Algorithmic Extensions of Dirac's Theorem*†

Fedor V. Fomin‡ fomin@ii.uib.no

Petr A. Golovach‡ petr.golovach@ii.uib.no

Danil Sagunov§¶ danilka.pro@gmail.com

Kirill Simonov‖ kirillsimonov@gmail.com

Abstract

In 1952, Dirac proved the following theorem about long cycles in graphs with large minimum vertex degrees: Every n-vertex 2-connected graph G with minimum vertex degree $\delta \geq 2$ contains a cycle with at least $\min\{2\delta, n\}$ vertices. In particular, if $\delta \geq n/2$, then G is Hamiltonian. The proof of Dirac's theorem is constructive, and it yields an algorithm computing the corresponding cycle in polynomial time. The combinatorial bound of Dirac's theorem is tight in the following sense. There are 2-connected graphs that do not contain cycles of length more than $2\delta + 1$. Also, there are non-Hamiltonian graphs with all vertices but one of degree at least $n/2$. This prompts naturally to the following algorithmic questions. For $k \geq 1$,

(A) How difficult is to decide whether a 2-connected graph contains a cycle of length at least $\min\{2\delta + k, n\}$?

(B) How difficult is to decide whether a graph G is Hamiltonian, when at least $n - k$ vertices of G are of degrees at least $n/2 - k$?

The first question was asked by Fomin, Golovach, Lokshtanov, Panolan, Saurabh, and Zehavi. The second question is due to Jansen, Kozma, and Nederlof. Even for a very special case of $k = 1$, the existence of a polynomial-time algorithm deciding whether G contains a cycle of length at least $\min\{2\delta + 1, n\}$ was open. We resolve both questions by proving the following algorithmic generalization of Dirac's theorem: If all but k vertices of a 2-connected graph G are of degree at least δ, then deciding whether G has a cycle of length at least $\min\{2\delta + k, n\}$ can be done in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$.

The proof of the algorithmic generalization of Dirac's theorem builds on new graph-theoretical results that are interesting on their own.

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‡Department of Informatics, University of Bergen, Norway.

§St. Petersburg Department of V.A. Steklov Institute of Mathematics, Russia

¶JetBrains Research, Saint Petersburg, Russia

‖Algorithms and Complexity Group, TU Wien, Austria

1 Introduction

The fundamental theorem of Dirac from 1952 guarantees the existence of a Hamiltonian cycle in a graph with a large minimum vertex degree.

THEOREM 1.1. (DIRAC [DIR52, THEOREM 3]) *If every vertex of an n-vertex graph G is of degree at least $n/2$, then G is Hamiltonian, that is, contains a Hamiltonian cycle.*

Theorem 1.1 follows from a more general statement of Dirac about long cycles in a graph.

THEOREM 1.2. (DIRAC [DIR52, THEOREM 4]) *Every n-vertex 2-connected graph G with minimum vertex degree $\delta(G) \geq 2$, contains a cycle with at least $\min\{2\delta(G), n\}$ vertices.*

Both Dirac's theorems were the first instances of results that developed into one of the core areas in Extremal Graph Theory. One of the main questions in this research domain is to establish vertex degree characterization of Hamiltonian graphs and conditions enforcing long paths or cycles in graphs. The (very) incomplete list of results in this area includes the classical theorems of Erdős and Gallai [EG59], Ore [Ore60], Bondy and Chvátal [BC76], Pósa [P6́2], Meyniel [Mey73], and Bollobás and Brightwell [BB93], see also the Wikipedia entry on the Hamiltonian path.[1] The chapters of Bondy [Bon95] and Bollobás [Bol95] in the Handbook of Combinatorics, as well as Chapter 3 in the Extremal Graph Theory book [Bol78] provide excellent introduction to this important part of graph theory. The survey of Li [Li13] is a comprehensive (but a bit outdated) overview of the area. After almost 70 years, the field remains active, see for example the very recent proof of the Woodall's conjecture by Li and Nung [LN21].

Computing long cycles and paths is also an important topic in parameterized complexity. It served as a test-bed for developing several fundamental algorithmic techniques including the color coding of Alon, Yuster and Zwick [AYZ95], the algebraic approaches of Koutis and Williams [Kou08, Wil09], matroids-based methods [FLPS16], and the determinants-sum technique of Björklund from his FOCS 2010 Test of Time Award paper [Bjö14]. We refer to [FK13], [KW16], and [CFK$^+$15a, Chapter 10] for an overview of algorithmic ideas and techniques developed for computing long paths and cycles in graphs.

Despite the tremendous progress in graph-theoretical and algorithmic studies of longest cycles, all the developed tools do not answer the following natural and "innocent" question. By Theorem 1.2, deciding whether a 2-connected graph G contains a cycle of length at least $\min\{2\delta(G), n\}$ can be trivially done in polynomial time by checking degrees of all vertices in G.

