Local and union boxicity

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ABSTRACT

The boxicity box(H) of a graph H is the smallest integer d such that H is the intersection of d interval graphs, or equivalently, that H is the intersection graph of axis-aligned boxes in $\mathbb{R}^d$. These intersection representations can be interpreted as covering representations of the complement H^c of H with co-interval graphs, that is, complements of interval graphs. We follow the recent framework of global, local and folded covering numbers (Knauer and Ueckerdt, 2016) to define two new parameters: the local boxicity box$_\ell(H)$ and the union boxicity $\overline{\text{box}}(H)$ of H. The union boxicity of H is the smallest d such that H^c can be covered with d vertex–disjoint unions of co-interval graphs, while the local boxicity of H is the smallest d such that H^c can be covered with co-interval graphs, at most d at every vertex.

We show that for every graph H we have box$_\ell(H) \leq \overline{\text{box}}(H) \leq$ box(H) and that each of these inequalities can be arbitrarily far apart. Moreover, we show that local and union boxicity are also characterized by intersection representations of appropriate axis-aligned boxes in $\mathbb{R}^d$. We demonstrate with a few striking examples, that in a sense, the local boxicity is a better indication for the complexity of a graph, than the classical boxicity.

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1. Introduction

All graphs considered in this article are finite, undirected, simple (have neither loops nor multiple edges), and have at least one edge. An *interval graph* is an intersection graph of intervals on the real line.[1] Such a set $\{I(v) \subseteq \mathbb{R} \mid v \in V(H)\}$ of intervals with $vw \in E(H) \Leftrightarrow I(v) \cap I(w) \neq \emptyset$ is called an *interval representation of H*. A box in $\mathbb{R}^d$, also called a *d-dimensional box*, is the Cartesian product of d intervals. The *boxicity* of a graph H, denoted by box(H), is the least integer d such that H is the intersection graph of d-dimensional boxes, and a corresponding set $\{B(v) \subseteq \mathbb{R}^d \mid v \in V(H)\}$ is a *box representation of H*. The boxicity was introduced by Roberts [17] in 1969 and has many applications in as diverse areas as ecology and operations research [4].

As two d-dimensional boxes intersect if and only if each of the d corresponding pairs of intervals intersect, we have the following more graph theoretic interpretation of the boxicity of a graph; also see Fig. 1(a).

Theorem 1 (*Roberts [17]*). *For a graph H we have* box(H) $\leq d$ *if and only if* $H = G_1 \cap \cdots \cap G_d$ *for some interval graphs* $G_1, \ldots, G_d$.

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[1] Throughout, we shall just say "intervals" and drop the suffix "on the real line". Intervals may be open, half-open, or closed (even though restricting to one kind does not affect the notion of an interval graph) and bounded or unbounded.
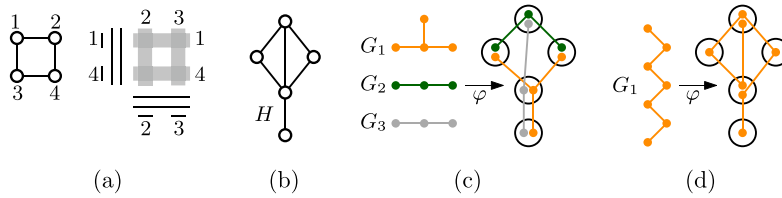
Fig. 1. (a) The 4-cycle as the intersection of two interval graphs. (b) Example graph H. (c) An injective covering of H that is 3-global and 2-local. (d) A (non-injective) 1-global 2-local covering of H.

I.e., the boxicity of a graph H is the least integer d such that H is the intersection of d interval graphs. For a graph $H = (V, E)$ we denote its complement by $H^c = (V, \binom{V}{2} - E)$. Then by De Morgan's law we have

$$H = G_1 \cap \cdots \cap G_d \iff H^c = G_1^c \cup \cdots \cup G_d^c, \tag{1}$$

i.e., box(H) is the least integer d such that the complement H^c of H is the union of d co-interval graphs $G_1^c, \ldots, G_d^c$, where a *co-interval graph* is the complement of an interval graph.[2] In other words, box(H) $\leq d$ if H^c can be covered with d co-interval graphs. Strictly speaking, we have to be a little more precise here. In order to use De Morgan's law, we should guarantee that $G_1, \ldots, G_d$ in (1) all have the same vertex set. To this end, if G is a subgraph of H, let $\bar{G} = (V(H), E(G))$ be the graph obtained from G by adding all vertices in $V(H) - V(G)$ as isolated vertices. (We use $\bar{G}$ to denote a graph obtained from $G \subseteq H$ by adding vertices of H not in G either as isolated or universal vertices.) Clearly we have

$$H^c = G_1^c \cup \cdots \cup G_d^c \implies H^c = \bar{G}_1^c \cup \cdots \cup \bar{G}_d^c \implies H = \bar{G}_1 \cap \cdots \cap \bar{G}_d$$

for any graph H and any set of subgraphs $G_1, \ldots, G_d$ of H. Now whenever G is a co-interval graph, then so is $\bar{G}$, implying that box(H) is the least integer d such that H^c can be covered with d co-interval graphs.

