

Integer Plane Multiflow Maximisation

One-Quarter-Approximation and Gaps.

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Abstract In this paper, we bound the integrality gap and the approximation ratio for maximum plane multiflow problems and deduce bounds on the flow-cut-gap. We consider instances where the union of the supply and demand graphs is planar and prove that there exists a multiflow of value at least half the capacity of a minimum multicut. We then show how to convert any multiflow into a half-integer flow of value at least half the original multiflow. Finally, we round any half-integer multiflow into an integer multiflow, losing at most half the value thus providing a $1/4$ -approximation algorithm and integrality gap for maximum integer multiflows in the plane.

Keywords Multicommodity Flow · multiflow · multicut · Network Design · Planar Graphs · flow-cut, integrality gap · approximation algorithm.

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1 Introduction

Given an undirected graph $G = (V, E)$ with edge capacities $c : E \rightarrow \mathbb{R}^+$, and some pairs of vertices specified as edges of the graph $H = (V, F)$, the

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maximum-multiflow problem with input (G, H, c) , asks for the maximum flow that can be routed in G , simultaneously, between the endpoints of edges in F , while respecting the capacities c .

This is one of many widely studied variants of the multiflow problem. Other popular variants include demand flows, all or nothing flows, unsplittable flows etc. In this paper, we are primarily interested in the integer version of this problem, but would also be considering the half-integer and fractional versions. When capacities are 1, the capacity constraint specialises to edge-disjointness, whence the *maximum edge disjoint paths problem (MEDP)* between given pairs of vertices is a special case. MEDP is NP-Hard even for very restricted settings like when G is a tree [11].

The edges in F are called *demand edges* (sometimes commodities), those in E are called *supply edges*; accordingly, $H = (V, F)$ is the *demand graph*, and $G = (V, E)$ is the *supply graph*. If $G + H = (V, E \cup F)$ is planar we call the problem a *plane multiflow problem*. Plane multiflows have been studied for the past forty years, starting with Seymour [27]. A flow of maximum value can be computed in (strongly) polynomial time ([25], 70.6, page 1225) using linear programming. If the flow on every path is integer or half-integer, we say that the flow is integer or half-integer, respectively. Middendorf and Pfeiffer [21] showed that the problem of finding edge disjoint paths between endpoints of demand edges is NP-hard even in the special case when $G + H$ is planar, and this also implies NP-hardness of the problem of finding maximum integer flow in this special case.

A *multicut* for (G, H) is a set of edges $M \subseteq E$ whose removal disconnects the endpoints of every demand edge. An inclusion-wise minimal multicut defines a partition of V such that all demand edges have endpoints in different sets of the partition; the supply edges with endpoints in different sets of the partition form the multicut. The capacity of a multicut is the sum of capacities of edges in the multicut. It is easy to see that the value of any feasible multiflow is less than or equal to the capacity of any multicut. Klein, Mathieu and Zhou [16] prove that computing the minimum multicut is NP-hard if $G + H$ is planar, and they also provide a PTAS for this problem.

How large is the ratio of the minimum multicut to the maximum multiflow for instances in a given class? This question has been considered for many graph classes and in this paper we refer to the ratio as the *flow-cut gap*¹. The integer (resp. half-integer) flow-cut gap is the ratio of the minimum multicut to the maximum integer (resp. half-integer) multiflow. The integer flow-cut gap is 1 when G is a path and 2 when G is a tree. For arbitrary (G, H) , the flow-cut gap is $\theta(\log |F|)$ [10]. Building on decomposition theorems from Klein, Plotkin and Rao [15], Tardos and Vazirani [28] showed a flow-cut gap of $O(r^3)$ for graphs which do not contain a $K_{r,r}$ minor; note that for $r = 3$ this includes the class of planar graphs. A long line of impressive work, culminated in [26] proving a constant approximation ratio for maximum half-

¹ A reader familiar with the literature on multiflows with specified demands will note that the *flow-cut gap* usually refers to the ratio of the sparsest cut to the maximum fraction of demands that can be concurrently routed.

integer flows, which together with [28] implies a constant half-integer flow-cut gap for planar supply graphs. A simple topological obstruction proves that the integer flow-cut gap for planar supply graphs, even when all demand edges are on one face of the graph, also called *Okamura-Seymour instances*, is $\Omega(|F|)$ [11].

Besides flow-cut gaps, this paper also considers the (half)-integrality gap of multiflow problems. The (half)-integrality gap for an instance is the ratio of the maximum (half)-integer flow to the maximum fractional flow and the (half)-integrality gap for a class is the maximum (half)-integrality gap for any instance in that class. A ρ -approximation algorithm ($\rho \in \mathbb{R}$) for a maximisation (resp. minimisation) problem is a polynomial algorithm which outputs a solution of value at least (resp. at most) ρ times the optimum; ρ is also called the *approximation ratio* (or guarantee).

Our first result (Section 3, Theorem 1) is an upper bound of 2 for the flow-cut gap (i.e. multicut/multiflow ratio) for plane instances. We prove this by relating multicuts to 2-edge-connectivity-augmentation in the planar dual, and using the algorithm of Williamson, Goemans, Mihail and Vazirani [30] for this problem. We next show (Section 4, Theorem 3) how to obtain a half-integer flow from a given (fractional) flow in plane instances, by reducing the problem to a linear program with a particular combinatorial structure, and solving it in integers. Finally (Section 5, Theorem 4), given any feasible half-integer flow, we show how to extract an integer flow of value at least half of the original, in polynomial time, using the 4-color theorem for planar graphs [23].

Our results imply a half-integrality gap of $1/2$ and an integrality gap of $1/4$ for maximum multiflows in plane instances. These together with the flow-cut gap of 2 imply a half-integer flow-cut gap of 4 and an integer flow-cut gap of 8 for plane instances. Our proofs are constructive and lead to a 2-approximation algorithm for minimum multicut, a $1/2$ -approximation algorithm for maximum half-integer flows and a $1/4$ -approximation algorithm for maximum integer flows. In section 6 we provide an example which shows a lower bound of $3/2$ on the flow-cut gap and a lower bound of 2 on the half-integer flow-cut gap. Figure 1 provides a summary of our results.

Independent of this work Huang, Mari, Mathieu, Vygen [13] gave constant bounds on the flow-cut and integrality gaps for multiflows on plane instances. Although their bounds are not as sharp as the ones in this paper, they propose an interesting new rounding method, and prove a new complexity result (stated in the next section). In another paper Huang, Mari, Mathieu, Vygen [14] generalize our results to prove a $O(g^{3.5} \log g)$ integer flow-cut gap when $G + H$ can be embedded on an orientable surface of genus g .