> **Question 1:** Is there a polynomial time algorithm to decide whether a 2-connected graph G contains a cycle of length at least $\min\{2\delta(G) + 1, n\}$?

The methods developed in the extremal Hamiltonian graph theory do not answer this question. The combinatorial bound in Theorem 1.2 is known to be sharp; that is, there exist graphs that have no cycles of length at least $\min\{2\delta(G) + 1, n\}$. Since the extremal graph theory studies the existence of a cycle under certain conditions, such type of questions are beyond its applicability. The techniques of parameterized algorithms do not seem to be much of use here either. Such algorithms compute a cycle of length at least k in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$, which in our case is $2^{\mathcal{O}(\delta(G))} \cdot n^{\mathcal{O}(1)}$. Hence when $\delta(G)$ is, for example, at least $n^{1/100}$, these algorithms do not run in polynomial time.

Similarly, the existing methods do not answer the question about another "tiny algorithmic step" from Dirac's theorem, what happens when all vertices of G but one are of large degree?

> **Question 2:** Let v be a vertex of the minimum degree of a 2-connected graph G. Is there a polynomial time algorithm to decide whether G contains a cycle of length at least $\min\{2\delta(G - v), n\}$?

(We denote by $G - v$ the induced subgraph of G obtained by removing vertex v.) Note that graph $G - v$ is not necessarily 2-connected and we cannot apply Theorem 1.2 to it.

[1] https://en.wikipedia.org/wiki/Hamiltonian_path

The incapability of existing techniques to answer Questions 1 and 2 was the primary motivation for our work. We answer both questions affirmatively and in a much more general way. Our result implies that in polynomial time one can decide whether G contains a cycle of length at least $2\delta(G-B)+k$ for $B \subseteq V(G)$ and $k \geq 0$ as long as $k+|B| \in \mathcal{O}(\log n)$. (We denote by $G-B$ the induced subgraph of G obtained by removing vertices of B.) To state our result more precisely, we define the following problem.

LONG DIRAC CYCLE parameterized by $k+|B|$

Input:	Graph G with vertex set $B \subseteq V(G)$ and integer $k \geq 0$.
Task:	Decide whether G contains a cycle of length at least $\min\{2\delta(G-B), |V(G)|-|B|\}+k$.

In the definition of LONG DIRAC CYCLE we use the minimum of two values for the following reason. The question whether an n-vertex graph G contains a cycle of length at least $2\delta(G-B)+k$ is meaningful only for $\delta(G-B) \leq n/2$. Indeed, for $\delta(G-B) > n/2$, G does not contain a cycle of length at least $2\delta(G-B)+k > n$. However, even when $\delta(G-B) > n/2$, deciding whether G is Hamiltonian, is still very intriguing. By taking the minimum of the two values, we capture both interesting situations.

The main result of the paper is the following theorem providing an algorithmic generalization of Dirac's theorem.

THEOREM 1.3. (**Main Theorem**) *On an n-vertex 2-connected graph G,* LONG DIRAC CYCLE *is solvable in time* $2^{\mathcal{O}(k+|B|)} \cdot n^{\mathcal{O}(1)}$.

In other words, LONG DIRAC CYCLE is fixed-parameter tractable parameterized by $k+|B|$ and the dependence on the parameters is single-exponential. This dependence is asymptotically optimal up to the Exponential Time Hypothesis (ETH) of Impagliazzo, Paturi, and Zane [IPZ01]. Solving LONG DIRAC CYCLE in time $2^{o(k)} \cdot n^{\mathcal{O}(1)}$ even with $B=\emptyset$ yields recognizing in time $2^{o(n)}$ whether a graph is Hamiltonian. A subexponential algorithm deciding Hamiltonicity would fail ETH. We show that solving LONG DIRAC CYCLE in time $2^{o(|B|)} \cdot n^{\mathcal{O}(1)}$ even for $k=1$ would contradict ETH as well. It is also NP-complete to decide whether a 2-connected graph G has a cycle of length at least $(2+\varepsilon)\delta(G)$ for any $\varepsilon > 0$.

The 2-connectivity requirement in the statement of the theorem is important—without it LONG DIRAC CYCLE is already NP-complete for $k=|B|=0$. Indeed, for an n-vertex graph G construct a graph H by attaching to each vertex of G a clique of size $n/2$. Then H has a cycle of length at least $2\delta(H) \geq n$ if and only if G is Hamiltonian. However, when instead of a cycle we are looking for a long path, the 2-connectivity requirement could be omitted. More precisely, consider the following problem.

LONG DIRAC PATH parameterized by $k+|B|$

Input:	Graph G with vertex set $B \subseteq V(G)$ and integer $k \geq 0$.
Task:	Decide whether G contains a path of length at least $\min\{2\delta(G-B), |V(G)|-|B|-1\}+k$.

