that the matrix is defined by to a finite field of order p, we obtain analogue results for modular counting. For the latter, cancellations are implied by automorphisms of order p, but intriguingly we find that these present the only obstacle to translating our results from exact counting to modular counting. If we restrict our attention to reduced objects without automorphisms of order p, we obtain results analogue to those for exact counting. This is underscored by a confluent reduction that allows this restriction by constructing a reduced object for any given object. We emphasize the strength of the categorial perspective by applying the duality principle, which yields immediate consequences for the dual problem of counting the number of morphisms from a fixed object.

In the second part of this thesis, we focus on graphs and the problem $\#_p\textsc{Hom}(H)$. We conjecture that automorphisms of order p capture all possible cancellations and that, for a reduced graph H, the problem $\#_p\textsc{Hom}(H)$ features the complexity dichotomy analogue to the one given for exact counting by Dyer and Greenhill. This serves as a generalization of the conjecture by Faben and Jerrum for the modulus 2. The criterion for tractability is that H is a collection of complete bipartite and reflexive complete graphs. From the findings of part one, we show that the conjectured dichotomy implies dichotomies for all quantum homomorphism problems, in particular counting vertex surjective homomorphisms and compactions modulo p. Since the tractable cases in the dichotomy are solved

by trivial computations, the study of the intractable cases remains. As an initial problem in a series of reductions capable of implying hardness, we employ the problem of counting weighted independent sets in a bipartite graph modulo prime p. A dichotomy for this problem is shown, stating that the trivial cases occurring when a weight is congruent modulo p to 0 are the only tractable cases. We reduce the possible structure of H to the bipartite case by a reduction to the restricted homomorphism problem $\#_p\textsc{Hom}^{\text{bip}}(H)$ of counting modulo p the number of homomorphisms between bipartite graphs that maintain a given order of bipartition. This reduction does not have an impact on the accessibility of the technical results, thanks to the generality of the findings of part one. In order to prove the conjecture, it suffices to show that for a connected bipartite graph that is not complete, $\#_p\textsc{Hom}^{\text{bip}}(H)$ is $\#_p\mathsf{P}$-hard. Through a rigorous structural study of bipartite graphs, we establish this result for the rich class of bipartite graphs that are $(K_{3,3}\backslash\{e\}$, *domino*)-free. This overcomes in particular the substantial hurdle imposed by squares, which leads us to explore the global structure of H and prove the existence of explicit structures that imply hardness.