Graph covering parameters

In the general graph covering problem, one is given an input graph H, a so-called covering class $\mathcal{G}$ and a notion of how to cover H with one or more graphs from $\mathcal{G}$. The most classic notion of covering, which also corresponds to the boxicity as discussed above, is that H shall be the union of $G_1, \ldots, G_t \in \mathcal{G}$, i.e., $V(H) = \bigcup_{i \in [t]} V(G_i)$ and $E(H) = \bigcup_{i \in [t]} E(G_i)$. (Here and throughout the paper, for a positive integer t we denote $[t] = \{1, \ldots, t\}$.) The *global covering number*, denoted by $c_g^{\mathcal{G}}(H)$, is then defined to be the minimum t for which such a cover exists. Many important graph parameters can be interpreted as a global covering number, e.g., the arboricity [15] when $\mathcal{G}$ is the class of forests, the track number [9] when $\mathcal{G}$ is the class of interval graphs (Note that this is the smallest number of interval graphs covering a graph H, which is very different from the track-number of H as defined in [5].), and the thickness [1,14] when $\mathcal{G}$ is the class of planar graphs, just to name a few.

Most recently, Knauer and Ueckerdt [10] suggested the following unifying framework for three kinds of covering numbers, differing in the underlying notion of covering. A *graph homomorphism* is a function $\varphi : V(G) \rightarrow V(H)$ with the property that if $uv \in E(G)$ then $\varphi(u)\varphi(v) \in E(H)$, i.e., φ maps vertices of G (not necessarily injectively) to vertices of H such that edges are mapped to edges. For abbreviation we shall simply write $\varphi : G \rightarrow H$ instead of $\varphi : V(G) \rightarrow V(H)$. Whenever G' is a subgraph of G, $\varphi(G')$ denotes the (not necessarily induced) subgraph H' of H with $V(H') = \{\varphi(v) \mid v \in V(G')\}$ and $E(H') = \{\varphi(u)\varphi(v) \mid uv \in E(G')\}$. A *copy* of a graph G' in H is a (not necessarily induced) subgraph H' of H that is isomorphic to G'.

For an input graph H, a covering class $\mathcal{G}$ and a positive integer t, a *t-global $\mathcal{G}$-cover* of H is an edge-surjective homomorphism $\varphi : G_1 \cup \cdots \cup G_t \rightarrow H$ such that $G_i \in \mathcal{G}$ for each $i \in [t]$. Here $\cup$ denotes the vertex–disjoint union of graphs. We say that φ is *injective* if its restriction to G_i is injective for each $i \in [t]$. A $\mathcal{G}$-cover is called *s-local* if $|\varphi^{-1}(v)| \leq s$ for every $v \in V(H)$.

Hence, if φ is a $\mathcal{G}$-cover of H, then

φ is *t-global* if it uses only t graphs[3] from the covering class $\mathcal{G}$,
φ is *injective* if $\varphi(G_i)$ is a copy of G_i in H for each $i \in [t]$,
φ is *s-local* if for each $v \in V(H)$ at most s vertices are mapped onto v.

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[2] Precisely, G is a co-interval graph if there is a set $\{I(v) \subseteq \mathbb{R} \mid v \in V(G)\}$ of intervals with $vw \in E(G) \Leftrightarrow I(v) \cap I(w) = \emptyset$. Equivalently, these are the comparability graphs of interval orders.

[3] More precisely, φ uses a multiset of size t consisting of graphs from $\mathcal{G}$, as the same graph may be used more than once.

For a covering class $\mathcal{G}$ and an input graph H the *global covering number* $c_g^{\mathcal{G}}(H)$, the *local covering number* $c_{\ell}^{\mathcal{G}}(H)$, and the *folded covering number* $c_f^{\mathcal{G}}(H)$ are then defined as follows; see also Fig. 1(b)–(d):

$$c_g^{\mathcal{G}}(H) = \min\{t : \text{there exists a } t\text{-global injective } \mathcal{G}\text{-cover of } H\}$$
$$c_{\ell}^{\mathcal{G}}(H) = \min\{s : \text{there exists an } s\text{-local injective } \mathcal{G}\text{-cover of } H\}$$
$$c_f^{\mathcal{G}}(H) = \min\{s : \text{there exists a } 1\text{-global } s\text{-local } \mathcal{G}\text{-cover of } H\}.$$

Intuitively speaking, for $c_{\ell}^{\mathcal{G}}(H)$ we want to represent the input graph H as the union of graphs from the covering class $\mathcal{G}$, where the number of graphs we use is not important. Rather we want to "use" each vertex of H in only few of these subgraphs. For $c_f^{\mathcal{G}}(H)$ it is convenient to think of the "inverse" mapping for φ. If $\varphi : G_1 \rightarrow H$ is a 1-global $\mathcal{G}$-cover of H, then the preimage under φ of a vertex $v \in V(H)$ is an independent set S_v in G_1. Moreover, for every $u, v \in V(H)$ we have $uv \in E(H)$ if and only if there is at least one edge between S_u and S_v in G_1. So G_1 is obtained from H by a series of *vertex splits*, where splitting a vertex v into an independent set S_v is such that for each edge vw incident to v there is at least one edge between w and S_v after the split. Now $c_f^{\mathcal{G}}(H)$ is the smallest s such that each vertex can be split into at most s vertices so that the resulting graph G_1 lies in the covering class $\mathcal{G}$.