Theorem 1.3 yields the following.

COROLLARY 1.1. *On a connected n-vertex graph G,* LONG DIRAC PATH *is solvable in time* $2^{\mathcal{O}(k+|B|)} \cdot n^{\mathcal{O}(1)}$.

Indeed, when G is connected, the graph $G+v$, obtained by adding a vertex v and making it adjacent to all vertices of the graph, is 2-connected. The minimum vertex degree of $G+v$ is equal to $\delta(G)+1$, and G has a path of length at least t if and only if $G+v$ has a cycle of length at least $t+2$.

Theorem 1.3 answers several open questions from the literature. Fomin, Golovach, Lokshtanov, Panolan, Saurabh and Zehavi in [FGL$^+$20a] asked about the parameterized complexity of problems (with parameter k) where for a given (2-connected) graph G and $k \geq 1$, the task is to check whether G has a path (cycle) with at least $2\delta(G)+k$ vertices. By Theorem 1.3 and Corollary 1.1 (the case $B=\emptyset$), both problems are fixed-parameter tractable.

Jansen, Kozma, and Nederlof in [JKN19] conjectured that if at least $n-k$ vertices of graph G are of degree at least $n/2-k$, then deciding whether G contains a Hamiltonian cycle can be done in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$. Theorem 1.3

resolves this conjecture. Indeed, if G is Hamiltonian, it is 2-connected. Then let B, $|B| \leq k$, be the set of vertices such that every vertex from $V(G)\setminus B$ is of degree (in G) at least $n/2-k$. Then $\delta(G-B) \geq n/2-k-|B| \geq n/2-2k$. If $n-|B| \leq n-2k-2|B|$, we put $k' = |B|$, otherwise we put $k' = n-2\delta(G-B)$. Note that because $2\delta(G-B) \geq n-4k$, in both cases we have that $k' \leq 4k$. Also by the choice of k', $\min\{2\delta(G-B), n-|B|\}+k' = n$ and hence G has a cycle of length at least $\min\{2\delta(G-B), n-|B|\}+k'$ if and only if G is Hamiltonian. By Theorem 1.3, deciding whether G has a cycle of length at least $\min\{2\delta(G-B), n-|B|\}+k'$ can be done in time $2^{\mathcal{O}(k'+|B|)} \cdot n^{\mathcal{O}(1)} = 2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$. Interestingly, while the conjecture of Jansen, Kozma and Nederlof follows from the statement of Theorem 1.3, to prove the theorem, we need to resolve this conjecture directly.

We state Theorem 1.3 for the decision variant of the problem. However, the proof is constructive and the corresponding cycle can be found within the same running time. Note that standard self-reduction arguments are not applicable here because deleting or contracting edges could change the minimum vertex degree.

Related work. Until very recently, graph-theoretical and algorithmic studies of the longest paths and cycles coexisted in parallel universes without almost any visible interaction. In 1992, Häggkvist [H9̈2], as a corollary of his structural theorem, provided an algorithm that decides in time $n^{\mathcal{O}(k)}$ whether a graph with the minimum vertex degree at least $n/2-k$ is Hamiltonian. In 2019, Jansen, Kozma, and Nederlof in [JKN19] gave two algorithms of running times $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ that decide whether the input graph G is Hamiltonian when either the minimum vertex degree of G is at least $n/2-k$ or at least $n-k$ vertices of G are of degree at least $n/2$. The first result of Jansen, Kozma, and Nederlof strongly improves the algorithm of Häggkvist. However, the methods they use, like the structural theorem of Häggkvist [H9̈2], are specific for Hamiltonicity and are not applicable for the more general problem of computing the longest cycle. Second, their parameterized algorithms work only in one of the scenarios: either when all vertices are of degree at least $n/2-k$ or when at least $n-k$ vertices are of degree at least $n/2$. Whether both scenarios could be combined, that is, the existence of a parameterized algorithm deciding Hamiltonicity when $n-k$ vertices are of degree at least $n/2-k$, was left open.

Fomin, Golovach, Lokshtanov, Panolan, Saurabh and Zehavi in [FGL+20a] gave an algorithm that in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ decides whether a 2-connected graph G contains a cycle of length at least $d+k$, where d is the degeneracy of G. Since the minimum vertex degree $\delta(G)$ does not exceed the degeneracy of G, this result also implies an algorithm for finding a cycle of length at least $\delta(G)+k$ in 2-connected graphs.

None of the works [JKN19] and [FGL+20a] could be used to address Questions 1 and 2, the very special cases of Theorem 1.3.