Garg and Kumar [8] modified the primal-dual algorithm of Williamson et al. [30] to compute a multicut of capacity at most twice a half-integer flow. This improves the half-integer flow-cut gap to 2 and the integer flow-cut gap to 4 for plane instances. However, their results do not imply better bounds on the (half)-integrality gap.

2 Preliminaries

Let $(G, H, c), G = (V, E), H = (V, F), c : E \rightarrow \mathbb{R}^+$ be an instance of a plane multiflow maximisation problem. Let $\mathcal{P}_e (e \in F)$ be the set of *simple* paths in G between the endpoints of e , and $\mathcal{P} := \cup_{e \in F} \mathcal{P}_e$. For $P \in \mathcal{P}_e$, the edge e is said to be the *demand-edge* of P , denoted by e_P . A *multiflow*, or for simplicity a *flow* in this paper, is a function $f : \mathcal{P} \rightarrow \mathbb{R}^+$. For a path $P \in \mathcal{P}$, we refer to $f(P)$ as the *flow on P* . The flow f is called *feasible*, if $\sum_{\{P \in \mathcal{P} : e \in P\}} f(P) \leq c(e)$ for all $e \in E$. The *value* of a flow f is defined as $|f| := \sum_{P \in \mathcal{P}} f(P)$.

In this paper we are primarily concerned with finding a flow of maximum value and do not place any upper bound on the flow between endpoints of a particular demand edge e . Such an upper bound can be realized by replacing edge e with two edges in series - a demand edge (which we continue to call e) and a supply edge of capacity $d(e)$. In the instance so obtained the maximum flow between endpoints of e cannot exceed $d(e)$. We will use this transformation later in this paper and refer to it as *capping demands*. Middendorf and Pfeiffer [21] proved that finding edge disjoint paths between endpoints of demand edges in a plane instance is NP-hard. By setting all capacities to 1 and capping demands to 1, we obtain an instance of a plane multiflow maximisation problem which has an integer flow of value $|F|$ if and only if there are edge disjoint paths between endpoints of demand edges. This establishes NP-hardness of the integer plane multiflow maximization problem.

Demand flows and the Cut Condition An instance of the *demand flow* problem is defined by the quadruple (G, H, c, d) , where G, H, c are as before, *demands* $d : F \rightarrow \mathbb{Z}^+$ are given, and we are looking for a feasible (sometimes in addition integer or half-integer) flow f satisfying $\sum_{P \in \mathcal{P}_e} f(P) = d(e)$ for all $e \in F$.

In a graph $G = (V, E)$, for $S \subset V, E' \subset E$, we denote by $\delta_{E'}(S)$ the set of edges of E' with exactly one endpoint in S . A *cut* in G is a partition of the vertex set $(S, V \setminus S)$. Note that $\delta_E(S) = \delta_E(V \setminus S)$ are the edges in the cut and $S, V \setminus S$ are called the *shores* of the cut. We adopt the usual way of extending a function on single elements to subsets. For instance, $d(F') := \sum_{e \in F'} d(e)$ is the *demand* of the set $F' \subset F$.

A necessary condition for the existence of a feasible multiflow satisfying all demands is the so called *Cut Condition*: for every $S \subseteq V, c(\delta_E(S)) \geq d(\delta_F(S))$, that is, the capacity of each cut must be at least as large as its demand. The cut condition is not sufficient for a flow in general, but Seymour [27] showed that it is sufficient provided $G + H$ is planar. We call $G + H$ *Eulerian* when capacities and demands are integer, and their sum, on the edges incident to any vertex, is even. Seymour in fact showed that the cut condition is sufficient for a half-integer flow in plane instances with integer capacities and demands. The same also holds true for Okamura-Seymour instances [22]. Moreover, if $G + H$ is Eulerian then the cut condition is sufficient for the existence of integer multiflows in both of these cases. There are more examples, unrelated to planarity, where the cut condition is sufficient to satisfy all de-

mands, and with an integer flow, for instance when all demand edges can be covered by at most two vertices.

Planar Duality Following Schrijver [25, p. 27] let $(G + H)^* = (V^*, E^* \cup F^*)$ be the planar-dual of $G + H$. Note that V^* corresponds to faces in the planar embedding of $G + H$ and each edge $e \in E$ (resp. $f \in F$) corresponds to an edge $e^* \in E^*$ (resp. $f^* \in F^*$) joining the two faces that share e (resp. f). The *cost* of an edge in E^* equals the capacity of the corresponding edge in E ; we overload notation and let $c : E^* \rightarrow \mathbb{Z}^+$ be this cost function. Let r^* be the vertex in $(G + H)^*$ corresponding to the infinite face of the planar embedding of $G + H$.

For $X \subseteq E$ denote $X^* := \{e^* : e \in X\}$. A *circuit* is a connected subgraph with all vertices having degree two. An important fact about planar duality we use is that C is an inclusion-wise minimal cut in $G + H$ if and only if C^* is a circuit in $(G + H)^*$. This correspondence between cuts in a planar graph and circuits in the dual allows one to transform any fact on cuts to circuits in the dual and vice versa. For example fractional, half-integer or integer packings of cuts in $(G + H)^*$, where each cut contains exactly one edge of F^* correspond to a fractional, half-integer or integer multiflow in $G + H$.

Seymour's proof on the sufficiency of the cut condition for plane instances is based on a nice correspondence to other combinatorial problems through planar duality. The cut condition can be checked in polynomial time as it reduces to the problem of checking whether F^* is a minimum cost T -join (see eg. [25]) in $(G + H)^*$, where T is the set of odd degree vertices of F^* , edges in E^* have cost equal to the capacity of the corresponding edge in E , and edges of F^* have cost equal to the demand of the corresponding edge in F . However, checking if demands can be routed integrally is NP-hard [21].

It is tempting to reduce multiflow maximisation to demand flows and use the sufficiency of the cut condition to compute a maximum (half-)integer multiflow. Given a multiflow f assign every edge $e \in F^*$ a length equal to the negative of the flow between the endpoints of e . Let every edge in E^* be assigned a length equal to its capacity. Since f is feasible, all circuits in $(G + H)^*$ have non-negative length. Thus the problem of finding a maximum (half-)integer multiflow can be viewed as finding an assignment of negative lengths to edges in F^* of maximum total absolute value so that all circuits in $(G + H)^*$ have non-negative length. Note however, that since the Eulerian condition need not be satisfied, an integer solution to this problem only implies a half-integer multiflow². Further, finding such an integer solution when $G + H$ is planar is NP-hard as was proved recently [13].

Laminar Families and Flows Let \mathcal{S} be a collection of subsets of a ground set X . The family \mathcal{S} is *laminar* if for all $S_1, S_2 \in \mathcal{S}$, either $S_1 \cap S_2 = \emptyset$ or $S_1 \subset S_2$ or $S_2 \subset S_1$. If for all $S_1, S_2 \in \mathcal{C} \subseteq \mathcal{S}$ either $S_1 \subset S_2$ or $S_2 \subset S_1$ then \mathcal{C} is a *chain*.