It is known that if the covering class $\mathcal{G}$ is closed under certain graph operations, we can deduce inequalities between the folded, local and global covering numbers. For a graph class $\mathcal{G}$ we define the following.

- $\mathcal{G}$ is *homomorphism-closed* if for any connected $G \in \mathcal{G}$ and any homomorphism $\varphi : G \rightarrow H$ into some graph H we have that $\varphi(G) \in \mathcal{G}$.
- $\mathcal{G}$ is *hereditary* if for any $G \in \mathcal{G}$ and any induced subgraph G' of G we have that $G' \in \mathcal{G}$.
- $\mathcal{G}$ is *union-closed* if for any $G_1, G_2 \in \mathcal{G}$ we have that $G_1 \cup G_2 \in \mathcal{G}$.

Proposition 2 (*Knauer–Ueckerdt [10]*). *For every input graph H and every covering class $\mathcal{G}$ we have*

(i) $c_{\ell}^{\mathcal{G}}(H) \leq c_g^{\mathcal{G}}(H)$, *and if $\mathcal{G}$ is union-closed, then* $c_f^{\mathcal{G}}(H) \leq c_{\ell}^{\mathcal{G}}(H)$,
(ii) *if G is hereditary and homomorphism-closed, then* $c_f^{\mathcal{G}}(H) \geq c_{\ell}^{\mathcal{G}}(H)$.

Boxicity variants

Let us put the boxicity into the graph covering framework by Knauer and Ueckerdt [10] as described above. To this end, let $\mathcal{C}$ denote the class of all co-interval graphs. Then we have $\text{box}(H) = c_g^{\mathcal{C}}(H^c)$ and we can investigate the new parameters

$$\text{box}_f(H) := c_f^{\mathcal{C}}(H^c) \quad \text{and} \quad \text{box}_{\ell}(H) := c_{\ell}^{\mathcal{C}}(H^c).$$

Clearly, if H is an interval graph, i.e., $H^c \in \mathcal{C}$, then $\text{box}_f(H) = \text{box}_{\ell}(H) = \text{box}(H) = 1$. As it turns out, if H is not an interval graph, then $\text{box}_f(H)$ is not very meaningful.

Theorem 3. *For every graph H we have $\text{box}_f(H) = 1$ if $H^c \in \mathcal{C}$ and $\text{box}_f(H) = \infty$ otherwise.*

Basically, Theorem 3 says that if H^c is not a co-interval graph, there is no way to obtain a co-interval graph from H^c by vertex splits. For example, if H has an induced 4-cycle and hence H^c has two independent edges, then $H^c \notin \mathcal{C}$ and whatever vertex splits are applied, the result will always have two independent edges, i.e., not be a co-interval graph. To overcome this issue, it makes sense to define $\overline{\mathcal{C}}$ to be the class of all vertex–disjoint unions of co-interval graphs[4] and consider the parameters

$$\overline{\text{box}}(H) := c_g^{\overline{\mathcal{C}}}(H^c), \quad \overline{\text{box}}_{\ell}(H) := c_{\ell}^{\overline{\mathcal{C}}}(H^c), \quad \overline{\text{box}}_f(H) := c_f^{\overline{\mathcal{C}}}(H^c).$$

We have defined in total six boxicity-related graph parameters, one of which (namely $\text{box}_f(H)$) turned out to be meaningless by Theorem 3. Somehow luckily, three of the remaining five parameters always coincide.

Theorem 4. *For every graph H we have $\text{box}_{\ell}(H) = \overline{\text{box}}_{\ell}(H) = \overline{\text{box}}_f(H)$.*

With Theorems 3 and 4 we are left with three boxicity-related parameters, one of which is the well-known boxicity itself. For the other two, which we refer to as the *local boxicity* $\text{box}_{\ell}(H)$ and the *union boxicity* $\overline{\text{box}}(H)$, we can use again De Morgan's law to give the following definition. The *join* (also known as the *Zykov sum*) of (vertex–disjoint) graphs $G_1, \ldots, G_t$ is the graph $G = G_1 + \cdots + G_t$ obtained from the disjoint union of $G_1, \ldots, G_t$ and adding all edges between those graphs, i.e., $G = (G_1^c \cup \cdots \cup C_t^c)^c$.

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[4] Equivalently, $\overline{\mathcal{C}}$ is the smallest union-closed class which contains $\mathcal{C}$.

Definition 5 (*Local and Union Boxicity*)**.** The *local boxicity* of a graph H, denoted by $\text{box}_\ell(H)$, is the smallest number k such that H is the intersection of t interval graphs, for some integer t, with each vertex of H being non-universal in at most k of these graphs.

The *union boxicity* of H, denoted by $\overline{\text{box}}(H)$, is the smallest number k such that H is the intersection of k graphs, each of which is the join of some number of interval graphs.