More generally, Theorem 1.3 fits into a popular trend in parameterized complexity called "above guarantee" parameterization. The general idea of this paradigm is that the natural parameterization of, say, a maximization problem by the solution size is not satisfactory if there is a lower bound for the solution size that is sufficiently large. For example, there always exists a satisfying assignment that satisfies half of the clauses or there is always a max-cut containing at least half the edges. Thus nontrivial solutions occur only for the values of the parameter that are above the lower bound. This indicates that for such cases, it is more natural to parameterize the problem by the difference of the solution size and the bound. Since the work of Mahajan and Raman [MR99] on Max Sat and Max Cut, the above guarantee approach was successfully applied to various problems, see e.g. [AGK+10, CJM+13, GP16a, GKLM11, GvIMY12, GP16b, GRSY07, LNR+14, MRS09]. In particular, [BCDF19] and [FGL+20b] study the longest path above the shortest s,t-path and the girth of a graph.

Organization of the paper. This proceedings version of the paper contains the detailed overview of the proof of the main result, Theorem 1.3, given in the following section; and a conclusion. Due to its extensive volume, the complete formal proof of Theorem 1.3 and further discussion of results is deferred to the full version of the paper on arXiv [FGSS20].

2 Overview of the proof

The original proof of Dirac is not constructive because it does not provide any procedure for constructing a cycle of length at least $2\delta(G)$. There are algorithmic proofs of Dirac's theorem; see, e.g., the thesis of Locke [Loc83]. The idea of Locke's proof that also provides a polynomial-time algorithm for constructing a cycle of length at least $2\delta(G)$ is to start from some cycle and to grow by inserting new vertices and short paths. Thanks to the conditions on the graph's degrees, such a procedure always constructs a cycle of the required length. On a very general level, our proof of Theorem 1.3 uses the same strategy. For an instance (G, B, k) of Long Dirac Cycle, we try to grow a cycle iteratively. However, enlarging the cycle by "elementary" improvements could get stuck with a cycle of length significantly smaller than $\min\{2\delta(G-B), |V(G)|-|B|\}+k$. It appears that the cycles that cannot be improved by "elementary" operations induce a very particular structure in a graph. These structural theorems play a crucial role in our algorithm.

The main technical contribution is the new graph decomposition that we call Dirac decomposition. Dirac decomposition is defined for a cycle C in G. Let C be a cycle of length less than $2\delta(G-B)+k$. Informally, the components of Dirac decomposition are connected components in $G-V(C)$. (For an intuitive description of the decomposition, we will assume that $B=\emptyset$. Handling vertices of B requires more technicalities—we have to refine the graph and work with its refinement.) Since G is 2-connected, we can reach C by a path starting in such a component in G. One of the essential properties of Dirac decomposition is a limited number of vertices in $V(C)$ that have neighbors outside of C. In fact, we can choose two short paths P_1 and P_2 in C (and short means that their total length is of order k) such that all connections between connected components of $G-V(C)$ and C go through $V(P_1)\cup V(P_2)$. The second important property is that each connected component of $G-(V(P_1)\cup V(P_2))$ is connected with P_i in G in a very restricted way: The maximum matching size between its vertex set and the vertex set of P_i is at most one.

Dirac decomposition appears to be very useful for algorithmic purposes. For a cycle C and a vertex set B, given a Dirac decomposition for C and B, in time $2^{\mathcal{O}(k+|B|)}\cdot n^{\mathcal{O}(1)}$ we either solve the problem or succeed in enlarging C. We also provide an algorithm that either constructs a Dirac decomposition in polynomial time or obtains additional structural information that again can be used either to solve the problem or to enlarge the cycle. More precisely, first, we need to eliminate the "extremal" cases. When $\delta(G-B)\in\mathcal{O}(k)$, the classical result of Alon, Yuster, and Zwick [AYZ95] solves the problem in time $2^{\mathcal{O}(k+|B|)}\cdot n^{\mathcal{O}(1)}$. Another extremal case is when $|B|\le k$ and $\delta(G-B)\ge \frac{n}{2}-k$. In that case, for solving Long Dirac Cycle, we have to decide in time $2^{\mathcal{O}(k)}\cdot n^{\mathcal{O}(1)}$ whether G is almost Hamiltonian, i.e., a cycle in G that cover all but $\mathcal{O}(k)$ vertices. The existence of such an algorithm for Hamiltonian cycles was conjectured in [JKN19] and Theorem 2.4 settles this conjecture. We give an overview of the proof of Theorem 2.4 later in this section. If we are in none of the extremal cases, then in polynomial time we can either (a) enlarge the cycle C, or (b) compute a vertex cover of $G-B$ of size at most $\delta(G-B)+2k$, or (c) compute a Dirac decomposition. In cases (a) and (c), we can proceed iteratively. For the case (b) we give an algorithm that solves the problem in time $2^{\mathcal{O}(k+|B|)}\cdot n^{\mathcal{O}(1)}$ (Theorem 2.3).

