

Algorithmic Aspects of Upper Domination: A Parameterised Perspective

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Abstract. This paper studies UPPER DOMINATION, *i.e.*, the problem of computing the maximum cardinality of a minimal dominating set in a graph, with a focus on parameterised complexity. Our main results include W[1]-hardness for UPPER DOMINATION, contrasting FPT membership for the parameterised dual CO-UPPER DOMINATION. The study of structural properties also yields some insight into UPPER TOTAL DOMINATION. We further consider graphs of bounded degree and derive upper and lower bounds for kernelisation.

1 Introduction

Domination, independence and irredundance are basic concepts in graph theory and most of the overall six respective minimisation and maximisation problems, which are related via the so-called domination chain (see [16]), are very well-studied. Especially for parameterised complexity, MINIMUM DOMINATION and MAXIMUM INDEPENDENT SET and their respective parameterised duals are sort of fundamental. With the exception of UPPER DOMINATION, all problems of the domination chain are known to be complete for either W[1] or W[2] while their corresponding parameterised dual is in FPT. This paper therefore studies the so far neglected parameter $\Gamma(G)$, which denotes the maximum cardinality of a minimal dominating set in G . More precisely, we discuss the following problems:

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UPPER DOMINATION**Input:** A graph $G = (V, E)$, a non-negative integer k .**Question:** Is $\Gamma(G) \geq k$?**CO-UPPER DOMINATION****Input:** A graph $G = (V, E)$, a non-negative integer ℓ .**Question:** Is $\Gamma(G) \geq |V| - \ell$?

Notice that CO-UPPER DOMINATION could be also addressed as MINIMUM MAXIMAL NONBLOCKER or as MINIMUM MAXIMAL STAR FOREST; see [1] for further discussion. From the perspective of classical complexity theory, both problems are trivially equivalent and were shown to be NP-complete quite some time ago [7]. Aside from this, very little is known, especially with respect to parameterised complexity. From this perspective, k and ℓ turn out to be the natural parameters, which turn them into dual problems in the parameterised complexity sense of this word. As we will only consider this natural parameterisation, we refrain from explicitly mentioning the parameter throughout this paper. Slightly abusing notation, we will therefore use the names UPPER DOMINATION and CO-UPPER DOMINATION to also refer to the parameterised problems.

In Sect. 2, we link minimal dominating sets to a decomposition of the vertex set that turns out to be a crucial tool for deriving our combinatorial and computational results. Section 3 then discusses properties of upper dominating sets from a parameterised point of view and reveals $W[1]$ -hardness for UPPER and UPPER TOTAL DOMINATION. Conversely, CO-UPPER DOMINATION is shown to be in FPT, which we prove by providing both a kernelisation and a branching algorithm. In Sect. 4, we consider graphs of bounded degree and derive kernelisations for UPPER and CO-UPPER DOMINATION for this restricted graph class. This section also includes an exact $O^*(1.3481^n)$ -algorithm for UPPER TOTAL DOMINATION restricted to subcubic graphs which builds on the decomposition derived in Sect. 2. We further discuss general questions of exact algorithms for UPPER DOMINATION, as well as some related questions for total domination variants (see [17]) in Sect. 5. For reasons of space, proofs and other details are omitted in this extended abstract.

Basic Notions. Throughout this paper, we only deal with undirected simple graphs $G = (V, E)$. The number of vertices $|V|$ is also known as the order of G . $N(v)$ denotes the open neighbourhood of v in a graph G , and $N[v]$ is the closed neighbourhood of v in G , i.e., $N[v] = N(v) \cup \{v\}$. These notions can be easily extended to vertex sets X , e.g., $N(X) = \bigcup_{x \in X} N(x)$. The cardinality of $N(v)$ is also known as the degree of v , denoted as $\deg(v)$. The maximum degree in a graph is written as Δ . A graph of maximum degree three is called subcubic.

Given a graph $G = (V, E)$, a subset S of V is a *dominating set* if every vertex $v \in V \setminus S$ has at least one neighbour in S , i.e., if $N[S] = V$. A dominating set is called *minimal* if no proper subset of it is a dominating set. Likewise, a vertex set I is *independent* if $N(I) \cap I = \emptyset$. An independent set is maximal if no proper superset is independent. In the following we use classical notations: $\alpha(G)$ denotes the cardinality of a maximum independent set in $G = (V, E)$ and $\tau(G) := |V| - \alpha(G)$ is the cardinality of a minimum vertex cover.