² Let $G = C_4$ (a circuit on 4 vertices) and $H = 2K_2$ (complement of G) so $G + H = K_4$. The supply edges have capacity 1 and the demands are capped at 1. An integer solution which routes 1 unit of each demand can only route flow half-integrally.

A chain is *full* (in \mathcal{S}) if for all $X, Y, Z \in \mathcal{S}$, $X \subseteq Y \subseteq Z$ and $X, Z \in \mathcal{C}$ implies $Y \in \mathcal{C}$. Edmonds and Giles [4] showed that the sets of any laminar family can be represented as vertices of a rooted directed tree (arborescence) such that full chains correspond to directed paths.

With every path $P \in \mathcal{P}$ in G on which flow can be routed we associate a set of vertices $\phi(P) \subseteq V^* \setminus \{r^*\}$ defined as follows: if C is the circuit $P \cup \{e_P\}$ then $\phi(P)$ is the unique set of vertices in $V^* \setminus \{r^*\}$ such that $\delta(\phi(P)) = C^*$. Given a multiflow f in (G, H) , let $\mathcal{S}(f) = \{A \subseteq V^* \setminus \{r^*\}, f(\phi^{-1}(A)) > 0\}$. Thus, $\mathcal{S}(f)$ is the collection of subsets of $V^* \setminus \{r^*\}$, corresponding to paths $P \in \mathcal{P}$ such that $f(P) > 0$.

If $\mathcal{S}(f)$ is laminar then we say that multiflow f is *laminar*.

Lemma 1 *For every feasible multiflow f there exists a laminar feasible multiflow f' such that $|f'| = |f|$, and f' can be found in polynomial time.*

Proof Given a instance (G, H, c, d) of the demand flow problem where $G + H$ is planar, Matsumoto et al. [20] give a $O(n^{5/2} \log n)$ algorithm which computes a laminar flow satisfying all demands or shows that the cut condition is violated (Theorem 4, [20]). Given any feasible fractional flow f , for every demand edge we can compute the total flow routed between its endpoints and use the algorithm in [20] to compute a laminar flow, f' , in $O(n^{5/2} \log n)$ time. \square

3 Multi-Cuts and 2-Edge Connectivity Augmentation

In this section we show that computing a multicut when $G + H$ is planar is equivalent to solving a 2-edge connectivity augmentation problem. This equivalence allows us to prove that the the flow-cut gap is at most two for plane instances.

Given $G = (V, E)$, $H = (V, F)$, a *2-connector for H in G* is a set of edges $Q \subseteq E$ such that each $e \in F$ is contained in a circuit of $Q \cup F$. If $H = (V, F)$ contains a circuit, we can contract all edges of this circuit in both G, H to obtain a smaller equivalent problem. Hence, it is no loss of generality to assume that H is a forest.

The following lemma shows a one to one correspondence between a multicut in (G, H) and a 2-connector for (V^*, F^*) in (V^*, E^*) .

Lemma 2 *The edge-set $Q \subseteq E$ is a multicut for (G, H) if and only if Q^* is a 2-connector for (V^*, F^*) in (V^*, E^*) .*

Proof The edge-set $Q \subseteq E$ forms a multicut in G if and only if for every demand edge $(u, v) \in F$, the endpoints u, v are in different connected components of $(V, E \setminus Q)$. This implies that for all $(u, v) \in F$ there exists a inclusion-wise minimal set of edges $C \subseteq Q \cup F$ such that C is a $u - v$ cut in $G + H$. However, C is an inclusion-wise minimal cut in $G + H$ if and only if $C^* \subseteq Q^* \cup F^*$ is a circuit in $G^* + H^*$. Hence $Q \subseteq E$ forms a multicut in G , if and only if for

all $f^* \in F^*$ there exists a circuit C^* in $Q^* \cup F^*$ such that $f^* \in C^*$. This implies Q^* is a 2-connector for (V^*, F^*) , in (V^*, E^*) .

We now prove the converse. Let Q^* be a 2-connector for (V^*, F^*) in (V^*, E^*) . We show that the dual edges corresponding to Q^* , ie. Q , form a multicut in G . Consider an edge $f^* \in F^*$. Since Q^* is a 2-connector for (V^*, F^*) , there exists a circuit $C^* \subseteq Q^* \cup F^*$ containing f^* . The corresponding dual edges C form a cut separating the endpoints of f in $G + H$. This argument applies to each edge in F^* and shows that Q is a multicut for the instance (G, H) . \square

Given graphs $G = (V, E), H = (V, F)$ and edge costs $c : E \rightarrow \mathbb{R}^+$, the *2-edge-connectivity Augmentation Problem (2ECAP)* is to find a minimum cost 2-connector for H in G . The 2ECAP is a NP-hard network design problem and similar problems have been studied extensively. In a network design problem given graph $G = (V, E)$ with edge costs $c : E \rightarrow \mathbb{R}^+$, we find a minimum-cost subgraph such that the number of edges crossing each cut $(S, \bar{S}), S \subset V$ is at least a specified requirement, $r(S)$ where $r : 2^V \rightarrow \mathbb{Z}^+$ is the requirement function. A network design problem can be formulated as an integer program.

$$\begin{aligned} & \text{minimize} && \sum_{e \in E} c(e)x(e) \\ & \text{subject to} && \\ & \sum_{e \in \delta_E(S)} x(e) \geq r(S) && S \subseteq V \\ & x(e) \in \{0, 1\} && e \in E \end{aligned}$$

Relaxing the integrality constraint on $x(e)$ to $0 \leq x(e) \leq 1$ gives a linear program whose (LP)-dual is:

$$\begin{aligned} & \text{maximize} && \sum_{S \subseteq V} r(S)y(S) \\ & \text{subject to} && \\ & \sum_{S: e \in \delta_E(S)} y(S) \leq c(e) && e \in E \\ & y(S) \geq 0 && S \in V \end{aligned}$$

A requirement function $r : 2^V \rightarrow \{0, 1\}$ is called *uncrossable* if $r(V) = 0$ and for any $A, B \subseteq V$, $r(A) = r(B) = 1$ implies either $r(A \cap B) = r(A \cup B) = 1$ or $r(A \setminus B) = r(B \setminus A) = 1$. For uncrossable r , Williamson et al. [30] gave a primal-dual 2-approximation algorithm (the WGMV algorithm) for the above integer program. The WGMV algorithm returns a feasible solution x_{WGMV} to the integer program and a feasible dual solution y_{WGMV} , such that

$$\sum_{e \in E} c(e)x_{\text{WGMV}}(e) \leq 2 \sum_{S \subseteq V} r(S)y_{\text{WGMV}}(S)$$

To frame 2ECAP as a network design problem we view 2ECAP as finding a minimum cost set of edges $Q \subseteq E$ such that for all $S \subseteq V$, $|\delta_F(S)| = 1$ implies $|\delta_Q(S)| \geq 1$. This suggests defining a requirement function, $r_F : 2^V \rightarrow \{0, 1\}$ as $r_F(S) = 1$ if and only if $|\delta_F(S)| = 1$. 2ECAP is then a network design problem with the 0-1 requirement function r_F .

