On the Complexity Landscape of the Domination Chain

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Abstract. In this paper, we survey and supplement the complexity landscape of the domination chain parameters as a whole, including classifications according to approximability and parameterised complexity. Moreover, we provide clear pointers to yet open questions. As this posed the majority of hitherto unsettled problems, we focus on Upper Irredundance and Lower Irredundance that correspond to finding the largest irredundant set and resp. the smallest maximal irredundant set. The problems are proved NP-hard even for planar cubic graphs. While Lower Irredundance is proved not $c\log(n)$-approximable in polynomial time unless NP $\subseteq$ DTIME$(n^{\log\log n})$, no such result is known for Upper Irredundance. Their complementary versions are constant-factor approximable in polynomial time. All these four versions are APX-hard even on cubic graphs.

1 Introduction

The well-known domination chain

$$\mathrm{ir}(G) \le \gamma(G) \le i(G) \le \alpha(G) \le \Gamma(G) \le \mathrm{IR}(G)$$

links parameters related to the fundamental notions of independence, domination and irredundance in graphs. It was introduced in [12,22], is thoroughly discussed in the textbook [34] and studied further in many ways, [11,21,39,43] showing only a small selection. These studies cover both combinatorial and computational aspects. We focus on the latter aspects in this paper. In this chain, $\gamma(G)$ and $\Gamma(G)$ are the minimum and maximum cardinalities over all minimal dominating sets in G, $\alpha(G)$ is the maximum cardinality of an independent set, $i(G)$ is the minimum cardinality over all maximal independent sets in G. The less known irredundance parameters are explained below.

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With $n(G)$ being the order (number of vertices) of G, we can write co$-\zeta(G) = n(G) - \zeta(G)$. Then, we state the following complementary domination chain:

$$\mathrm{co} - \mathrm{IR}(G) \leq \mathrm{co} - \Gamma(G) \leq \mathrm{co} - \alpha(G) \leq \mathrm{co} - i(G) \leq \mathrm{co} - \gamma(G) \leq \mathrm{co} - \mathrm{ir}(G)\,.$$

Sometimes, the complement problems have received their own names, like Nonblocker, Maximum Enclaveless Set, or Maximum Spanning Star Forest, which all refer to the complement problem of Minimum Domination, or, most likely better known, Minimum Vertex Cover which refers to the complement problem of Maximum Independent Set. We will also use $\tau(G)$ instead of co $-\ \alpha(G)$ to refer to this graph parameter.

Throughout this paper, we will use rather standard terminology from graph theory. For any subset $S \subseteq V$ and $v \in S$ we define the private neighbourhood of v with respect to S as $pn(v, S) := N[v] - N[S - \{v\}]$. Any $w \in pn(v, S)$ is called a *private neighbour of v (with respect to S)*. S is called *irredundant* if every vertex in S has at least one private neighbour, i.e., if $|pn(v, S)| > 0$ for every $v \in S$. A maximal irredundant set is also known as an *upper irredundant set*. $\mathrm{IR}(G)$ denotes the cardinality of the largest irredundant set in G, while $\mathrm{ir}(G)$ is the cardinality of the smallest maximal irredundant set in G that is the smallest upper irredundant set in G. The domination chain is largely due to the following two combinatorial properties: (1) Every maximal independent set is a minimal dominating set. (2) A dominating set $S \subseteq V$ is minimal if and only if $|pn(v, S)| > 0$ for every $v \in S$. Observe that v can be a private neighbour of itself, i.e., a dominating set is minimal if and only if it is also an irredundant set. Actually, every minimal dominating set is also a maximal irredundant set.

For any $\varepsilon > 0$, a graph $G = (V, E)$ is called *everywhere-ε-dense* if every vertex in G has at least $\varepsilon|V|$ neighbours and *average-ε-dense* if $|E| \geq \varepsilon n^2$, for $0 < \varepsilon < 1/2$.

We first present some combinatorial bounds for $\mathrm{IR}(G)$. The same kind of bounds have been derived for $\Gamma(G)$ in [6]. Some proofs are omitted due to space restrictions.