The most critical and challenging component of the proof is the algorithmic properties of Dirac decomposition. We use these properties of Dirac decomposition to show that an enlargement of a cycle C of length at most $2\delta(G-B)+k-1$ can be done in a very particular way. By an extension of Dirac's existential theorem, we can assume that C is of length at least $2\delta(G-B)$. The most interesting and not-trivial situation that could occur is that for some vertices $x\in V(P_1)$ and $y\in V(P_2)$, we replace the shortest (x,y)-path in C by a detour with a particular property. This detour leaves x, moves to a vertex s of some 2-connected component of $G-V(C)$, visits some vertices in this component, leaves it from a vertex t, and goes to vertex y. Since the length of the longest (x,y)-path in C is at least $\delta(G-B)$, to decide whether such a detour exists, it is sufficient to solve the following problem. For vertices s,t of a 2-connected graph G, decide whether G contains an (s,t)-path of length at least $\delta(G-B)+k$. We give an algorithm solving this problem in time $2^{\mathcal{O}(k+|B|)}\cdot n^{\mathcal{O}(1)}$ (Theorem 2.2). The combinatorial bound that an (s,t)-path of length $\delta(G)$ always exists if G is 2-connected, is the classical theorem of Erdős and Gallai [EG59, Theorem 1.16]. Because of that, we name the problem of computing an (s,t)-path of length at least $\delta(G-B)+k$ by the Long Erdős-Gallai (s,t)-Path problem. Long Erdős-Gallai (s,t)-Path is an interesting problem on its own, and to prove Theorem 2.2, we use another structural result which we call Erdős-Gallai decomposition. Similar to Dirac decomposition, this decomposition is very useful from the algorithmic perspective. We define this decomposition, provide efficient algorithms for constructing it, and use it to solve Long Erdős-Gallai (s,t)-Path. Another interesting component of the solution to Long Erdős-Gallai (s,t)-Path is the algorithm for computing the longest cycle passing through two specified vertices (Theorem 2.1). We are not aware of the previous work in parameterized algorithms on this natural problem.

Figure 1 displays the most important steps of the proof and the dependencies between them. In the remaining part of this section, we highlight the ideas behind each of the auxiliary steps (Theorems 2.1, 2.2, 2.3, and 2.4) in the proofs of Theorem 1.3 and algorithmic properties of Dirac decomposition.

The first auxiliary problem whose solution we use in the proof of Theorem 2.2 is the following.

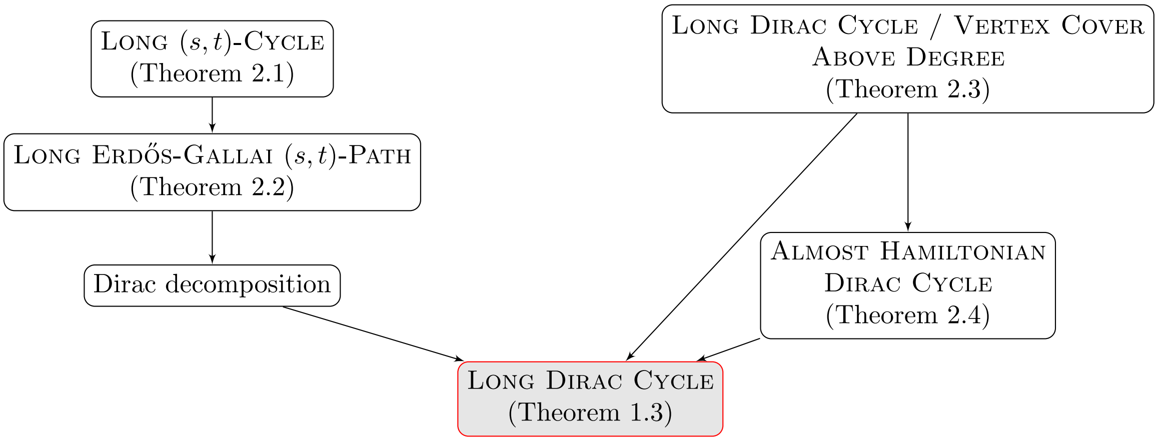

Figure 1: The main steps and connections in the proof of Theorem 1.3.

Long (s,t)-Cycle parameterized by k

Input: Graph G with two vertices $s, t \in V(G)$ and integer $k \geq 0$.
Task: Decide whether there is a cycle in G of length at least k that passes through s and t.

When $s \neq t$, an equivalent formulation is to decide whether G contains two internally disjoint (s,t)-paths of total length at least k. We prove that this problem is fixed-parameter tractable.

Theorem 2.1. Long (s,t)-Cycle *is solvable in time* $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$.