Lemma 3 *Given $H = (V, F)$, r_F as defined above is uncrossable.*

Proof We prove the claim by case analysis. Suppose $r_F(A) = r_F(B) = 1$ and $f_1 \in F$ be the unique edge incident to A and $f_2 \in F$ the unique edge incident to B . We consider two cases:

Case 1: $f_1 = f_2$. Either f_1 has endpoints in $A \setminus B$ and $B \setminus A$ or in $A \cap B$ and $\overline{A \cup B}$. In the former case, $r_F(A \setminus B) = r_F(B \setminus A) = 1$ and in the latter case $r_F(A \cap B) = r_F(A \cup B) = 1$.

Case 2: $f_1 \neq f_2$.

2.1 Both f_1 and f_2 have an endpoint in $A \cap B$. Then the other endpoints are in $B \setminus A$ and $A \setminus B$ which implies $r_F(A \setminus B) = r_F(B \setminus A) = 1$.

2.2 Neither f_1 nor f_2 have an endpoint in $A \cap B$. Then both f_1, f_2 have an endpoint in $V \setminus A \cup B$ which implies $r_F(A \setminus B) = r_F(B \setminus A) = 1$.

2.3 Exactly one of f_1, f_2 , say f_1 , has an endpoint in $A \cap B$. Then the other endpoint of f_1 is in $B \setminus A$. Then f_2 has endpoints in $B \setminus A$ and $\overline{A \cup B}$ which implies $r_F(A \cap B) = r_F(A \cup B) = 1$.

In all cases we concluded that either $r_F(A \setminus B) = r_F(B \setminus A) = 1$ or $r_F(A \cap B) = r_F(A \cup B) = 1$ which proves the claim. \square

The above lemma implies that one can use the WGMV algorithm to obtain a 2-approximation algorithm for the 2ECAP problem which in turn leads to a 2-approximation algorithm for the multicut. We formalise this in the following Theorem.

Theorem 1 *Let (G, H, c) be a plane multifold problem. Then there exists a feasible multifold f and a multicut Q , such that $c(Q) \leq 2|f|$, and both f and Q can be computed in polynomial time.*

Proof We run the WGMV algorithm on the 2ECAP instance $(V^*, E^*), (V^*, F^*)$ and let $Q = \{e \in E, x_{\text{WGMV}}(e^*) = 1\}$. To simplify presentation we let r denote the requirement function r_{F^*} . By the feasibility of x_{WGMV} it follows that for all $S \subseteq V^*$, $|\delta_{Q^*}(S)| \geq r(S)$. Hence if $|\delta_{F^*}(S)| = 1$ then $|\delta_{Q^*}(S)| \geq 1$ which implies Q^* is a 2-connector of (V^*, F^*) in (V^*, E^*) and which in turn implies that Q is a multicut in (G, H) .

Consider a set $S \subseteq V^*$ with $r(S) = 1$. Then $|\delta_{F^*}(S)| = 1$ and let $e^* = \delta_{F^*}(S)$. This implies there exists a circuit C in $G + H$ containing e such that $C^* \subseteq \delta(S)$. Hence $C \setminus F$ is a path in G ; denote this by P_S . Define a multifold f in $G + H$ by setting for all $S \subseteq V^*$, $f(P_S) = y_{\text{WGMV}}(S)$. The feasibility of f follows from the feasibility of the dual solution y_{WGMV} . Since $|f| = \sum_{S \subseteq V^*} r(S)y_{\text{WGMV}}(S)$, we get

$$c(Q) = \sum_{e \in E} c(e)x_{\text{WGMV}}(e^*) \leq 2 \sum_{S \subseteq V^*} r(S)y_{\text{WGMV}}(S) = 2|f|,$$

which proves the Lemma. \square

If y_{WGMV} is half-integer (assuming integer edge-costs), the multifold f is half-integer and this implies a half-integer flow-cut gap of 2. The WGMV algorithm does not necessarily produce half-integer dual solutions, but it can be modified to do so, thus establishing the upper bound of 2 on the half-integer flow-cut gap [8].

4 From Fractional to Half-Integer Flow

We assume edge capacities are integers and show how to round a laminar flow f into a laminar integer flow f' , $|f'| \geq |f|$, where f' violates edge capacities by at most 1. As a corollary we obtain a laminar half-integer flow $f_{1/2}$ which respects edge capacities and $|f_{1/2}| \geq |f|/2$.

Let us denote by $\underline{1}$ the all 1 function on E .

Theorem 2 *Let (G, H, c) be a plane multiflow problem, where $c : E \rightarrow \mathbb{Z}^+$. Given a laminar multiflow f , one can in polynomial time compute a laminar integer multiflow f' which is feasible for the capacity function $c + \underline{1}$ and $|f'| \geq |f|$.*

Proof Let $\mathcal{S}(f) = \mathcal{L}$. Since f respects capacity constraints, for every edge $e \in E$, $\sum_{L \in \mathcal{L}, e^* \in \delta(L)} f(L) \leq c(e)$. Conversely any solution x to the following linear program is a feasible multiflow in (G, H, c)

$$\begin{aligned} \text{[LP1]} \quad & \text{maximize} \quad \sum_{L \in \mathcal{L}} x(L) \\ & \text{subject to} \\ & \sum_{L \in \mathcal{L}, e^* \in \delta(L)} x(L) \leq c(e) \quad \forall e \in E \\ & x(L) \geq 0 \quad \forall L \in \mathcal{L} \end{aligned}$$

For $a \neq b \in V^*$, let $\mathcal{L}(a, b) = \{L \in \mathcal{L} : a \in L, b \notin L\}$. The sets in \mathcal{L} which contribute to the constraint corresponding to edge e in [LP1] are exactly the sets in $\mathcal{L}(u, v)$ and $\mathcal{L}(v, u)$ where $e^* = (u, v)$. Let $d(u, v) = \sum_{L \in \mathcal{L}(u, v)} f(L)$ and since f is a feasible multiflow $d(u, v) + d(v, u) \leq c(e)$. Consider the linear program

$$\begin{aligned} \text{[LP2]} \quad & \text{maximize} \quad \sum_{L \in \mathcal{L}} x(L) \\ & \text{subject to} \\ & \sum_{L \in \mathcal{L}(u, v)} x(L) \leq \lceil d(u, v) \rceil \quad \forall (u, v) \in E^* \\ & \sum_{L \in \mathcal{L}(v, u)} x(L) \leq \lceil d(v, u) \rceil \quad \forall (u, v) \in E^* \\ & x(L) \geq 0 \quad \forall L \in \mathcal{L} \end{aligned}$$