Lemma 1. *For any connected graph G with $n > 0$ vertices we have:*

$$\alpha(G) \ \leq \ IR(G) \ \leq \ \ \max\left\{\alpha(G), \frac{n}{2} + \frac{\alpha(G)}{2} - 1\right\} \tag{1}$$

Lemma 2. *For any connected graph G with $n > 0$ vertices, minimum degree δ and maximum degree Δ, we have:*

$$\alpha(G) \ \leq \ IR(G) \ \leq \ \max\left\{\alpha(G),\ \frac{n}{2} + \frac{\alpha(G)(\Delta - \delta)}{2\Delta} - \frac{\Delta - \delta}{\Delta}\right\} \tag{2}$$

This lemma generalises [35, Proposition 12], which states the property for Δ-regular graphs, where, in particular, $\delta = \Delta$. Equation 1 immediately yields:

Lemma 3. *Let G be a connected graph. Then,*

$$\frac{\tau(G)}{2} + 1 \leq co - IR(G) \leq \tau(G) \tag{3}$$

2 The Complexity of the Domination Chain

We are studying algorithmic and complexity aspects of the domination chain parameters in this paper. For the basic definitions on classical complexity, approximation and parameterised algorithms we refer to standard texts like [5,26]. For providing hardness proofs in the area of approximation algorithms, L-reductions and E-reductions have become a kind of standard. An optimisation problem APX-hard under L-reduction has no polynomial-time approximation scheme if P $\neq$ NP. The notion of an E-reduction was introduced by Khanna et al. [37].

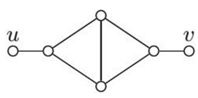
We have summarised what is known (and what is done in this paper) in Tables 1 and 2. Clearly, there is no need to repeat classical complexity results in Table 2. However, observe that the status of parameterised complexity and approximation of these problems and their complementary versions indeed differ. The hitherto unsolved questions regarding Upper Domination have been tackled and largely resolved in [6], which can be seen as a kind of companion paper to this one. Notice that in Table 1, the optimisation problems that correspond to the first three listed graph parameters are minimisation problems (in particular Lower Irredundance wich corresponds to find $\mathrm{ir}(G)$), while the last three are maximisation problems (in particular Upper Irredundance wich corresponds to find $\mathrm{IR}(G)$); this split is indicated by the double lines; this is reversed in Table 2. Also, when considering these problems as parameterised problems, we only consider the standard parameterisation, which is a lower bound on the entity to be maximised or an upper bound on the entity to be minimised. In order to distinguish the problem parameters of the two tables, we use k in Table 1 and ℓ in Table 2. The purpose of this paper is to survey the state of art and to solve most of what was still open until now.

3 On the Classical Complexity of Irredundant Set Problems

In this section, we prove that Lower Irredundance and Upper Irredundance (also their complementary versions) are NP-hard on planar cubic graphs.

Theorem 1. Lower Irredundance *is NP-hard on planar cubic graphs.*

Proof. We use the same construction as in [39], where Minimum Domination on planar cubic graphs is reduced to Minimum Independent Domination, that is: Given a planar cubic graph $G = (V, E)$, construct G' from G by replacing every $(u, v) \in E$ by the following planar cubic subgraph with four new vertices:



The argumentation [39] shows that $i(G') = \gamma(G) + |E|$ which automatically gives us $\mathrm{ir}(G') \leq \gamma(G) + |E|$. One can also proof that $\mathrm{ir}(G') \geq \gamma(G) + |E|$ which