For any subset $S \subseteq V$ and $v \in S$ we define the private neighbourhood of v with respect to S as $pn(v, S) := N[v] \setminus N[S \setminus \{v\}]$. Any $w \in pn(v, S)$ is called a *private neighbour of v with respect to S* . If the set S is clear from the context, we will omit the “with respect to S ” part. A dominating set $S \subseteq V$ is minimal if and only if $|pn(v, S)| > 0$ for every $v \in S$. Observe that v can be a private neighbour of itself.

Parameterised Complexity. We mainly refer to the textbooks [8,10] in the area. Important notions that we will make use of include the parameterised complexity classes FPT, W[1] and W[2], parameterised reductions and kernelisation. In this area, it has also become customary not only to suppress constants (as in the O notation), but also even polynomial-factors, leading to the so-called O^* -notation.

2 Graph Decompositions for Minimal Dominating Sets

The following exposition is crucial for the development of the algorithms we derive in this paper and also for the general investigation of properties of minimal dominating sets. Any minimal dominating set D for a graph $G = (V, E)$ can be associated with a partition of the set of V into four sets F, I, P, O given by: $I := \{v \in D : v \in pn(v, D)\}$, $F := D \setminus I$, $P \in \{B \subseteq N(F) \setminus D : |pn(v, D) \cap B| = 1 \text{ for all } v \in F\}$ and $O := V \setminus (D \cup P)$. This representation is not necessarily unique since there might be different choices for the sets P and O , but for every partition of this kind, the following properties hold:

1. Every vertex $v \in F$ has at least one neighbour in F , called a friend.
2. The set I is an independent set in G .
3. The subgraph induced by the vertices $F \cup P$ has an edge cut set separating F and P that is, at the same time, a perfect matching; hence, P can serve as the set of **private neighbours** for F .
4. The neighbourhood of a vertex in I is always a subset of O , which are otherwise the **outsiders** (Fig. 1).

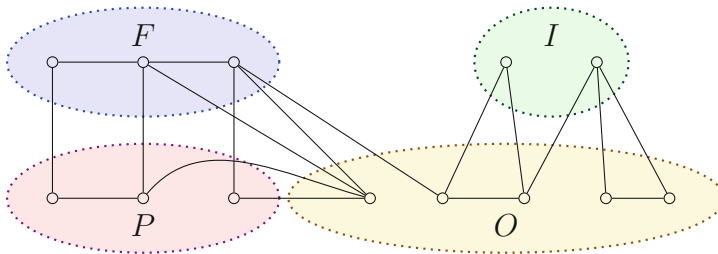
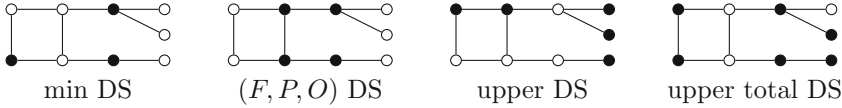


Fig. 1. Illustration of the FIPO structure imposed by minimal dominating sets

This partition is also related to a different characterisation of $\Gamma(G)$ in terms of so-called upper perfect neighbourhoods [16]. Observe two important special

cases of the partition (F, I, P, O) : If $F = \emptyset$, then I is an independent dominating set. If $I = \emptyset$, then F is a minimal total dominating set, *i.e.*, a set $S \subseteq V$ such that $V = N(S)$ and $N(S') \neq V$ for all $S' \subset S$. Both notions have been thoroughly studied in the literature. Observe that finding a maximum cardinality minimal dominating set for which $I = \emptyset$ holds in an (F, I, P, O) partitioning (called (F, P, O) -DOMINATION set in the following) is not equivalent to the problem UPPER TOTAL DOMINATION, which asks for a maximum cardinality minimal total dominating set. The following example illustrates the differences between optimal solutions (illustrated by the black vertices) for MINIMUM, (F, P, O) -, UPPER and UPPER TOTAL DOMINATION:



From the domination chain we know $\alpha(G) \leq \Gamma(G)$ for all graphs G , which is simply due to the fact that any maximal independent set is also a minimal dominating set. Considering the partition (F, I, P, O) for a minimal dominating set S for a graph G of order $n > 0$, we immediately know that $|I| \leq \alpha(G)$. Further, we know $|F| = |P|$ and hence $|F| = 1/2(n - |I| - |O|) \leq 1/2(n - \alpha(G))$. With $|S| = |F| + |I|$, we see that $|S| \leq 1/2(n + \alpha(G))$ and since this inequality holds for all minimal dominating sets S , we can conclude:

$$\alpha(G) \leq \Gamma(G) \leq \frac{n}{2} + \frac{\alpha(G)}{2} \tag{1}$$

3 Fixed Parameter Tractability

In this section we will investigate the fixed parameter tractability of UPPER DOMINATION, its dual and related problems. The problems MINIMUM DOMINATION, MINIMUM INDEPENDENT DOMINATION and MAXIMUM INDEPENDENT SET were among the first problems conjectured not to be in FPT [9]. In fact, aside from UPPER DOMINATION, all other problems from the domination chain are now known to be complete for either $W[1]$ or $W[2]$ (see [2, 11] for UPPER and LOWER IRREDUNDANCE respectively). It is perhaps not very surprising that UPPER DOMINATION is also unlikely to belong to FPT, and it looks rather unexpected that this question has been open for such a long time. We show that UPPER DOMINATION is $W[1]$ -hard by a reduction from MULTICOLOURED CLIQUE, a problem introduced in [13, 20] to facilitate $W[1]$ -hardness proofs. While the construction used in our reduction itself is not very complicated, proving its correctness turns out to be quite complex and technical.

Theorem 1. UPPER DOMINATION is $W[1]$ -hard.

Proof. (Sketch) Let $G = (V, E)$ be a graph with k different colour-classes given by $V = V_1 \cup V_2 \cup \dots \cup V_k$. MULTICOLOURED CLIQUE asks if there exists a clique $C \subseteq V$ in G such that $|V_i \cap C| = 1$ for all $i = 1, \dots, k$. For this problem, one

can assume that each set V_i is an independent set in G , since edges between vertices of the same colour-class have no impact on the existence of a solution. MULTICOLOURED CLIQUE is known to be W[1]-complete, parameterised by k . We construct a graph $G' = (V', E')$ by: $V' := V \cup \{v_e : e \in E\}$ and

$$E' := \bigcup_{i=1}^k V_i \times V_i \cup \bigcup_{i=1}^k \bigcup_{j=1}^k \{(v_e, v_{e'}) : e, e' \in (V_i \times V_j) \cap E\} \\ \cup \bigcup_{i=1}^k \bigcup_{j=1}^k \{(v_{(u,w)}, x) : (u, w) \in (V_i \times V_j) \cap E, x \in ((V_i \cup V_j) \setminus \{u, w\})\}.$$

It can be shown that there exists a minimal dominating set S of cardinality $k + \frac{1}{2}(k^2 - k)$ for G' if and only if $|S \cap V_i| = 1$ for all $i = 1, \dots, k$ and $|S \cap V_{i,j}| = 1$ for all $i \neq j$, where $V_{i,j} := \{v_e : e \in E \cap (V_i \times V_j)\}$. With this property, it is easy to see that S is minimal if and only if $S \cap V$ is a clique in the original graph; observe that if S contains two vertices v_i and v_j from V_i and V_j , respectively, which are not adjacent in G , then these already dominate all vertices of $V_{i,j}$ in G' . Overall, it can be shown that G' has an upper dominating set of cardinality $k + \frac{1}{2}(k^2 - k)$ if and only if G is a “yes”-instance for MULTICOLOURED CLIQUE, which proves W[1]-hardness for UPPER DOMINATION, parameterised by $\Gamma(G')$. \square

We want to point out that the above reduction also works for the restriction of UPPER DOMINATION to solutions for which I is empty:

Corollary 1. *(F, P, O)-DOMINATION, that is the restriction of UPPER DOMINATION to solutions S such that $V = N(S)$, is W[1]-hard.*

This result means that if we consider somehow splitting the problem UPPER DOMINATION into the subproblems of computing the independent vertices I and (F, P, O)-DOMINATION, we end up with two W[1]-hard problems. Considering UPPER TOTAL DOMINATION, the construction in the proof of Theorem 1 is not very helpful, since unfortunately any set S with $|S \cap V_i| = 1$ for all $i = 1, \dots, k$ and $|S \cap V_{i,j}| = 1$ for all $i \neq j$, regardless of the structure of the original graph G , is a minimal total dominating set for G' . We can however use a much simpler construction to show W[1]-hardness for UPPER TOTAL DOMINATION, a result which cannot be inferred from the known NP-hardness of the problem, see [12].

Theorem 2. UPPER TOTAL DOMINATION is W[1]-hard.