Note that f is a feasible solution to this linear program. Let f' be an optimum solution; then $\sum_{L \in \mathcal{L}} f'(L) \geq \sum_{L \in \mathcal{L}} f(L) = |f|$. By assigning a flow value $f'(L)$ to the path $P \in \mathcal{P}$ where $\phi(P) = L$, we get a multiflow in (G, H) which need not be feasible since it might violate capacity constraints. However, since $\lceil d(u, v) \rceil + \lceil d(v, u) \rceil \leq c(e) + 1$, we have that for all $e \in E$, $\sum_{L \in \mathcal{L}, e^* \in \delta(L)} f'(L) \leq c(e) + 1$, and so the violation of any capacity constraint is by at most 1. Thus f' is a feasible multiflow for $(G, H, c + \underline{1})$.

We now argue that f' is an integer solution. The sets of \mathcal{L} appearing in any constraint in LP2 correspond to full chains in \mathcal{L} and hence correspond to directed paths in a rooted arborescence representing \mathcal{L} . This implies that the constraint matrix of [LP2] is a network matrix and as such, is totally unimodular, which implies [LP2] has an integer optimum which can be computed in polynomial time. \square

The rooted arborescence representation of \mathcal{L} and its consequence for the integer optimum of linear programs through total unimodularity are highly nontrivial but well-known tools of combinatorial optimization developed by Edmonds and Giles [4]. The existence of an integer optimum solution was proved by Tutte [29] (see also [6, 1.4.1, Section 4.2.2] and [25, Theorem 13.21, pages 213-216]) and Hoffman and Kruskal [12], [24, Theorem 19.3 (ii), p. 269] showed that the integer optimum can be computed in polynomial time. Our variant of the matrix representation is similar to the “one-way cut incidence matrix” of [18, Theorem 5.28].

For integer capacities c , $(c + \underline{1})/2 \leq c$. Hence $f_{1/2} = f'/2$ is a half-integer feasible flow and $|f_{1/2}| \geq |f|/2$, proving the following Theorem.

Theorem 3 *Let (G, H, c) be a plane multifold problem, where $c : E \rightarrow \mathbb{Z}^+$. Given a laminar multifold f , one can in polynomial time compute a half-integer laminar multifold $f_{1/2}$ such that $|f_{1/2}| \geq |f|/2$.*

It is interesting to contrast Theorem 2 with similar results for demand flows in non-Eulerian plane instances satisfying the cut condition. Korach and Penn [17] proved that all demands with the exception of at most one unit on each bounded face of G can be routed integrally. Frank and Szegedi [7] extended this result to prove that if for every set $S \subset V$, $c(\delta_E(S)) - d(\delta_F(S))$ exceeds the number of faces of G containing a demand edge in $\delta_F(S)$, then all demands can be routed integrally.

Recall that [8] sharpens Theorem 1 by showing a half-integer multifold f of value at least half the capacity of a multicut and hence at least half the maximum fractional flow. While [8] provides an alternate argument to show that the half-integrality gap is $1/2$ we believe that the sharper result in Theorem 2 is of independent interest.

5 From Half-Integer to Integer Flow

In this section, we show how to round a laminar half-integer flow to an integer one, losing at most one half of the flow value. Note that a laminar half-integer flow is provided by Theorem 3.

In order to round the half-integer flow to an integer flow we will need to find a stable-set of size $n/4$ in a planar auxiliary graph on n vertices. The maximum stable set problem is NP-hard, but there is a PTAS for it in planar graphs [2], which, combined with the 4-color theorem [1] provides a stable-set of size $n/4$. An alternative is to use the 4-coloring algorithm of Robertson, Sanders, Seymour and Thomas [23] which directly provides a 4-coloring of a planar graph in polynomial time, and the largest color class is clearly of size at least $n/4$.

Theorem 4 *Let (G, H, c) be a plane multifold problem, where $G = (V, E)$, $H = (V, F)$ and $c : E \rightarrow \mathbb{Z}^+$. Given a laminar, half-integer multifold $f_{1/2}$, one can in polynomial time compute a laminar integer multifold f' such that $|f'| \geq |f_{1/2}|/2$.*

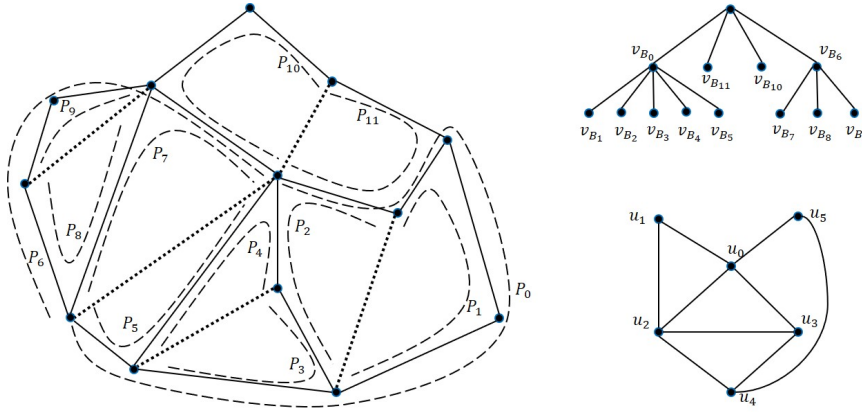


Fig. 1: The graph $G + H$ with edges of H dotted and G solid. The paths along which half-integer flow is routed are shown dashed. The tree T is on the top right while the auxiliary graph U corresponding to vertex v_{B_0} is at the bottom right.

Proof Let T be a rooted arborescence whose vertices are the sets in the laminar family $\mathcal{L} = \mathcal{S}(f_{1/2})$. Let $v_A \in T$ be the node corresponding to $A \in \mathcal{L}$. Note that $(v_A, v_B) \in T$ if $A \subset B$ and there does not exist an $X \in \mathcal{L}$ such that $A \subset X \subset B$. Thus $A \subset B$ if and only if v_A is a descendant of v_B .

Let $v_{B_1}, v_{B_2}, \dots, v_{B_k}$ be the children of a node $v_{B_0} \in T$. The sets B_1, B_2, \dots, B_k and $V^* \setminus B_0$ are disjoint. Construct an auxiliary graph, U , which has a vertex u_i for each set $B_i, 0 \leq i \leq k$. Vertices $u_i, u_j, 0 \leq i, j \leq k$ are adjacent if $\delta_{E^*}(B_i), \delta_{E^*}(B_j)$ are not disjoint.