Table 1. Status of various problems related to the domination chain

	ir	γ	i	α	Γ	IR
exact $\mathcal{O}^*()$	1.99914^n [11]	1.4864^n [36]	1.3351^n [14]	1.2002^n [44]	1.7159^n [6]	1.9369^n [11]
$\in$ FPT?	W[2]-C [11]	W[2]-C [25]	W[2]-C [25]	W[1]-C [25]	W[1]-H [6]	W[1]-C [27]
non-apx rat.	$c\log(n)$ Th.5	$c\log(n)$ [29]	$n^{1-\varepsilon}$ [33]	$n^{1-\varepsilon}$ [45]	$n^{1-\varepsilon}$ [6]	?
degree restrictions						
apx-ratio	$\frac{3}{2}\Delta$ [23]	$\log(\Delta)+1$ [19]	$\Delta+1$ Obs.4	$\frac{\Delta+3}{5}$ [7]	$\frac{6\Delta^2+2\Delta-3}{10\Delta}$ [6] & Obs.2	
kernel	$\frac{3}{2}\Delta k$ Obs.5	$(\Delta+1)k$ Obs. 4		Δk Obs. 3		
dense-apx	?	APX-H [32]	not $n^{1-\varepsilon}$ Th.9	not $n^{1-\varepsilon}$ Pr.1	not $n^{1-\varepsilon}$ Co.5	APX-H Th.8
cubic graphs						
+planar	NP-C Th.1	NP-C [31]	NP-C [39]	NP-C [31]	NP-C [6]	NP-C Th.2
$\in$ PTAS?	APX-C Co.2	APX-C [2]	APX-H Co.4	APX-C [2]	APX-C [6]	APX-C Co.3

means that Minimum Domination on G has a solution of cardinality at most k if and only if Lower Irredundance on G' has a solution of cardinality at most $k + |E|$. $\square$

Interesting side note to this proof is that ir, γ and i coincide on G'. Since especially ir and i are known to differ arbitrarily even on cubic graphs [46], this is obviously due to the special structure of G'. It contains induced $K_{1,3}$ (every original vertex with its neighbourhood), so the result for ir $= \gamma = i$ from [28] does not apply. This makes this construction an interesting candidate to study the characterisation of the graph class for which ir $= i$. With a different construction, we can show the same type of result for Upper Irredundance.

Theorem 2. Upper Irredundance *is NP-hard on planar cubic graphs.*

4 A Special Flavour of Minimax/Maximin Problems

Half of the parameters in the domination chain can be defined as either, in case of minimax problems, looking for the smallest of all (inclusion-wise) maximal vertex sets with a certain property ($i(G)$ is the size of the smallest maximal independent set; similarly, ir(G) is defined), or, in case of maximin problems, looking for the largest of all minimal vertex sets with a certain property ($\Gamma(G)$ is an example). Also, the complementary problems share this flavour; for instance, co $- i(G)$ can be seen as looking for the largest of all minimal vertex covers.

Typical exact algorithms for maximisation problems fix certain subsets to be part of the solution. In the decision variant, when a parameter value that lower-bounds the size of the solution is part of the input, we might have a

Table 2. Status of various problems related to the complementary domination chain

	co − ir	co − γ	co − i	τ	co − Γ	co − IR
apx-rat.	2 Obs.1	$\frac{240}{193}$ [4]	$\sqrt{n}$ [13]	2 (folklore)	4 [6]	4 Th.6
non-apx rat.	?	$\frac{260}{259}$ [41]	$n^{\frac{1}{2}-\varepsilon}$ [13]	2 (UGC) [38]	?	?
kernel	$2\ell - 1$ [11]	$\frac{5}{3}\ell + 3$ [24]	ℓ^2 [30](Sec.4.3)	2ℓ [26]	$\ell^2 + \ell$ [6]	3ℓ [11]
FPT-$\mathcal{O}^*()$	3.841^ℓ [11]	2.0226^ℓ [24]	1.5874^ℓ [13]	1.2738^ℓ [18]	4.3077^ℓ [6]	2.8752^ℓ [11]
degree restrictions						
$3 \leq \Delta \leq d$	APX-C Co.2	APX-C [8]	1.5d-apx [13]	APX-C [42]	APX-C [6]	
dense	?	?	APX-C Th.9	APX-C [20]	APX-C Co.5	APX-C Th.8

sufficient number of vertices in our partial solution and now want to (rather immediately) announce that a sufficiently large solution exists. This is not a problem for determining $\alpha(G)$ or $\mathrm{IR}(G)$, but this may become problematic in the case of maximin problems. In the following we consider the extension-problem for the other two maximin problems related to the domination-chain: co $- i(G)$ and co $- \mathrm{ir}(G)$. The first one can formally be stated as follows:

Minimal Vertex Cover Extension
Input: A graph $G = (V, E)$, a set $S \subseteq V$.
Question: Does G possess a minimal vertex cover S' with $S' \supseteq S$?