The three parameters boxicity, local boxicity and union boxicity are non-trivial and reflect different aspects of the graph, as will be investigated in more detail in this paper. Proposition 2 and Theorem 4 give $\text{box}_\ell(H) = \overline{\text{box}}(H) = c_\ell^{\overline{C}}(H^c) \leq c_g^{\overline{C}}(H^c) = \overline{\text{box}}(H)$ for every input graph H. As $C \subset \overline{C}$ we have $\overline{\text{box}}(H) = c_g^{\overline{C}}(H^c) \leq c_g^{C}(H^c) = \text{box}(H)$ for every input graph H. Thus for every graph H we have

$$\text{box}_\ell(H) \leq \overline{\text{box}}(H) \leq \text{box}(H). \tag{2}$$

A graph H is an interval graph if and only if $\text{box}(H) = 1$, and in this case (2) gives $\text{box}_\ell(H) = \overline{\text{box}}(H) = 1$ as well. It will follow from our results (c.f. Theorem 7) that $\text{box}_\ell(H) = 1$ implies $\overline{\text{box}}(H) = 1$. However, this is the only case in which we can generally bound a more restricted boxicity variant from above in terms of a more relaxed variant. This is formalized in the following theorem.

Theorem 6. *For every positive integer k there exist graphs H_k, H_k', H_k'' with*

(i) $\text{box}_\ell(H_k) \geq k$,
(ii) $\text{box}_\ell(H_k') = 2$ *and* $\overline{\text{box}}(H_k') \geq k$,
(iii) $\overline{\text{box}}(H_k'') = 1$ *and* $\text{box}(H_k'') = k$.

We also give geometric interpretations of the local and union boxicity of a graph H in terms of intersecting high-dimensional boxes. For positive integers k, d with $k \leq d$ we call a d-dimensional box $B = I_1 \times \cdots \times I_d$ k-local if for at most k indices $i \in \{1, \ldots, d\}$ we have $I_i \neq \mathbb{R}$. Thus a k-local d-dimensional box is the Cartesian product of d intervals, at least $d - k$ of which are equal to the entire real line $\mathbb{R}$. Note that when B_1 is a k_1-local d_1-dimensional box and B_2 is a k_2-local d_2-dimensional box, then the Cartesian product $B_1 \times B_2$ is a $(k_1 + k_2)$-local $(d_1 + d_2)$-dimensional box.

Theorem 7. *Let H be a graph and $k \geq 1$ be an integer.*

(i) *We have $\overline{\text{box}}(H) \leq k$ if and only if there exist positive integers $d_1, \ldots, d_k$ such that H is the intersection graph of Cartesian products of k boxes, where the ith box is 1-local d_i-dimensional, $i = 1, \ldots, k$.*
(ii) *We have $\text{box}_\ell(H) \leq k$ if and only if there exists a positive integer d such that H is the intersection graph of k-local d-dimensional boxes.*

Of course, the most natural way to prove $\text{box}(H) \leq d$ for a graph H is to explicitly define an intersection representation with d-dimensional boxes. One standard approach in the literature is to split H into few induced subgraphs $H_1, \ldots, H_t$ of H in such a way that **(a)** every pair of vertices in H appear together in H_i for at least one $i \in [t]$, and **(b)** each H_i, $i \in [t]$, has small boxicity. Then, adding all vertices in $V(H) - V(H_i)$ as universal vertices to a box representation of H_i and taking the Cartesian product of all t box representations prove $\text{box}(H) \leq \sum_{i \in [t]} \text{box}(H_i)$. With Theorem 7 we can conclude in such cases that $\text{box}_\ell(H) \leq \max\{\text{box}(H_i) \mid i \in [t]\}$, which is usually significantly less than $\sum_{i \in [t]} \text{box}(H_i)$.

Let us restrict here to one such case, which is comparably simple. For a graph H the *acyclic chromatic number*, denoted by $\chi_a(H)$, is the smallest k such that there exists a proper vertex coloring of H with k colors in which any two color classes induce a forest. In other words, an acyclic coloring has no monochromatic edges and no bicolored cycles. Esperet and Joret [6] have recently shown that for any graph H with $\chi_a(H) = k$ we have $\text{box}(H) \leq k(k - 1)$. Indeed, their proof (which we include here for completeness) gives an intersection representation of H with $2(k - 1)$-local $k(k - 1)$-dimensional boxes, implying the following theorem.

Theorem 8. *For every graph H we have $\text{box}_\ell(H) \leq 2(\chi_a(H) - 1)$.*

Proof. Let c be an acyclic coloring of H with k colors. For any pair $\{i, j\}$ of colors consider the subgraph $G_{i,j}$ induced by the vertices of colors i and j. As $G_{i,j}$ is a forest, we have $\text{box}(G_{i,j}) \leq 2$ (this follows from [18] but can also be seen fairly easily). Moreover, since H is the union of all $G_{i,j}$, the complement H^c of H is the intersection of the complements of all $\bar{G}_{i,j}$ (here $\bar{G}_{i,j}$ contains vertices in H not in $G_{i,j}$ as isolated vertices).