For $0 \leq i \leq k$, let $\phi(P_i) = B_i$ and $C_i = P_i \cup \{e_{P_i}\}$. Thus C_i is a circuit in $G + H$ corresponding to vertex u_i in U . Since B_i, B_j are disjoint, circuits C_i, C_j correspond to disjoint regions in a planar embedding of $G + H$. Further u_i, u_j are adjacent if and only if C_i, C_j share an edge. The planar embedding of $G + H$ thus yields a planar embedding of U and hence U is planar. Thus given a color for vertex u_0 we can color the remaining vertices of U with at most 4 colors so that no two adjacent vertices have the same color.

We use this observation to color the nodes of T with 4 colors by starting at the root, building an auxiliary graph on the root and its children and coloring it with 4 colors. We continue this process down the tree and at each step we color the children of a node which has been assigned a color in a previous step. Our coloring has the property that two nodes v_A, v_B which are siblings or parent/child in T have different colors if $\delta_{E^*}(A), \delta_{E^*}(B)$ are not disjoint.

The coloring of nodes of T yields a coloring on sets in \mathcal{L} . Let $X \subset \mathcal{L}$ be the largest color class; then $|X| \geq |\mathcal{L}|/4$. For all $A \in X$ we set $f'(\phi^{-1}(A)) = 1$ and for $A \in \mathcal{L} \setminus X$ we set $f'(\phi^{-1}(A)) = 0$. Thus $|f'| = |X| \geq |\mathcal{L}|/4 = |f_{1/2}|/2$.

Recall that for $u \neq v \in V^*$, $\mathcal{L}(u, v) = \{L \in \mathcal{L} : u \in L, v \notin L\}$. To prove that f' is feasible we note that $f(e) = (|\mathcal{L}(u, v)| + |\mathcal{L}(v, u)|)/2$ where $e^* = (u, v)$.

Consecutive sets in the chains $\mathcal{L}(u, v)$, $\mathcal{L}(v, u)$ have different colors. Let A (resp. B) be the maximal set in $\mathcal{L}(u, v)$ (resp. $\mathcal{L}(v, u)$). Then v_A, v_B are siblings in T and $e^* \in \delta(A) \cap \delta(B)$ and hence A, B have different colors. Thus

$$f'(e) = |X \cap (\mathcal{L}(u, v) \cup \mathcal{L}(v, u))| \leq \lceil (|\mathcal{L}(u, v)| + |\mathcal{L}(v, u)|) / 2 \rceil = \lceil f(e) \rceil \leq c(e)$$

which implies that f' does not violate capacity constraints. \square

A similar rounding argument appeared in the work of Fiorini, Hardy, Reed and Vetta [5] in the somewhat different context of proving an upper bound of Král and Voss [19] for the ratio between “minimum size of an odd cycle edge transversal” versus the “maximum odd cycle edge packing” using the 4-color theorem [1].

6 Lower Bounds

In the previous sections we showed upper bounds on the flow-cut gap, and lower bounds on the (half)-integrality gap for multiflow maximization. In this Section we present some examples which establish lower/upper bounds on these gaps.

The plane instance $G = C_4, H = 2K_2$, with edge capacities 1 and demands capped at 1 shows an integrality gap of $1/2$ since the maximum integer flow in this instance is 1 while the maximum fractional flow (which is also half-integer) is 2. The maximum fractional flow need not be half-integer as shown by the example in Figure 2. This example shows a half-integrality gap of $6/7$, a flow-cut gap of $9/7$, and a half-integer flow-cut gap of $3/2$. The gap values

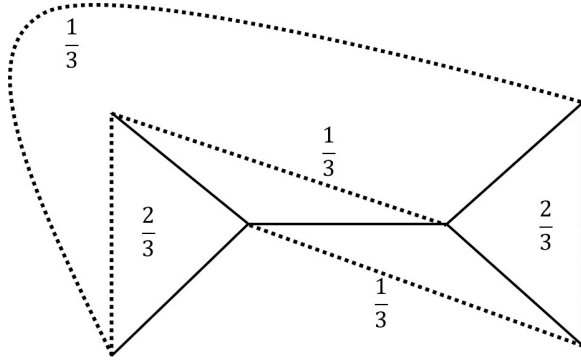


Fig. 2: Supply edges have capacity 1 and are solid while demand edges are dotted. The maximum (fractional) flow is $7/3$, while the maximum half-integer (or integer) flow equals 2. The minimum multicut is 3.

of Figure 2 are not best possible. Cheriyan et.al. [3] defined a class of instances

to show integrality gap results for the Tree Augmentation Problem. We observe that these instances provide asymptotically the best possible values for some of the gaps.

Let $G_k = (V_k, E_k)$, $H_k = (V_k, F_k)$, $k \geq 3$ be an instance of the multi-flow problem defined as follows: $V_k = \{a_1, a_2, \dots, a_k\} \cup \{b_1, b_2, \dots, b_k\}$, $E_k = \{(a_i, b_i) | i \in [1, k]\} \cup \{(a_i, a_{i+1}) | i \in [1, k-1]\}$ and $F_k = \{(b_i, b_{i+1}) | i \in [1, k-1]\} \cup \{(b_i, a_{i+2}) | i \in [1, k-2]\}$. The capacity of all edges in E_k is 1 (see Fig. 3).

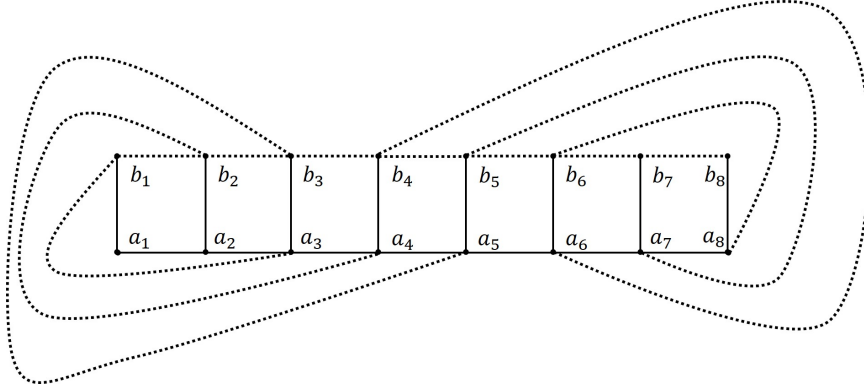


Fig. 3: G_8 : the capacity of supply edges is 1; demand edges are dotted.