Observe that this extension problem can also be seen as a kind of subset problem for independent sets by rephrasing the question to: Is there a maximal independent set S' for G with $S' \subseteq V - S$? In more general terms, one can view the extension-version of some maximin problem as exclusion-version of the complementary minimax problem.

Theorem 3. Minimal Vertex Cover Extension *is NP-hard even restricted to planar cubic graphs.*

Proof. Consider the following simple reduction from satisfiability: For a formula $c_1 \wedge \cdots \wedge c_m$ over variables $x_1, \ldots, x_n$, let $G = (V, E)$ be the graph with vertices $v_i, \bar{v}_i$ for every $i = 1, \ldots, n$ and $c_1, \ldots, c_m$ and edges connecting every clause with its literals and connecting v_i with $\bar{v}_i$ for every i. For this graph, the set $S = \{c_1, \ldots, c_m\}$ can be extended to a minimal vertex cover if and only if the formula $c_1 \wedge \cdots \wedge c_m$ is satisfiable. A more sophisticated construction yields a planar cubic graph G as input for Minimal Vertex Cover Extension. □

The maximin problem co$-$ir(G) can also be considered with respect to extension. Since complements of irredundant sets are rather uncomfortable, we describe this problem in terms of the complementary problem $\mathrm{ir}(G)$:

Minimal Co-Irredundant Extension
Input: A graph $G = (V, E)$, a set $S \subseteq V$.
Question: Does G possess a maximal irredundant set S' with $S' \subseteq V - S$?

Theorem 4. Minimal Co-Irredundant Extension *is NP-hard.*

5 Approximation Results

In this section, after studying the approximation on general graphs, we consider bounded degree graphs and cubic graphs.

Theorem 5. *For any $c > 0$, there is no $c\log(n)$-approximation for* Lower Irredundance *unless $NP \subseteq \mathrm{DTIME}(n^{\log\log n})$.*

For the little studied complement of Lower Irredundance we observe:

Observation 1. *For any graph G without isolated vertices one can compute a minimal dominating set of cardinality at most $\frac{n}{2}$ in polynomial time for an arbitrary spanning forest of G. The complement of this dominating set is consequently a 2-approximation for* Co-Lower Irredundance.

Using Lemma 3, one can use known exact or approximation algorithms for Minimum Vertex Cover and also results from parameterized approximation such as [15] to deduce:

Theorem 6. Co-Upper Irredundance *can be approximated with factor 4 in polynomial, factor 3 in $O^*(1.2738^{\tau(G)})$ and factor 2 in $O^*(1.2738^{\tau(G)})$ or $O^*(1.2002^n)$ time.*

There is a kind of methodology to link optimisation problems related to the domination chain to those related to the complementary domination chain, which can be stated as follows.

Theorem 7. *Assume that the optimisation problem associated to some graph parameter ζ of the domination chain is APX-hard on cubic graphs. Then, the optimisation problem associated to the complement problem of ζ is also APX-hard on cubic graphs.*