While the first idea to design an algorithm claimed in Theorem 2.1 would be to use the color coding technique of Alon, Yuster and Zwick [AYZ95], this idea does not work directly. The reason is that color coding can be used only to find in the claimed running time the cycle whose length is of order of k. However, it is quite possible that the lengths of all solutions are much larger than k; in such situation color coding cannot be applied directly. Our approach in proving Theorem 2.1 builds on ideas from [FLP+18, Zeh16], where a parameterized algorithms for finding a directed (s,t)-path and a directed cycle of length at least k were developed. The main idea of the proof is the following. First, we use color coding to verify whether the considered instance has a solution composed by two (s,t)-paths of total length at most $3k$. If the instance has a solution, we return it and stop. Otherwise, we conclude that the total length of the paths of every solution is at least $3k+1$. This allows to use structural properties of paths. Let P_1 and P_2 be the (s,t)-paths of a solution of minimum total length. Then there are vertices x_1 and x_2 on P_1 and P_2, respectively, such that (i) the total length of the (s,x_1)-subpath P_1' of P_1 and the (s,x_2)-subpath P_2' of P_2 is exactly k, (ii) either $x_1 = s$ or the length of the (x_1,t)-subpath P_1'' of P_1 is at least k, and, symmetrically, (iii) either $x_2 = s$ or the length of the (x_2,t)-subpath P_2'' of P_2 is at least k. Then P_1'' and P_2'' are internally disjoint paths that are shortest disjoint paths avoiding $V(P_1') \cup V(P_2') \setminus \{x_1, x_2\}$. We use the method of random separation to distinguish the following three sets: $V(P_1') \cup V(P_2) \setminus \{x_1, x_2\}$, the last $\min\{k, |V(P_1)| - 2\}$ internal vertices of P_1'', and the last $\min\{k, |V(P_2)| - 2\}$ internal vertices of P_2''. This allows to highlight the crucial parts of the shortest solution and then find a solution.

The second problem whose solution we use in the proof of Theorem 1.3, comes from another classical theorem due to Erdős and Gallai from [EG59, Theorem 1.16], see also [Loc85]. For every pair of vertices s, t of a 2-connected graph G, there is a path of length at least $\min_{v \in V(G)\setminus\{s,t\}} \deg v$. The proof of this result is constructive, and it implies a polynomial time algorithm that finds such a path. We define Long Erdős-Gallai (s,t)-Path as follows.

Long Erdős-Gallai (s,t)-Path parameterized by $k + |B|$

Input: Graph G with vertex set $B \subseteq V(G)$, two vertices $s, t \in V(G)$ and integer $k \geq 0$.
Task: Decide whether G contains an (s,t)-path of length at least $\delta(G - B) + k$.

Our following result plays an important role in the proof of the algorithmic properties of Dirac decomposition.

THEOREM 2.2. LONG ERDŐS-GALLAI (s,t)-PATH *is solvable in time* $2^{\mathcal{O}(k+|B|)} \cdot n^{\mathcal{O}(1)}$ *on* 2*-connected graphs.*

Similar to LONG DIRAC PATH, the requirement that the input graph is 2-connected is important. It is easy to prove that LONG ERDŐS-GALLAI (s,t)-PATH is NP-complete for $k = |B| = 0$ when the input graph is not 2-connected.

To prove Theorem 2.2, we apply the following strategy. We take an (s,t)-path P and try to extend it as much as possible. The principal tool in enlarging the path P is the proof of the extension of the theorem of Erdős and Gallai that takes into account the vertices of B. In the extremal case, when we cannot extend the path anymore, we obtain a graph decomposition whose properties become useful from the algorithmic perspective. We call this decomposition by the name of Erdős-Gallai decomposition and prove that, in that case, the graph can be decomposed in a very particular way. Roughly speaking, after a certain refinement of the graph, the (s,t)-path P consists of a prefix P_1 and a suffix P_2 with the following properties. These parts of the path are sufficiently far from each other in P. Moreover, all components of the graph $G - V(P_1 \cup P_2)$, we call them Erdős-Gallai component, are connected to P_1 and P_2 in a very restricted way. Such a graph-theoretical insight helps us to characterize how a long (s,t)-path traverses through an Erdős-Gallai component. This property allows us to design the recursive algorithm that proves the theorem.

The next auxiliary result required for proof of Theorem 1.3, concerns LONG DIRAC CYCLE parameterized by the vertex cover of a graph. It is well-known, see e.g., [CFK+15b], that a longest path in a graph G could be found in time $2^{\mathcal{O}(t)}n^{\mathcal{O}(1)}$, where t is the size of the minimum vertex cover of G. However, we need a much more refined result for the proof of the main theorem, where the parameter is not just the size of the vertex cover, but the difference between that size and $\delta(G-B)$. We define the following parameterized problem.

LONG DIRAC CYCLE / VERTEX COVER ABOVE DEGREE parameterized by $p + |B|$

Input:	Graph G with vertex set $B \subseteq V(G)$, vertex cover S of G of size $\delta(G-B)+p$ and integer $k \geq 0$.
Task:	Decide whether G contains a cycle of length at least $2\delta(G-B)+k$.