Theorem 5 *The graph $G_k + H_k$ is planar for all $k \geq 3$, and the following hold:*

1. *The minimum multicut has capacity $k - 1$.*
2. *The maximum multifold has value $2(k - 1)/3$.*
3. *The maximum half-integer multifold has value $k/2$.*
4. *The maximum integer multifold has value $\lfloor k/2 \rfloor$.*

Proof The minimum multicut has capacity at most $k - 1$, since deleting the $k - 1$ edges on the path (a_1, a_k) , disconnects the endpoints of all demand edges. We prove by induction that for an arbitrary multicut, C , $|C| \geq k - 1$. For $k = 3$ the statement can be easily checked. Deleting vertices b_1, a_1 and the incident supply and demand edges gives a graph (G', H') which is isomorphic to G_{k-1}, H_{k-1} . Since removing edges (a_1, b_1) and (a_1, a_2) can only separate demands $b_1 b_2$ and $b_1 a_3$, $C' := C \setminus \{(a_1, b_1), (a_1, a_2)\}$ is a multicut in (G', H') .

By the induction hypothesis, C' has capacity at least $k - 2$. We consider two cases.

1. Both the edges $(a_2, b_2), (a_2, a_3)$ are in C' . Since $C' \setminus \{(a_2, b_2)\}$ is also a multicut in (G', H') , our induction hypothesis implies that capacity of C' is at least $k - 1$.
2. At most one of $(a_2, b_2), (a_2, a_3)$ is in C' . This implies at least one of $(a_1, b_1), (a_1, a_2)$ must be in C , since otherwise C does not disconnect b_1 from b_2 or from a_3 . Since capacity of C' is at least $k - 2$, capacity of C is at least $k - 1$.

For the second assertion of the Theorem note that the supply edges form a tree and so \mathcal{P} , the set of paths between endpoints of demand edges contains exactly one path P_e for each demand edge e . The flow f defined as $f(P_{b_{k-1}b_k}) = 2/3$ and $f(P) = 1/3$ for each other path $P \in \mathcal{P}$, has value $2(k-1)/3$. Since a supply edge is on at most 3 paths of \mathcal{P} , f is feasible. Note that the total capacity of supply edges is $2k-1$ and every path in \mathcal{P} uses exactly 3 supply edges. Hence no flow can have value more than $(2k-1)/3$. A more careful analysis shows that f with value $2(k-1)/3$ is in fact maximum.

We prove the third and fourth assertions by induction. Note that in any feasible flow f in (G_k, H_k) the flow on edges (a_1, b_1) and (a_1, a_2) is equal; let $\alpha(f, k)$ denote this quantity. We prove that a half-integer flow f , has value at most $k/2$ if $\alpha(f, k) = 1$ and value at most $(k-1)/2$ if $\alpha(f, k) \leq 1/2$. It is easy to check this for $k = 3$. Let f be an arbitrary half-integer flow in G_k, H_k and let f' be the flow induced on (G', H') . The following three cases complete the argument.

1. $\alpha(f, k) = 0$. Since $\alpha(f', k-1) \leq 1$, $|f'| \leq (k-1)/2$ which implies $|f| \leq (k-1)/2$.
2. $\alpha(f, k) = 1/2$. Then $\alpha(f', k-1) \leq 1/2$ and so $|f'| \leq (k-2)/2$ which implies $|f| \leq (k-1)/2$.
3. $\alpha(f, k) = 1$. Then either f routes $1/2$ units for demands b_1b_2 and b_1a_3 or 1 unit for one of the demands b_1b_2, b_1a_3 . In both cases $\alpha(f', k-1) \leq 1/2$ which implies $|f'| \leq (k-2)/2$. Hence $|f| \leq k/2$.

A flow f which routes $1/2$ units for demands $b_i b_{i+1}, 1 \leq i \leq k-1$ satisfies $\alpha(f, k) = 1/2$ and has value $(k-1)/2$. Augmenting f by routing $1/2$ unit of demand $b_1 a_3$ yields a flow f' satisfying $\alpha(f', k) = 1$ with value $k/2$.

This proves that the maximum half-integer flow in $(G_k, H_k) = k/2$. It also implies that the maximum integer flow is at most $\lfloor k/2 \rfloor$. An integer flow of this value can be obtained by sending unit flow for demands $b_i b_{i+1}, 1 \leq i \leq k-1, i \equiv 1 \pmod{2}$. \square

7 Conclusions

This paper establishes bounds on the integrality gap of multiflows, develops approximation algorithms for them, and bounds the flow-cut gap. Applying the best bounds for each variant, the main facts can be summarized in the following theorem; the pointers to the proofs are included thereafter.

Theorem 6 *There exists a $1/4$ -approximation algorithm for maximum integer multifold a $1/2$ -approximation algorithm for maximum half-integer multifold and a 2 -approximation algorithm for minimum multicut in plane instances. In plane instances the flow-cut gap is at most 2, the half-integer flow-cut gap is at most 4 and the integer flow-cut gap is at most 8. Furthermore, the maximum multifold problem in plane instances has a half-integrality gap of at least $1/2$ and an integrality gap of at least $1/4$.*

	max fractional flow	max half-integer flow	max integer flow
max (fractional) flow	1	$(\frac{1}{2}, \frac{3}{4})$	$(\frac{1}{4}, \frac{1}{2})$
min multicut	$(\frac{3}{2}, 2]$	$[2, 4)$	$[2, 8)$
Complexity	P	?	NP-hard

Table 1: Gaps, complexity, and approximation ratios for plane multiflows and multicut. The first row shows the integrality gaps, the second row the flow-cut gaps and the last row the complexity.

Half-integrality gap We do not know the complexity of finding a maximum half-integer multiflow in plane instances. However, a $1/2$ -approximation algorithm follows by starting with a maximum fractional flow f_{OPT} (which can be computed in polynomial time by solving a Linear Program), converting f_{OPT} into a laminar flow f (where $|f| = |f_{\text{OPT}}|$) using Lemma 1 and then converting f into a half-integer flow $f_{1/2}$ (where $|f_{1/2}| = |f|/2$) using Theorem 3. This implies that the half-integrality gap, which is the ratio of the maximum half-integer flow to the maximum flow, is at least $1/2$ for plane instances. The half-integrality gap is $3/4$ for the example in Theorem 5 and finding a tight bound on the half-integrality gap for plane instances remains an interesting open question.

Integrality gap Computing a maximum integer flow is NP-hard and a $1/4$ -approximation can be obtained by converting the half-integer flow $f_{1/2}$ computed above (where $|f_{1/2}| \geq |f_{\text{OPT}}|/2$) to an integer flow f_1 using Theorem 4 such that $|f_1| \geq |f_{1/2}|/2$. This also proves that the integrality gap for maximum flow in plane instances is at least $1/4$. The instance $(C_4, 2K_2)$ with unit capacities and demands capped to 1 shows that the integrality gap can be as small as $1/2$. The ratio of the maximum integer flow to the maximum half-integer flow is at least $1/2$ by Theorem 4 and the example of $(C_4, 2K_2)$ shows this is tight.