Proof. (Sketch) We reduce from MULTICOLOURED INDEPENDENT SET. Let $G = (V, E)$ be a graph with k different colour-classes given by $V = V_1 \cup V_2 \cup \dots \cup V_k$. We construct a graph $G' = (V', E')$ as follows: Starting from G , we add k vertices $C = \{c_1, \dots, c_k\}$ and turn each vertex set $V_j \cup \{c_j\}$ into a clique. We claim that G admits a multicoloured independent set (of size k) if and only if G' has a minimal total dominating set with $2k$ vertices. \square

We do not know if UPPER DOMINATION belongs to $W[1]$, but we can at least place it in $W[2]$, the next level of the W -hierarchy. We obtain this result by describing a suitable multi-tape Turing machine that solves this problem, see [4].

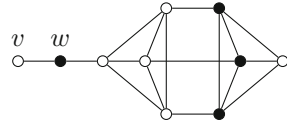
Proposition 1. UPPER DOMINATION belongs to $W[2]$.

Proof. Recall how MINIMUM DOMINATION can be seen to belong to $W[2]$ by providing an appropriate multi-tape Turing machine [4]. First, the k vertices that should belong to the dominating set are guessed, and then this guess is verified in k further (deterministic) steps using n further tapes in parallel, where n is the order of the input graph. We only need to make sure that the guessed set of vertices is minimal. To this end, we copy the guessed vertices k times, leaving one out each time, and we also guess one vertex for each of the $k - 1$ -element sets that is not dominated by this set. Such a guess can be tested in the same way as sketched before using parallel access to the $n + 1$ tapes. The whole computation takes $O(k^2)$ parallel steps of the Turing machine, which shows the claim. \square

Let us notice that very similar proofs also show membership in $W[2]$ and hardness for $W[1]$ for the question whether, given some hypergraph G and parameter k , there exists a minimal hitting set of G with at least k vertices. This also means that UPPER TOTAL DOMINATION belongs to $W[2]$.

In the context of parameterised complexity, we would like to point out another difference between UPPER DOMINATION and MINIMUM DOMINATION. Despite its $W[2]$ -hardness, there is at least a reduction-rule for MINIMUM DOMINATION, which deals with vertices of degree one, as they can be assumed not to be contained in a minimum dominating set. One might suspect that any upper dominating set would conversely always choose to contain degree-one vertices.

As the example on the right illustrates, there can not be such a rule for UPPER DOMINATION, since the degree-one vertex v is never part of a maximum solution; in fact, the black vertices form the unique optimal solution for this graph.



Another interesting question is to consider the dual parameter ℓ , that is to decide the existence of an upper dominating set of size at least $n - \ell$. This is in fact the natural parameterisation for CO-UPPER DOMINATION.

Theorem 3. CO-UPPER DOMINATION is in FPT. More precisely, it admits a kernel of at most $\ell^2 + \ell$ vertices and at most ℓ^2 edges.

Proof. Let $G = (V, E)$ be an input graph of order n . Consider a vertex $v \in V$ with $deg(v) > \ell$ and any minimal dominating set D with partition (F, I, P, O) :

- If $v \in I$, all neighbours of v have to be in O which means $|O| \geq |N(v)| > \ell$.
- If $v \in F$, exactly one neighbour p of v is in P and $N[v] \setminus \{p\} \subseteq F \cup O$, which gives $|O| + |P| = |O| + |F| \geq |N[v] \setminus \{p\}| > \ell$.
- If $v \in P$, exactly one neighbour p of v is in F and $N[v] \setminus \{p\} \subseteq P \cup O$, so $|O| + |P| > \ell$.

We always have either $v \in O$ or $|O| + |P| > \ell$, which means a “no”-instance for CO-UPPER DOMINATION. Consider the graph G' built from G by deleting the vertex v and all its edges. For any minimal dominating set D for G with partition (F, I, P, O) such that $v \in O$, D is also minimal for G' , since $pn(w, D) \supseteq \{w\}$ for all $w \in I$ and $|pn(u, D) \cap P| = 1$ for all $u \in F$. Also, any set $D' \subset V \setminus \{v\}$ which does not dominate v has a cardinality of at most $|V \setminus N[v]| < n - \ell$, so if G' has a dominating set D' of cardinality at least $n - \ell$, $N(v) \cap D' \neq \emptyset$; hence, D' is also dominating for G . These observations allow us to successively reduce (G, ℓ) to (G', ℓ') with $\ell' = \ell - 1$, as long as there are vertices v with $\deg(v) > \ell$. Any isolated vertex in the resulting graph G' originally only has neighbours in O which means it belongs to I in any dominating set D with partition (F, I, P, O) and can hence be deleted from G' without affecting the existence of an upper dominating set with $|P| + |O| \leq \ell'$.