Now take an intersection representation of $G_{i,j}$ with 2-dimensional boxes and extend it to one for $\bar{G}_{i,j}$ by putting the box $\mathbb{R}^2$ for each vertex colored neither i nor j. Then the Cartesian product of all these $\binom{k}{2}$ box representations is an intersection representation of H with $2(k-1)$-local $k(k-1)$-dimensional boxes. This proves that $\text{box}(H) \leq k(k-1)$ and $\text{box}_\ell(H) \leq 2(k-1)$, as desired. $\square$

Organization of the paper.

In Section 2 we prove Theorem 3, i.e., that $\text{box}_f(H)$ is meaningless, and Theorem 4, i.e., that three of the remaining five boxicity variants coincide. In Section 3 we consider the problem of separation for boxicity and its local and union variants, that is, we give a proof of Theorem 6. In Section 4 we describe and prove the geometric interpretations of local and union boxicity from Theorem 7. Finally, we give some concluding remarks and open problems in Section 5.

2. Local and union boxicity

Recall that a graph class $\mathcal{G}$ is homomorphism-closed if for every *connected* graph $G \in \mathcal{G}$ and any homorphism $\varphi : G \to H$ into some graph H we have $\varphi(G) \in \mathcal{G}$. Since φ is a homomorphism, $\varphi(G)$ arises from G by a series of "inverse vertex splits", i.e., an independent set in G is identified into a single vertex of $\varphi(G)$. If $\mathcal{G}$ is not only homomorphism-closed, but also closed under identifying non-adjacent vertices in *disconnected* graphs, then the folded covering number $c_f^{\mathcal{G}}$ turns out to be somewhat meaningless.

Lemma 9. *If a covering class $\mathcal{G}$ is closed under identifying non-adjacent vertices, then for every non-empty input graph H we have*

$$c_f^{\mathcal{G}}(H) < \infty \quad \Longleftrightarrow \quad H \in \mathcal{G} \quad \Longleftrightarrow \quad c_f^{\mathcal{G}}(H) = 1.$$

Proof. The right equivalence follows by definition of $c_f^{\mathcal{G}}(H)$.

The implication $H \in \mathcal{G} \Rightarrow c_f^{\mathcal{G}}(H) < \infty$ in the first equivalence is thereby obvious, and it is left to show that $c_f^{\mathcal{G}}(H) = 1$ whenever $c_f^{\mathcal{G}}(H) < \infty$. So let $\varphi : G_1 \to H$ be any 1-global cover of H. We prove the lemma by induction over $|V(G_1)|$, the number of vertices in G_1.

If $|V(G_1)| = |V(H)|$, i.e., no vertices are folded, then φ is injective and therefore $c_f^{\mathcal{G}}(H) = 1$. So assume that $|V(G_1)| > |V(H)|$ and let v, w be distinct vertices in G_1 with $\varphi(v) = \varphi(w)$. Consider the graph G_1' that we obtain by identifying v and w in G_1. Since $\varphi(v) = \varphi(w)$ is only possible if v and w are non-adjacent, and $\mathcal{G}$ is closed under identifying non-adjacent vertices we know that $G_1' \in \mathcal{G}$. Now the 1-global $\mathcal{G}$-cover $\varphi : G_1 \to H$ induces a 1-global $\mathcal{G}$-cover $\varphi' : G_1' \to H$ by $\varphi = \varphi' \circ \psi$, where $\psi : G_1 \to G_1'$ identifies v and w in G_1 and fixes all other vertices. As $|V(G_1')| = |V(G_1)| - 1$, we can apply induction to φ' to conclude that $c_f^{\mathcal{G}}(H) = 1$. $\square$

Lemma 10. *Let $\mathcal{C}$ be the class of all co-interval graphs and $\overline{\mathcal{C}}$ be the class of all vertex–disjoint unions of co-interval graphs. Then*

  (i) *$\mathcal{C}$ and $\overline{\mathcal{C}}$ are hereditary,*
 (ii) *$\mathcal{C}$ is closed under identifying non-adjacent vertices, and*
(iii) *$\overline{\mathcal{C}}$ is homomorphism-closed.*

Proof.