Proof. We claim that the reduction that acts as the identity on graph (instances) and complements solution sets is an L-reduction. Given a cubic graph $G = (V, E)$ of order n with $m = \frac{3}{2}n$ edges as an instance of the optimisation problem belonging to ζ (and also to the complement problem). Let us distinguish the two optima by writing $\mathrm{opt}_\zeta(G)$ and $\mathrm{opt}_{\mathrm{co}-\zeta}(G)$, respectively. Then, $\mathrm{opt}_{\mathrm{co}-\zeta}(G) = n - \mathrm{opt}_\zeta(G)$. Similarly, if S' is a solution to G in the complement problem, then $n - |S'|$ is the size of the solution $S := V \setminus S'$ of the original problem. Hence,

$$\left|\mathrm{opt}_\zeta(G) - |S|\right| = \left|(n - \mathrm{opt}_{\mathrm{co}-\zeta}(G)) - (n - |S'|)\right| = \left|\mathrm{opt}_{\mathrm{co}-\zeta} - |S'|\right| .$$

Moreover, as $\mathrm{ir}(G) \geq \frac{2n}{9}$ according to [23], which yields $\mathrm{opt}_\zeta(G) \geq \frac{2n}{9}$ by the domination chain,

$$\mathrm{opt}_{\mathrm{co}-\zeta}(G) \leq n \leq \frac{9}{2}\,\mathrm{opt}_\zeta(G),$$

which proves the claim. □

Theorem 3.3 in [2] shows that Minimum Domination, restricted to cubic graphs, is APX-hard. We can use Theorem 7 to immediately deduce:

Corollary 1. *The complement problem corresponding to* Minimum Domination *is APX-hard when restricted to cubic graph instances.*

This sharpens earlier results [8] that only considered the subcubic case.

Corollary 2. Lower Irredundance *restricted to cubic graphs is APX-hard. Similarly,* Co-Lower Irredundance *is APX-hard on cubic graphs.*

Proof. The reduction from Theorem 1 can be seen as an L-reduction from the APX-hard Minimum Domination problem on cubic graphs [2] to Lower Irredundance on cubic graphs. Observe that $\gamma(G) \geq \frac{n}{4}$ and $|E| = \frac{3}{2}n$ for any cubic graph G, which gives $\mathrm{ir}(G') = \gamma(G) + |E| \leq 7\gamma(G)$. Furthermore, any maximal irredundant set of cardinality val' for G' can be used to compute a dominating set for G of cardinality $val = val' - |E|$, which yields $val - \gamma(G) = val' - \mathrm{ir}(G')$. Together with Theorem 7 the result for Co-Lower Irredundance follows. □

The computations in the previous proof can be carried out completely analogously for Upper Irredundance and Co-Upper Irredundance.

Corollary 3. Upper Irredundance *is APX-hard on cubic graphs. Similarly,* Co-Upper Irredundance *is APX-hard on cubic graphs.*

Manlove's NP-hardness proof for Minimum Independent Domination on cubic planar graphs [39] turns out to be an L-reduction, so that with Theorem 7 we can conclude:

Corollary 4. Minimum Independent Domination *and* Maximum Minimal Vertex Cover *is APX-hard on cubic graphs.*

This improves on earlier results for Maximum Minimal Vertex Cover, for instance, the APX-hardness shown in [40] for graphs of maximum degree bounded by five.

6 Further Algorithmic Observations

Most of the previously collected results have been hardness results; here we complement some of them by simple algorithmic results.

Observation 2. *The approximation-results for* Upper Domination *restricted to graphs of bounded degree from* [6] *are based on Eq. 2 and the fact that every maximal independent set is an upper dominating set which is also true for* Upper Irredundance*. The approximation by a suitable independent set yields the same approximation-ratio here which especially means that* Upper Irredundance *can be approximated within factor at most* $\frac{6\Delta^2+2\Delta-3}{10\Delta}$ *for any graph* G *of bounded degree* Δ*.*

Observation 3. *With Brooks' Theorem one can always find an independent set of cardinality at least* $\frac{n}{\Delta}$ *for any graph* G *of bounded degree* Δ*. From a parameterised point of view, this immediately gives a* Δk*-kernel for* Maximum Independent Set*,* Upper Domination *and* Upper Irredundance *for the natural parameter* k *of these problems, since any bounded-degree graph with more than* Δk *vertices is a trivial "yes"-instance.*