Let (G', ℓ') be the instance obtained after the reduction above with $G' = (V', E')$ and let $n' = |V'|$. If there is an upper dominating set D for G' with $|D| \geq n' - \ell'$, any associated partition (F, I, P, O) for D satisfies $|P| + |O| \leq \ell'$. Since G' does not contain isolated vertices, every vertex in I has at least one neighbour in O . Also, any vertex in V' , and hence especially any vertex in O , has degree at most ℓ' , which means that $|I| \leq |N(O)| \leq \ell'|O|$. Overall:

$$|V'| \leq |I| + |F| + |P| + |O| \leq (\ell' + 1)|O| + 2|P| \leq \max_{j=0}^{\ell'} \{j(\ell' + 1), 2(\ell' - j)\},$$

and hence $|V'| \leq \ell'(\ell' + 1)$, or (G', ℓ') and consequently (G, ℓ) is a “no”-instance. Concerning the number of edges, we can derive a similar estimate. There are at most ℓ edges incident with each vertex in O . In addition, there is exactly one edge incident with each vertex in P that has not yet been accounted for, and, in addition, there could be $\ell - 1$ edges incident to each vertex in F that have not yet been counted. This shows the claim. \square

We just derived a kernel result for CO-UPPER DOMINATION, in fact a kernel of quadratic size in terms of the number of vertices and edges. This poses the natural question if we can do better also with respect to the question whether the brute-force search we could perform on the quadratic kernel is the best we can do to solve CO-UPPER DOMINATION in FPT time.

Proposition 2. CO-UPPER DOMINATION can be solved in time $O^*(4.3077^\ell)$.

Proof. (Sketch) This result can be shown by designing a branching algorithm that takes a graph $G = (V, E)$ and a parameter ℓ as input. Due to space restriction, we only describe here the rough ideas without any proof. As in Sect. 2, to each graph $G = (V, E)$ and (partial) dominating set, we associate a partition (F, I, P, O) . We consider $\kappa = \ell - (\frac{|F|}{2} + \frac{|P|}{2} + |O|)$ as a measure of the partition and for the running time of the algorithm. Note that $\kappa \leq \ell$. At each branching step, our algorithm picks some vertices from R (the set of yet undecided remaining vertices). They are either added to the current dominating set $D := F \cup I$ or to $\overline{D} := P \cup O$. Each time a vertex is added to P (resp. to O) the value of κ

decreases by $\frac{1}{2}$ (resp. by 1). Also, whenever a vertex x is added to F , the value of κ decreases by $\frac{1}{2}$.

Let us describe the two halting rules. First, whenever κ reaches zero, we are facing a “no”-instance. Then, if the set R of undecided vertices is empty, we check whether the current domination set D is minimal and of size at least $n - \ell$, and if so, the instance is a “yes”-instance. Then, we have a simple reduction rule: whenever the neighbourhood of an undecided vertex $v \in R$ is included in \overline{D} , we can safely add v to I . Finally, vertices are placed to F , I or \overline{D} according to three branching rules. The first one considers undecided vertices with a neighbour already in F (in such a case, v cannot belong to I). The second one considers undecided vertices with only one undecided neighbour (in such a case, several cases may be discarded as, *e.g.*, they cannot be both in I or both in \overline{D}). The third branching rule considers all the possibilities for an undecided vertex and due to the previous branching rules, it can be assumed that each undecided vertex has at least two undecided neighbours (which is nice since such vertices have to belong to \overline{D} whenever an undecided neighbour is added to I). \square

Of course, the question remains to what extent the previously presented parameterised algorithm can be improved on. In this context, we briefly discuss the issue of (parameterised) approximation for this parameter.

Theorem 4. CO-UPPER DOMINATION is 4-approximable in polynomial time 3-approximable with a running time in $O^*(1.0883^{\tau(G)})$ and 2-approximable in time $O^*(1.2738^{\tau(G)})$ or $O^*(1.2132^n)$.

Proof. First of all, observe by subtracting n from Eq. (1) that $\tau(G)$ relates to the co-upper domination number in the following way:

$$\frac{\tau(G)}{2} + 1 \leq n - \Gamma(G) \leq \tau(G) \quad (2)$$

Using any 2-approximation algorithm one can compute a vertex cover V' for G , and define $S' = V \setminus V'$. Let S be a maximal independent set containing S' . $V \setminus S$ is a vertex cover of size $|V \setminus S| \leq |V'| \leq 2\tau(G) \leq 4(n - \Gamma(G))$. Moreover, S is maximal independent and hence minimal dominating set which makes $V \setminus S$ a feasible solution for CO-UPPER DOMINATION with $|V \setminus S| \leq 4(n - \Gamma(G))$. The claimed running time for the factor-2 approximation stems from the best parameterised and exact algorithms for MINIMUM VERTEX COVER by [6] and [19], the factor-3 approximation from the parameterised approximation in [3]. \square

4 Graphs of Bounded Degree

In contrast to the case of general graphs, UPPER DOMINATION turns out to be easy (in the sense of parameterised complexity) for graphs of bounded degree.