  (i) Consider any graph $G \in \overline{\mathcal{C}}$. Then $G = G_1 \cup \cdots \cup G_t$ for some $G_1, \ldots, G_t \in \mathcal{C}$. If $G \in \mathcal{C}$, then $t = 1$. For $i \in [t]$ consider an intersection representation $\{I_i(v) \mid v \in V(G_i)\}$ of G_i^c with intervals. For any vertex set $S \subseteq V(G)$, consider the induced subgraphs when restricted to vertices in S, i.e., $G' = G[S]$ and $G_i' = G_i[V(G_i) \cap S]$ for $i \in [t]$. Note that $\{I_i(v) \mid v \in V(G_i) \cap S\}$ is an interval representation of $(G_i')^c$, i.e., $G_i' \in \mathcal{C}$. Hence $G' = G_1' \cup \cdots \cup G_t' \in \overline{\mathcal{C}}$ and $G' \in \overline{\mathcal{C}}$ if $t = 1$. This shows that $\mathcal{C}$ and $\overline{\mathcal{C}}$ are hereditary.
 (ii) Let $G \in \mathcal{C}$, x, y be two non-adjacent vertices in G and $\{I(v) \mid v \in V(G)\}$ be an intersection representation of G^c with intervals. Let G' be the graph obtained from G by identifying x and y into a single vertex z. Since $xy \in E(G^c)$ we have $I(x) \cap I(y) \neq \emptyset$ and hence $I(z) := I(x) \cap I(y)$ is a non-empty interval. As for any interval J we have $J \cap I(z) \neq \emptyset$ if and only if $J \cap I(x) \neq \emptyset$ and $J \cap I(y) \neq \emptyset$, we have that $\{I(v) \mid v \in V(G), v \neq x, y\} \cup \{I(z)\}$ is an intersection representation of $(G')^c$ and thus $G' \in \mathcal{C}$, as desired.
(iii) If $G \in \overline{\mathcal{C}}$ then $G = G_1 \cup \cdots \cup G_t$ for some $G_1, \ldots, G_t \in \mathcal{C}$. If x, y are two non-adjacent vertices in the same connected component, then x, y are in the same G_i, say G_1. By (ii) identifying x and y in G_1 gives a graph $G_1' \in \mathcal{C}$. Moreover, identifying x and y in G gives a graph $G' = G_1' \cup G_2 \cup \cdots \cup G_t$. As $G_1' \in \mathcal{C}$ we have $G' \in \overline{\mathcal{C}}$ and hence $\overline{\mathcal{C}}$ is homomorphism-closed. $\square$

Proof of Theorem 3. This is a direct corollary of Lemmas 9 and 10(ii). $\square$

Proof of Theorem 4. We have that $\overline{\mathcal{C}}$ is hereditary by Lemma 10(i), homomorphism-closed by Lemma 10(iii) and union-closed by definition. Hence by Proposition 2 we have $\overline{\text{box}}_f(H) = c_f^{\overline{\mathcal{C}}}(H^c) = c_\ell^{\overline{\mathcal{C}}}(H^c) = \overline{\text{box}}_\ell(H)$.

As $\mathcal{C} \subset \overline{\mathcal{C}}$ we clearly have $\overline{\text{box}}_\ell(H) = c_\ell^{\overline{\mathcal{C}}}(H^c) \leq c_\ell^{\mathcal{C}}(H^c) = \text{box}_\ell(H)$. Finally, consider any s-local t-global $\overline{\mathcal{C}}$-cover $\varphi : G_1 \cup \cdots \cup G_t \to H^c$. For $i = 1, \ldots, t$ we have $G_i \in \overline{\mathcal{C}}$ and hence G_i is the vertex–disjoint union of some graphs in $\mathcal{C}$. Thus we can interpret φ as an s-local t'-global $\mathcal{C}$-cover of H^c for some $t' \geq t$. This shows that $\text{box}_\ell(H) = c_\ell^{\mathcal{C}}(H^c) \leq c_\ell^{\overline{\mathcal{C}}}(H^c) = \overline{\text{box}}_\ell(H)$ and thus concludes the proof. $\square$

3. Separating the variants

Proof of Theorem 6.

(i) For a fixed integer $k \geq 1$ we consider an arbitrary graph F_k that is $2k$-regular and has girth at least 6 (i.e., its shortest cycle has length at least 6). Now let $\varphi : G_1 \cup \cdots \cup G_t \to F_k$ be an injective s-local $\mathcal{C}$-cover of F_k, i.e., each $\varphi(G_i)$, $i \in [t]$, is a subgraph of F_k and a disjoint union of co-interval graphs, every edge of F_k lies in at least one such $\varphi(G_i)$, and every vertex of F_k is contained in at most s such $\varphi(G_i)$. Such a cover exists as we could for example cover each edge of F_k with a separate K_2, which is a co-interval graph. We shall show that $s \geq k$, proving that $c_\ell^{\mathcal{C}}(F_k) \geq k$ and hence $\text{box}_\ell(H_k) \geq k$, where $H_k = F_k^c$ denotes the complement of F_k.

A co-interval graph G does not contain any induced matching on two edges. Hence G does not contain any induced cycle of length at least 6. (Moreover, as G is perfect, it also contains no induced cycles of length 5.) Since F_k has girth at least 6, this implies that every subgraph of F_k that is a co-interval graph is a forest. In particular, every $\varphi(G_i) \cong G_i$ has average degree less than 2, i.e., $\sum_{x \in V(G_i)} \deg_{G_i}(x) < 2|V(G_i)|$. We conclude that

$$2k \cdot |V(F_k)| = \sum_{v \in V(F_k)} \deg_{F_k}(v) \leq \sum_{i=1}^{t} \sum_{x \in V(G_i)} \deg_{G_i}(x)$$

$$< \sum_{i=1}^{t} 2|V(G_i)| \leq 2s \cdot |V(F_k)|,$$

where the first inequality holds since every edge of F_k is covered and the last inequality holds since every vertex is contained in at most s of the $\varphi(G_i)$, $i \in [t]$. From the above it follows that $s \geq k$, as desired.