Observation 4. *Bounded degree* Δ *implies* $\gamma \geq \frac{n}{\Delta+1}$*, which means that any greedy solution yields a* $(\Delta+1)$*-approximation for* Minimal Maximum Independent Set *(*$i(G)$ *in domination chain) and* Minimum Domination*. For standard parameterisation this also yields a* $(\Delta+1)k$ *kernel for these problems since graphs with more than* $(\Delta+1)k$ *vertices are trivial "no"-instances.*

Lower Irredundance is the only problem for which these consequences of bounded degree are less obvious. A more thorough investigation of lower irredundant sets in [23] yields the bound $\mathrm{ir}(G) \geq \frac{2n}{3\Delta}$.

Observation 5. *The bound from* [23] *implies that any greedy maximal irredundant set for a graph of bounded degree* Δ *is a* 1.5Δ*-approximation for* Lower Irredundance*. Parameterised by* $k = ir(G)$*, any graph with more than* $1.5\Delta k$ *vertices is a trivial "no"-instance which yields a* $1.5\Delta k$ *kernel.*

Notice that, although the kernel results indicated in the previous two observations look weak at first glance, they allow for lower bound results based on the assumption that $P \neq NP$ according to [17].

7 Consequences for Everywhere Dense Graphs

In [3], Arora et al. presented a unified framework for proving polynomial time approximation schemes for (average) dense graphs, mainly for Max Cut type problems, and for Min Bisection for everywhere dense graphs. Concerning the problems from the domination chain Minimum Vertex Cover and Minimum Domination were studied; in [20], Minimum Vertex Cover is proved APX-hard on everywhere dense graphs and in [32], it is proved that Minimum Domination is NP-hard on (average) dense graphs. We will show inapproximation results for more domination-chain problems on everywhere dense graphs. Interestingly, we can make use of our reductions for sparse (cubic) graphs:

Theorem 8. *For any* $\varepsilon > 0$*,* Upper Irredundance *and* Co-Upper Irredundance *are APX-hard for everywhere-*ε*-dense graphs.*

Proof. We construct an L-reduction from (Co-)Upper Irredundance on cubic graphs to (Co-)Upper Irredundance on everywhere-ε-dense graphs. Given a connected cubic graph $G = (V, E)$ on n vertices, we construct a dense graph G' by joining a clique C of $\lceil \frac{\varepsilon n - 3}{1-\varepsilon} \rceil$ new vertices to G. G' has minimum degree $\varepsilon n'$, where $n' = n + \lceil \frac{\varepsilon n-3}{1-\varepsilon} \rceil = \lceil \frac{\varepsilon n - 3 + n - \varepsilon n}{1-\varepsilon} \rceil = \lceil \frac{n-3}{1-\varepsilon} \rceil$ is the number of vertices of G'. Any vertex $v \in V$ has $3 + \lceil \frac{\varepsilon n-3}{1-\varepsilon} \rceil = \lceil \frac{\varepsilon n - 3 + 3 - 3\varepsilon}{1-\varepsilon} \rceil = \lceil \frac{\varepsilon(n-3)}{1-\varepsilon} \rceil$ many neighbours in G'. Any vertex in the added clique has an even higher degree if $n \geq 4$. As any maximal irredundant set of G' that contains a vertex of C is a singleton set, $\mathrm{opt}(G') = \mathrm{opt}(G)$ and, w.l.o.g., any maximum irredundant set in G' is a subset of V which makes it a maximal irredundant set of G.