Proposition 3. Fix $\Delta > 2$. UPPER DOMINATION has a problem kernel with at most Δk many vertices.

Proof. First, we can assume that the input graph G is connected, as otherwise we can apply the following argument separately on each connected component. Assume G is a cycle or a clique. Then, the problem UPPER DOMINATION can be optimally solved in polynomial time, *i.e.*, we can produce a kernel as small as we want. Otherwise, Brooks' Theorem yields a polynomial-time algorithm that produces a proper colouring of G with (at most) Δ many colours. Extend the biggest colour class to a maximal independent set I of G . As I is maximal, it is also a minimal dominating set. So, there is a minimal dominating set I of size at least n/Δ , where n is the order of G . So, $\Gamma(G) \geq n/\Delta$. If $k < n/\Delta$, we can therefore immediately answer YES. In the other case, $n \leq \Delta k$ as claimed. \square

With some more combinatorial effort, we obtain:

Proposition 4. *Fix $\Delta > 2$. CO-UPPER DOMINATION has a problem kernel with at most $(\Delta + 0.5)\ell$ many vertices.*

Proof. Consider any graph $G = (V, E)$. For any partition (F, I, P, O) corresponding to an upper dominating set $D = I \cup F$ for G , isolated vertices in G always belong to I and can hence be deleted in any instance of CO-UPPER DOMINATION without changing ℓ . For any graph G without isolated vertices, the set $P \cup O$ is a dominating set for G , since $\emptyset \neq N(v) \subset O$ for all $v \in I$ and $N(v) \cap P \neq \emptyset$ for all $v \in F$. Maximum degree Δ hence immediately implies $n = |N[P \cup O]| \leq (\Delta + 1)\ell$.

Since any connected component can be solved separately, we can assume that G is connected. For any $v \in P$, the structure of the partition (F, I, P, O) yields $|N[v] \cap D| = 1$, so either $|N[v]| = 1 < \Delta$ or there is at least one $w \in P \cup O$ such that $N[v] \cap N[w] \neq \emptyset$. For any $v \in O$, if $N[v] \cap F \neq \emptyset$, the F -vertex in this intersection has a neighbour $w \in P$, which means $N[w] \cap N[v] \neq \emptyset$. If $N(v) \subseteq I$ and $N[v] \neq V$, at least one of the I -vertices in $N(v)$ has to have another neighbour to connect to the rest of the graph. Since $N(I) \subseteq O$, this also implies the existence of a vertex $w \in O$, $w \neq v$ with $N[w] \cap N[v] \neq \emptyset$. Finally, if $N[v] \not\subseteq I \cup F$, there is obviously a $w \in P \cup O$, $w \neq v$ with $N[w] \cap N[v] \neq \emptyset$.

Assume that there is an upper dominating set with partition (F, I, P, O) such that $|P \cup O| = l \leq \ell$ and let v_1, \dots, v_l be the $l > 1$ vertices in $P \cup O$. By the above argued domination-property of $P \cup O$, we have:

$$n = \left| \bigcup_{i=1}^l N[v_i] \right| = \frac{1}{2} \sum_{i=1}^l |N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]| + \frac{1}{2} \sum_{i=1}^l |N[v_i] \setminus \bigcup_{j=i+1}^l N[v_j]|$$

Further, by the above argument about neighbourhoods of vertices in $P \cup O$, maximum degree Δ yields for every $i \in \{1, \dots, l\}$ either $|N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]| \leq \Delta$ or $|N[v_i] \setminus \bigcup_{j=i+1}^l N[v_j]| \leq \Delta$ which gives:

$$n = \frac{1}{2} \sum_{i=1}^l |N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]| + |N[v_i] \setminus \bigcup_{j=i+1}^l N[v_j]| \leq \frac{1}{2} l(2\Delta + 1) \leq (\Delta + 0.5)\ell.$$

Any graph with more than $(\Delta + 0.5)\ell$ vertices is consequently a “no”-instance which yields the stated kernelisation, as the excluded case $|P \cup O| = 1$ (or in other words $N[v] = V$ for some $v \in O$) can be solved trivially. \square