(ii) Our proof follows the ideas of Milans et al. [13], who consider $L(K_n)$, the line graph of K_n, and prove that $c_g^{\mathcal{I}}(L(K_n)) \to \infty$ for $n \to \infty$, while $c_\ell^{\mathcal{I}}(L(K_n)) = 2$ for every $n \in \mathbb{N}$, where $\mathcal{I}$ denotes the class of all interval graphs. However, instead of using the ordered Ramsey numbers (which is also possible in our case) we shall rather use the following hypergraph Ramsey numbers: Let K_n^3, $n \in \mathbb{N}$, denote the complete 3-uniform hypergraph on n vertices, i.e., $K_n^3 = ([n], \binom{[n]}{3})$. For an integer $k \geq 1$, the Ramsey number $R_k(K_6^3)$ is the smallest integer n such that every coloring of the hyperedges of K_n^3 with k colors contains a monochromatic copy of K_6^3. The hypergraph Ramsey theorem implies that $R_k(K_6^3)$ exists for every k [16].

Now for fixed $k \geq 1$, choose an integer $n = n(k) > R_k(K_6^3)$ and consider $L(K_n)$, the line graph of K_n. Let $\varphi : G_1 \cup \cdots \cup G_t \to L(K_n)$ be any injective t-global $\overline{\mathcal{C}}$-cover of $L(K_n)$ with each $G_i \in \overline{\mathcal{C}}$ being a disjoint union of co-interval graphs for $i \in [t]$ and some $t \in \mathbb{N}$. Again, such φ exists as we could cover each edge of $L(K_n)$ with a separate copy of $K_2 \in \overline{\mathcal{C}}$. We shall show that $t \geq k$, proving that $c_g^{\overline{\mathcal{C}}}(L(K_n)) \geq k$ and hence $\overline{\text{box}}(H_k') \geq k$, where $H_k' = (L(K_n))^c$ denotes the complement of $L(K_n)$.

Assume for the sake of contradiction that $t < k$. From the $\overline{\mathcal{C}}$-cover φ of $L(K_n)$, we define a coloring c of $E(K_n^3)$ with t colors. Given $x, y, z \in [n]$ with $x < y < z$, let $c(x, y, z) = \min\{i \in [t] \mid \{xy, yz\} \in E(\varphi(G_i))\}$ be the smallest index of a co-interval graph in $\{G_1, \ldots, G_t\}$ that covers the edge between xy and yz in $L(K_n)$. Since $n > R_k(K_6^3) \geq R_t(K_6^3)$ under c there is a monochromatic copy of K_6^3, say it is in color i and that its vertices are $x_1 < \cdots < x_6$. By definition of coloring c, this means that $\varphi(G_i)$ contains the path $x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_5, x_5 x_6$ in $L(K_n)$. However, the two edges $\{x_1 x_2, x_2 x_3\}$ and $\{x_4 x_5, x_5 x_6\}$ induce a matching in $L(K_n)$ and hence also in a connected component K of $\varphi(G_i) \cong G_i$. This is a contradiction to $G_i \in \overline{\mathcal{C}}$, i.e., K being a co-interval graph, and thus implies that $t \geq k$, as desired.

Finally, observe that for any $n \in \mathbb{N}$ the following is an injective 2-local $\mathcal{C}$-cover of $L(K_n)$: For each $i \in [n]$ let G_i be the clique in $L(K_n)$ formed by all edges incident to vertex i of K_n. Then $\{G_1, \ldots, G_n\}$ is a set of n co-interval graphs in $L(K_n)$ with the property that every edge of $L(K_n)$ lies in exactly one G_i and every vertex of $L(K_n)$ lies in exactly two G_i. This shows that $c_\ell^{\mathcal{C}}(L(K_n)) = \text{box}_\ell(H_k') \leq 2$.

(iii) For fixed $k \geq 1$ consider M_k, the matching on k edges. We shall show that $c_g^{\overline{\mathcal{C}}}(M_k) = 1$ and $c_g^{\mathcal{C}}(M_k) = k$, proving that $\overline{\text{box}}(H_k'') = 1$ and $\text{box}(H_k'') = k$, where $H_k'' = M_k^c$ is the complement of M_k. Indeed, as no co-interval graph contains an induced matching on two edges, any $\mathcal{C}$-cover of M_k contains at least k co-interval graphs to cover all k edges of M_k. Since K_2 is a co-interval graph, there actually is an injective k-global $\mathcal{C}$-cover of M_k. Thus, we have $c_g^{\mathcal{C}}(M_k) = \text{box}(H_k'') = k$.