Part of our work is devoted to the proof of the following theorem, which we need for both Theorem 2.4 and Theorem 1.3.

THEOREM 2.3. LONG DIRAC CYCLE / VERTEX COVER ABOVE DEGREE *is solvable in* $2^{\mathcal{O}(p+|B|)} \cdot n^{\mathcal{O}(1)}$ *running time.*

To prove Theorem 2.3, we establish the new structural result. It reduces the crucial case of the problem about the long cycle to a particular path cover problem. This equivalence becomes very handy because we can use color-coding to compute the particular path cover, and thus by the lemma, to compute a long path. In spirit, this result is close to the classical theorem of Nash-Williams [NW71], stating that a 2-connected graph G with $\delta(G) \geq (n+2)/3$ is either Hamiltonian or contains an independent set of size $\delta(G)+1$. An extension of this theorem is due to Häggkvist [H9̈2], which was used by Jansen, Kozma and Nederlof [JKN19] in their algorithm for Hamiltonicity below Dirac's condition. In our case, we cannot use the structural theorem of Häggkvist as a black box, and build on the new graph-theoretic lemma instead.

The last ingredient we need to prove Theorem 1.3, is its special case when the minimum degree of $\delta(G-B)$ is nearly $\frac{n}{2}$. Specifically, the problem is defined as follows.

ALMOST HAMILTONIAN DIRAC CYCLE parameterized by k

Input:	Graph G, integer $k \geq 0$ and vertex set $B \subset V(G)$, such that $|B| \leq k$ and $\delta(G-B) \geq \frac{n}{2} - k$.
Task:	Find the longest cycle in G.

Observe that for a 2-connected graph, extended Dirac's theorem always gives a cycle of length $2\delta(G-B) \geq n-2k$. Thus it is more natural to state the problem in the form above, as the length of the longest cycle is necessarily between $n-2k$ and n, which is at most $2\delta(G-B)+2k$. In other words, we look for an almost Hamiltonian cycle, in a sense that it does not cover only $\mathcal{O}(k)$ vertices. Now we state our result for ALMOST HAMILTONIAN CYCLE.

THEOREM 2.4. *Let G be a given 2-connected graph on n vertices and let k be a given integer. Let $B \subseteq V(G)$ be such that $|B| \leq k$ and $\delta(G-B) \geq \frac{n}{2} - k$. There is a $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ running time algorithm that finds the longest cycle in G.*

The key obstacle for proving the theorem is the low-degree set B, since for empty B, we could simply apply the Nash-Williams theorem [NW71] and obtain either a Hamiltonian cycle or an independent set of size $\delta(G)+1$, and in the latter case use our result for LONG DIRAC CYCLE / VERTEX COVER ABOVE DEGREE. Assume there exists a Hamiltonian cycle in G (for almost Hamiltonian cycles the algorithm is similar), it induces a certain path cover of the vertices of B, where the endpoints of paths belong to $V(G) \setminus B$, and their total length is $\mathcal{O}(k)$. Such a path cover can be found by color-coding and dynamic programming in time $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$. Now either the rest of the graph is not 2-connected, and we have a $\mathcal{O}(k)$-sized separator, or we can apply the Nash-Williams theorem and obtain a cycle covering everything except the path cover, or a large independent set. The latter case is dealt with by Theorem 2.3, and for the case of the small separator we design a special algorithm that leverages the fact that the resulting components are very dense. So the main case is when the graph splits into a long cycle and the path cover. Now we crucially use that the paths in the path cover start and end outside of B, thus the endpoints of a path have high degree, each of them sees roughly half of the vertices of the long cycle. This makes it "hard" to not be able to insert the path somewhere in the cycle and make it longer. However, this last intuitive idea is achieved by a very intricate case analysis that constitutes the most of technical difficulty of the proof. Also, in some of the cases, we cannot make the cycle longer nor conclude that it is impossible, but instead we are able to find either a small separator or a large independent set. Again, we settle these cases by using the respective specialized algorithms.

3 Conclusion

In this paper, we developed an algorithmic extension of the classical theorem of Dirac. Our main result, Theorem 1.3, is that LONG DIRAC CYCLE is solvable in $2^{\mathcal{O}(k+|B|)} \cdot n^{\mathcal{O}(1)}$ time on 2-connected graphs. An important step in the proof of Theorem 1.3 is Theorem 2.2: LONG ERDŐS-GALLAI (s,t)-PATH is solvable in $2^{\mathcal{O}(k+|B|)} \cdot n^{\mathcal{O}(1)}$ time on 2-connected graphs. We conclude with open questions for further research.