For Co-Upper Irredundance, we have $\mathrm{opt}(G') = \mathrm{opt}(G) + \lceil \frac{\varepsilon n-3}{1-\varepsilon} \rceil$ and, given any solution S' in G', we can transform it into a new one containing all new vertices and some vertices from V. The set $S' \cap V$ is a solution for G. In a cubic graph, the optimum value of the complement of an upper irredundant set is at least $n/4$ using inequality (3) and the fact that $\tau(G) \geq n/2$ (as G is connected and non-trivial) and thus $\mathrm{opt}(G) \geq n/4$. Thus $\mathrm{opt}(G') \leq \mathrm{opt}(G) + \frac{\varepsilon n-3}{1-\varepsilon} \leq \mathrm{opt}(G) + \frac{4\varepsilon\, \mathrm{opt}(G)-3}{1-\varepsilon} \leq \frac{1+3\varepsilon}{1-\varepsilon}\, \mathrm{opt}(G)$. □

Observe that the arguments and the computations of the previous proof are also valid for Co-Upper Domination. Since it is also APX-hard on cubic graphs [6] we can conclude the same result. Almost the same reduction is an E-reduction when we start with a general instance for Upper Domination (just adding more vertices in order to be sure that G' is everywhere-ε-dense). Since Upper Domination is not $n^{1-\delta}$-approximable for any $\delta > 0$, if P $\neq$ NP on general graphs [6] we can conclude the same result for everywhere-dense graphs.

Corollary 5. *For any $\varepsilon > 0$,* Co-Upper Domination *is APX-hard and* Upper Domination *is not $n^{1-\delta}$-approximable for any $\delta > 0$, if P $\neq$ NP, for everywhere-ε-dense graphs.*

The inapproximability result from [45] with the above reduction yields:

Proposition 1. *For any $\varepsilon > 0$,* Maximum Independent Set *is not $n^{1-\delta}$-approximable for any $\delta > 0$, if P $\neq$ NP, for everywhere-ε-dense graphs.*[1]

Theorem 9. *For any $\varepsilon > 0$,* Maximum Minimal Vertex Cover *is APX-hard and* Minimum Maximal Independent Set *is not $n^{1-\delta}$-approximable for any $\delta > 0$, if P $\neq$ NP, for everywhere-ε-dense graphs.*

Proof. We give an E-reduction from Minimum Maximal Independent Set on general graphs to Minimum Maximal Independent Set on everywhere-ε-dense graphs. Consider for a graph G the family $\{G^j : j \in \mathbb{N}\}$, recursively defined by $G^0 := G$ and $G^{j+1} := G^j + G^j$ ("+" denotes graph join). If the order of G is n, the order of G^j is $2^j n$ for every $j \in \mathbb{N}$. Also every $v \in G^j$ has degree at least $n(2^j - 1)$ which means that G^j is $(1 - 1/2^j)$-dense. Let V be the vertices

[1] We were informed about this fact by Marek Karpiński.

of G and $V \cup V'$ be the vertices of $G + G$. For any independent set S of $G + G$ either $S \subseteq V$ or $S \subseteq V'$, which means that independent sets in $G + G$ always yield equivalent independent sets in G and hence $i(G) = i(G + G)$. Inductively, this argument implies $i(G) = i(G^j)$ for all $j \in \mathbb{N}$. For $j = \lceil \log_2(1/(1 - \varepsilon)) \rceil$, the graph G^j hence yields the aforementioned E-reduction since any independent set in G^j yields an independent set in G of the same size.

Starting with a cubic graph G, G^j yields an L-reduction from Maximum Minimal Vertex Cover on cubic graphs, which is APX-hard by Corollary 4, to Maximum Minimal Vertex Cover on everywhere-ε-dense graphs, since for cubic graphs $\text{co} - i(G) \geq \frac{n}{2}$ and hence $\text{co} - i(G^j) < 2^j n \leq 2^{j+1}\text{co} - i(G)$. □

8 Summary, Open Problems and Prospects

We have presented a sketch of the complexity landscape of the domination chain. As can be seen from our tables, the status of most combinatorial problems has now been solved. However, there are still several question marks in these tables, and also the positive (algorithmic) results implicitly always ask for possible improvements.

For the investigation of complexity aspects of graph parameters, chains of inequalities like the domination chain help to unify proofs, but also to find spots that have not been investigated yet. Also, the idea of looking at the complementary chain should work out in each case. An example of a similar chain of parameters is the Roman domination chain [16]. Most of what we know is concerning Roman domination and its complementary version, which is also called the differential of a graph; see [1,8–10].

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