This implies that we have a $3k$ -size vertex kernel for UPPER DOMINATION, restricted to subcubic graphs, and a 3.5ℓ -size vertex kernel for CO-UPPER DOMINATION, again restricted to subcubic graphs. With [5, Theorem 3.1], we can conclude the following consequence:

Corollary 2. *Unless P equals NP , for any $\varepsilon > 0$, UPPER DOMINATION, restricted to subcubic graphs, does not admit a kernel with less than $(1.4 - \varepsilon)k$ vertices; neither does CO-UPPER DOMINATION, restricted to subcubic graphs, admit a kernel with less than $(1.5 - \varepsilon)\ell$ vertices.*

Exact Algorithms

Let us recall one important result on the pathwidth of subcubic graphs from [15].

Theorem 5. *Let $\epsilon > 0$ be given. For any subcubic graph G of order $n > n_\epsilon$, a path decomposition proving $\text{pw}(G) \leq n/6 + \epsilon$ is computable in polynomial time.*

This result immediately gives an $O^*(1.2010^n)$ -algorithm for solving MINIMUM DOMINATION on subcubic graphs. We will take a similar route to prove moderately exponential-time algorithms for UPPER DOMINATION.

Proposition 5. *UPPER DOMINATION on graphs of pathwidth p can be solved in time $O^*(7^p)$, given a corresponding path decomposition.*

We are considering all partitions of each bag of the path decomposition into 6 sets: F, F^*, I, P, O, O^* , where

- F is the set of vertices that belong to the upper dominating set and have already been matched to a private neighbour;
- F^* is the set of vertices that belong to the upper dominating set and still need to be matched to a private neighbour;
- I is the set of vertices that belong to the upper dominating set and is independent in the graph induced by the upper dominating set;
- P is the set of private neighbours that are already matched to vertices in the upper dominating set;
- O is the set of vertices that are not belonging neither to the upper dominating set nor to the set of private neighbours but are already dominated;
- O^* is the set of vertices not belonging to the upper dominating set that have not been dominated yet.

The upper bound on the running time can be improved for graphs of a certain maximum degree to $O^*(6^p)$ so that we can conclude:

Corollary 3. *UPPER DOMINATION on subcubic graphs of order n can be solved in time $O^*(1.3481^n)$, using the same amount of space.*

We like to point out that the idea from the pathwidth algorithm above can be adapted to work for treewidth.

Proposition 6. *UPPER DOMINATION on graphs of treewidth p can be solved in time $O^*(11^p)$, given a corresponding nice tree decomposition.*

5 Discussions and Open Problems

The motivation to study UPPER DOMINATION (at least for some of the authors) was based on the following observation based on enumeration; see [14].

Proposition 7. UPPER DOMINATION can be solved in time $O^*(1.7159^n)$ on general graphs of order n .

It is of course a bit nagging that there seems to be no better algorithm (analysis) than this enumeration algorithm for UPPER DOMINATION. Recall that the minimisation counterpart can be solved in better than $O^*(1.5^n)$ time [18, 21]. As this appears to be quite a tough problem, it makes a lot of sense to study it on restricted graph classes. This is what we did above for subcubic graphs, see Corollary 3. We summarise some open problems.

- Is UPPER DOMINATION in $W[1]$? Or, hard for $W[2]$?
- Can we improve on the 4-approximation of CO-UPPER DOMINATION?
- Can we find smaller kernels for UPPER or CO-UPPER DOMINATION on degree-bounded graphs?
- Can we find exact (*e.g.*, branching) algorithms that beat the enumeration or pathwidth-based ones for UPPER DOMINATION, at least on cubic graphs?

Also for UPPER TOTAL DOMINATION, the best exact algorithm seems to be based on enumeration. The $O^*(1.7159^n)$ bound from [14] is achieved by a branching algorithm that enumerates all minimal set covers of an instance $(\mathcal{U}, \mathcal{S})$, where \mathcal{S} is a collection of subsets over a universe \mathcal{U} and then uses a simple reduction from a dominating set instance to a set cover instance. It is implicit from the analysis (see Sect. 4 of [14]) that a SET COVER instance has at most $1.156154^{|\mathcal{U}|+2.720886|\mathcal{S}|}$ minimal set covers which can be enumerated in time $O^*(1.156154^{|\mathcal{U}|+2.720886|\mathcal{S}|})$. As an easy consequence, minimal total dominating sets of a graph $G = (V, E)$ can be enumerated in time $O^*(1.7159^n)$, by picking as the universe $\mathcal{U} = V$ and $\mathcal{S} = \{N(v) : v \in V\}$. This allows to conclude that UPPER TOTAL DOMINATION can be solved in the same time. Similarities to UPPER DOMINATION continue to some extent; however, the general picture is not very clear and still needs some research.