3.1 Open questions. Dirac's theorem is the first fundamental result in Extremal Hamiltonian Graph Theory. The area contains many deep and interesting theorems but it remains largely unexplored from the algorithmic perspective. Here we present several open questions hoping that these questions would trigger further research in this fascinating area.

The first question is from [FGL+20a]. Recall that the *average degree* of a graph G is

$$\frac{1}{|V(G)|} \sum_{v \in V(G)} \deg_G(v) = 2|E(G)|/|V(G)|.$$

The following was shown by Erdős and Gallai [EG59].

PROPOSITION 3.1. ([EG59]) *Every graph G with average degree at least $d \geq 2$ has a cycle of length at least d.*

Similarly, it can be shown that a graph G with average degree at least d has a path of length at least d. This leads to the following question.

OPEN QUESTION 1. (**Path/cycle above average degree**) *Given a 2-connected (connected, respectively) graph G and a nonnegative integer k, how difficult is to decide whether G has a cycle (a path, respectively) of length at least $2|E(G)|/|V(G)|+k$?*

We do not know whether the problem is FPT parameterized by k, W[1]-hard, or Para-NP. Even the simplest variant of the question: whether a path of length $2|E(G)|/|V(G)|+1$ could be computed in polynomial time, is open.

Our second open question concerns the problem of finding a cycle containing a specified set of vertices. The study of this problem can be traced back to another fundamental theorem of Dirac from 1960s about the existence of a cycle in h-connected graph passing through a given set of h vertices [Dir60]. According to Kawarabayashi [Kaw08] *"...cycles through a vertex set or an edge set are one of central topics in all of graph theory."* Such type of problems have been a popular and important topic in algorithms as well. See, e.g., Björklund, Husfeldt and Taslaman [BHT12] and Wahlström [Wah13], and Kawarabayashi [Kaw08].

In Extremal Hamiltonian Graph Theory, the following theorem of Egawa, Glas, and Locke [EGL91] is well-known.

THEOREM 3.1. ([EGL91]) *Let G be an h-connected graph, $h \geq 2$, with minimum degree d, and at least $2d-1$ vertices. Let X be a set of h vertices of G. Then G has a cycle C of length at least $2d$ such that every vertex of X is on C.*

This brings us to the following algorithmic problem.

OPEN QUESTION 2. (**Cycle above Egawa, Glas, and Locke condition**) *Given an h-connected graph G, a set of vertices $X \subseteq V(G)$ of size h, and a nonnegative integer k, how difficult is to decide whether G has a cycle of length at least $2\delta(G)+k$ containing every vertex of X?*

As for Open Question 1 the question is open even for $k = 1$.

Finally, let us mention the area of directed graphs; we refer to the book of Bang-Jensen and Gutin [BG09] and the survey of Bermond and Thomassen [BT81] for extremal theorems for directed graphs. In particular, the classical result of Ghouila-Houri [GH60] from 1960, generalizes Theorem 1.1. Recall that a digraph D is *strong* if for every two vertices u and v, D has directed (u, v) and (v, u)-path, and the degree $\deg_D(v)$ of a vertex v is the sum of its *in-degree* $\deg^-_D(v)$ and *out-degree* $\deg^+_D(v)$.

THEOREM 3.2. ([GH60]) *If for every vertex v of a strong digraph D with n vertices $\deg_D(v) \geq n$, then D has a Hamiltonian cycle.*

The following question is the variant of the question discussed by Jansen, Kozma and Nederlof in [JKN19] for undirected graphs.

OPEN QUESTION 3. (**Cycle above Ghouila-Houri condition**) *Given an n-vertex strong digraph D and a nonnegative integer k such that at least $n-k$ vertices have degree at least n, how difficult is to decide whether D is Hamiltonian?*

Again, the simplest variant—whether there is a polynomial time algorithm for for $k = 1$—is open. We also do not know the complexity of the problem when every vertex has degree at least $n-k$.

A digraph D with at least two vertices is 2-*connected* if it is strong and remains strong after deleting an arbitrary vertex. Thomassen in [Tho81] proved the following analog of Theorem 1.2.

THEOREM 3.3. ([THO81]) *Let D be a 2-connected digraph with at least $2d+1$ vertices such that $\deg^-_D(v) \geq d$ and $\deg^+_D(v) \geq d$ for every $v \in V(D)$. Then D contains a cycle of length at least $2d$.*

Whether Thomassen's theorem can be extended algorithmically is our last open question.

OPEN QUESTION 4. (**Cycle above Thomassen condition**) *What is the (parameterized) complexity of the following problem. Given a 2-connected digraph D such that $\deg^-_D(v) \geq d$ and $\deg^+_D(v) \geq d$ for every $v \in V(D)$, and a nonnegative integer k. Decide whether D contains a cycle of length at least $2d+k$.*

As in Questions 1–3, even the existence of a polynomial time algorithm for $k = 1$ in Question 4 is open.

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