Multicuts and flow-cut gap Finding the minimum multicut in plane instances is NP-hard but using the equivalence to 2-connectors (Lemma 2) one can obtain a PTAS [16]. The (fractional) flow-cut gap is at least $3/2$ by Theorem 5 and at most 2 by Theorem 1. The half-integer flow-cut gap is at least 2 by Theorem 5 and at most 4 since the half-integrality gap is at least $1/2$. We note that the improvement by Garg and Kumar [8] shows that the half-integer flow-cut gap is at most 2. For the integer flow-cut gap, Theorem 5 gives a lower bound of 2 and an upper bound of 8 follows from the fact that the integrality gap is at least $1/4$ and the (fractional) flow-cut gap is at most 2. The improvement of Garg and Kumar [8] together with Theorem 4 implies an upper-bound of 4 on the integer flow-cut gap.

References

1. Appel, K., Haken, W.: A proof of the four color theorem. *Discrete Mathematics* **16**, 179–180 (1976)
2. Baker, B.S.: Approximation algorithms for np-complete problems on planar graphs. *J. ACM* **41**(1), 153–180 (1994). DOI 10.1145/174644.174650. URL <http://doi.acm.org/10.1145/174644.174650>
3. Cheriyan, J., Karloff, H., Khandekar, R., Könemann, J.: On the integrality ratio for tree augmentation. *Operations Research Letters* **36**(4), 399–401 (2008)
4. Edmonds, J., Giles, R.: A min-max relation for submodular functions in graphs. *Ann. Discrete Math, I* pp. 185–204 (1977)
5. Fiorini, S., Hardy, N., Reed, B., Vetta, A.: Approximate min-max relations for odd cycles in planar graphs. *Mathematical Programming* **110**(1), 71–91 (1995)
6. Frank, A.: *Connections in Combinatorial Optimization*. Oxford University Press (2011)
7. Frank, A., Szegedi, Z.: A note on packing paths in planar graphs. *Mathematical Programming* **70**, 201–209 (1995)
8. Garg, N., Kumar, N.: Dual half-integrality for uncrossable cut cover and its application to maximum half-integral flow. In: *European Symposium on Algorithms* (2020)
9. Garg, N., Kumar, N., Sebő, A.: Integer plane multiflow maximisation: Flow-cut gap and one-quarter-approximation. In: *International Conference on Integer Programming and Combinatorial Optimization*, pp. 144–157. Springer (2020)
10. Garg, N., Vazirani, V.V., Yannakakis, M.: Approximate max-flow min-(multi) cut theorems and their applications. *SIAM Journal on Computing* **25**(2), 235–251 (1996)
11. Garg, N., Vazirani, V.V., Yannakakis, M.: Primal-dual approximation algorithms for integral flow and multicut in trees. *Algorithmica* **18**(1), 3–20 (1997)
12. Hoffman, A.J., J.B.Kruskal: Integral boundary points of convex polyhedra. *Linear Inequalities and Related Systems* pp. 223–246 (1956)
13. Huang, C.C., Mari, M., Mathieu, C., Schewior, K., Vygen, J.: An approximation algorithm for fully planar edge-disjoint paths. *arXiv preprint arXiv:2001.01715* (2020)
14. Huang, C.c., Mari, M., Mathieu, C., Vygen, J.: Approximating maximum integral multiflows on bounded genus graphs. *arXiv preprint arXiv:2005.00575* (2020)
15. Klein, P., Plotkin, S.A., Rao, S.: Excluded minors, network decomposition, and multicommodity flow. In: *Proceedings of the twenty-fifth annual ACM symposium on Theory of computing*, pp. 682–690. ACM (1993)
16. Klein, P.N., Mathieu, C., Zhou, H.: Correlation clustering and two-edge-connected augmentation for planar graphs. In: *32nd International Symposium on Theoretical Aspects of Computer Science (STACS 2015)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik (2015)
17. Korach, E., Penn, M.: Tight integral duality gap in the chinese postman problem. *Mathematical Programming* **55**, 183–191 (1992)
18. Korte, B., Vygen, J.: *Combinatorial optimization: Theory and Algorithms*, vol. 21. Springer-Verlag Berlin and Heidelberg, collection Algorithms and Combinatorics, fifth edition (2012)
19. Král, D., Voss, H.: Edge-disjoint odd cycles in planar graphs. *Journal of Combinatorial Theory, Series B* **90**, 107–120 (2004)
20. Matsumoto, K., Nishizeki, T., Saito, N.: Planar multicommodity flows, maximum matchings and negative cycles. *SIAM Journal on Computing* **15**(2), 495–510 (1986)
21. Middendorf, M., Pfeiffer, F.: On the complexity of the disjoint paths problem. *Combinatorica* **13**(1), 97–107 (1993)
22. Okamura, H., Seymour, P.D.: Multicommodity flows in planar graphs. *Journal of Combinatorial Theory, Series B* **31**(1), 75–81 (1981)
23. Robertson, N., Sanders, D.P., Seymour, P., Thomas, R.: Efficiently four-coloring planar graphs. In: *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, pp. 571–575 (1996)
24. Schrijver, A.: *Theory of Linear and Integer Programming*. Wiley and Chichester (1986)
25. Schrijver, A.: *Combinatorial optimization: polyhedra and efficiency*, vol. 24. Springer Science & Business Media (2003)
26. Seguin-Charbonneau, L., Shepherd, F.B.: Maximum edge-disjoint paths in planar graphs with congestion 2. In: *2011 IEEE 52nd Annual Symposium on Foundations of Computer Science*, pp. 200–209. IEEE (2011)

-
27. Seymour, P.D.: On odd cuts and plane multicommodity flows. *Proceedings of the London Mathematical Society* **3**(1), 178–192 (1981)
 28. Tardos, É., Vazirani, V.V.: Improved bounds for the max-flow min-multicut ratio for planar and k - r , r -free graphs. *Inf. Process. Lett.* **47**(2), 77–80 (1993). DOI 10.1016/0020-0190(93)90228-2. URL [https://doi.org/10.1016/0020-0190\(93\)90228-2](https://doi.org/10.1016/0020-0190(93)90228-2)
 29. Tutte, W.T.: Lectures on matroids. *J. of Research, Nat. Bureau of Standards* **B**(69) (1965)
 30. Williamson, D.P., Goemans, M.X., Mihail, M., Vazirani, V.V.: A primal-dual approximation algorithm for generalized steiner network problems. *Combinatorica* **15**(3), 435–454 (1995)