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References

1. Abu-Khzam, F.N., Bazgan, C., Chopin, M., Fernau, H.: Data reductions and combinatorial bounds for improved approximation algorithms. *J. Comput. Syst. Sci.* **82**(3), 503–520 (2016)
2. Binkele-Raible, D., Brankovic, L., Cygan, M., Fernau, H., Kneis, J., Kratsch, D., Langer, A., Liedloff, M., Pilipczuk, M., Rossmanith, P., Wojtaszczyk, J.O.: Breaking the 2^n -barrier for irredundance: two lines of attack. *J. Discrete Algorithms* **9**, 214–230 (2011)

3. Brankovic, L., Fernau, H.: A novel parameterised approximation algorithm for minimum vertex cover. *Theor. Comput. Sci.* **511**, 85–108 (2013)
4. Cesati, M.: The turing way to parameterized complexity. *J. Comput. Syst. Sci.* **67**, 654–685 (2003)
5. Chen, J., Fernau, H., Kanj, I.A., Xia, G.: Parametric duality and kernelization: lower bounds and upper bounds on kernel size. *SIAM J. Comput.* **37**, 1077–1108 (2007)
6. Chen, J., Kanj, I.A., Xia, G.: Improved upper bounds for vertex cover. *Theor. Comput. Sci.* **411**(40–42), 3736–3756 (2010)
7. Cheston, G.A., Fricke, G., Hedetniemi, S.T., Jacobs, D.P.: On the computational complexity of upper fractional domination. *Discrete Appl. Math.* **27**(3), 195–207 (1990)
8. Cygan, M., Fomin, F., Kowalik, L., Lokshtanov, D., Marx, D., Pilipczuk, M., Pilipczuk, M., Saurabh, S.: *Parameterized Algorithms*. Springer, Switzerland (2015)
9. Downey, R.G., Fellows, M.R.: Fixed parameter tractability and completeness. *Congressus Numerantium* **87**, 161–187 (1992)
10. Downey, R.G., Fellows, M.R.: *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, London (2013)
11. Downey, R.G., Fellows, M.R., Raman, V.: The complexity of irredundant set parameterized by size. *Discrete Appl. Math.* **100**, 155–167 (2000)
12. Fang, Q.: On the computational complexity of upper total domination. *Discrete Appl. Math.* **136**(1), 13–22 (2004)
13. Fellows, M.R., Hermelin, D., Rosamond, F., Vialette, S.: On the parameterized complexity of multiple-interval graph problems. *Theor. Comput. Sci.* **410**(1), 53–61 (2009)
14. Fomin, F.V., Grandoni, F., Pyatkin, A.V., Stepanov, A.A.: Combinatorial bounds via measure and conquer: bounding minimal dominating sets and applications. *ACM Trans. Algorithms* **5**(1), 1–17 (2008)
15. Fomin, F.V., Høie, K.: Pathwidth of cubic graphs and exact algorithms. *Inf. Process. Lett.* **97**, 191–196 (2006)
16. Haynes, T.W., Hedetniemi, S.T., Slater, P.J.: *Fundamentals of Domination in Graphs*. Monographs and Textbooks in Pure and Applied Mathematics, vol. 208. Marcel Dekker, New York (1998)
17. Hennings, M., Yeo, A.: *Total Domination in Graphs*. Springer, New York (2013)
18. Iwata, Y.: A faster algorithm for dominating set analyzed by the potential method. In: Marx, D., Rossmanith, P. (eds.) *IPEC 2011*. LNCS, vol. 7112, pp. 41–54. Springer, Heidelberg (2012)
19. Kneis, J., Langer, A., Rossmanith, P.: A fine-grained analysis of a simple independent set algorithm. In: Kannan, R., Narayan Kumar, K. (eds.) *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science*. LIPIcs, FSTTCS 2009, vol. 4, pp. 287–298. Schloss Dagstuhl – Leibniz-Zentrum für Informatik (2009)
20. Pietrzak, K.: On the parameterized complexity of the fixed alphabet shortest common supersequence and longest common subsequence problems. *J. Comput. Syst. Sci.* **67**(4), 757–771 (2003)
21. van Rooij, J.M.M., Bodlaender, H.L.: Exact algorithms for dominating set. *Discrete Appl. Math.* **159**(17), 2147–2164 (2011)