On the other hand, the class $\overline{\mathcal{C}}$ is union-closed and, since K_2 is a co-interval graph, $\overline{\mathcal{C}}$ contains all matchings. In particular $M_k \in \overline{\mathcal{C}}$ and therefore we have $c_g^{\overline{\mathcal{C}}}(M_k) = \overline{\text{box}}(H_k'') = 1$. □

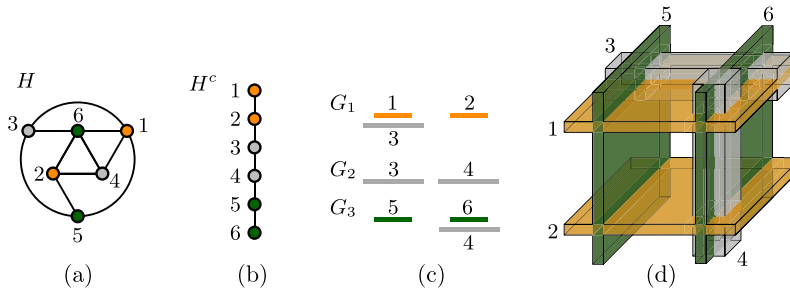## 4. Geometric interpretations

Lemma 11. *A graph H is the intersection graph of 1-local d-dimensional boxes if and only if H^c is the vertex–disjoint union of d co-interval graphs.*

Proof. For an illustration of the proof, see Fig. 2. First, if $\{B(v) \mid v \in V(H)\}$ is an intersection representation of H with 1-local boxes in $\mathbb{R}^d$, then for each $v \in V(H)$ let $B(v) = I_1(v) \times \cdots \times I_d(v)$. Without loss of generality assume that for every
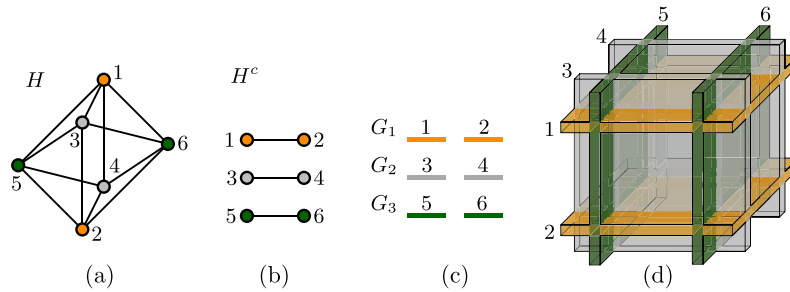
Fig. 2. (a) The octahedron H. (b) Its complement H^c. (c) H^c as the vertex–disjoint union of three co-interval graphs (given in their interval representation). (d) The corresponding intersection representation of H with 1-local 3-dimensional boxes. The two long sides of each box have actually infinite length.

Fig. 3. (a, b) A graph H and its complement H^c. (c) H^c can be covered using three co-interval graphs. (d) The resulting intersection representation. Note that the boxes are 3-dimensional as the cover uses three co-interval graphs and the boxes are 1-local and 2-local if the corresponding vertices are covered once $(1, 2, 5, 6)$ and twice $(3,4)$, respectively. The long sides of each box have actually infinite length.

$v \in V(H)$ there is some coordinate $i \in [d]$ for which $I_i(v) \neq \mathbb{R}$ is a bounded interval. For each $i \in [d]$ consider the set $V_i = \{v \in V(H) \mid I_i(v) \neq \mathbb{R}\}$ of those vertices v for which $B(v)$ is bounded in the ith coordinate. Then $V_1, \ldots, V_d$ is a partition of $V(H)$ and for each $i \in [d]$ the set $\{I_i(v) \mid v \in V_i\}$ is an intersection representation with intervals of some graph G_i with vertex set V_i. Then we have $H = \bar{G}_1 \cap \cdots \cap \bar{G}_d$, where $\bar{G}_i$ arises from G_i by adding all vertices of H not in G_i as universal vertices. Hence $H^c = \bar{G}_1^c \cup \cdots \cup \bar{G}_d^c = G_1^c \cup \cdots \cup G_d^c$. Thus H^c is the vertex–disjoint union of the d co-interval graphs, as desired.

Now let $H^c = G_1^c \cup \cdots \cup G_d^c$, where $G_i^c \in \mathcal{C}$ for $i = 1, \ldots, d$. Consider for each i an intersection representation $\{I_i(v) \mid v \in V(G_i)\}$ of the complement G_i of G_i^c with intervals. For $v \in V(H)$ we define

$$I_i'(v) = \begin{cases} I_i(v), & \text{if } v \in V(G_i) \\ \mathbb{R}, & \text{if } v \notin V(G_i). \end{cases}$$

Then $B(v) = I_1'(v) \times \cdots \times I_d'(v)$ is a 1-local d-dimensional box. Moreover, $\{B(v) \mid v \in V(H)\}$ is an intersection representation of H, which concludes the proof. $\square$

From Lemma 11 we easily derive Theorem 7, i.e., the geometric intersection representations characterizing the local and union boxicity, respectively.

Proof of Theorem 7.

(i) This follows easily from Lemma 11. Indeed, if $\overline{\mathrm{box}}(H) = c_g^{\overline{\mathcal{C}}}(H^c) \leq k$, then $H^c = G_1 \cup \cdots \cup G_k$ where for $i = 1, \ldots, k$ the graph $G_i \in \overline{\mathcal{C}}$ is the vertex–disjoint union of d_i co-interval graphs. By Lemma 11 G_i^c has an intersection representation with 1-local d_i-dimensional boxes. Similarly to the proof of Lemma 11, extending this 1-local box representation of G_i^c to all vertices of H by adding a box $\mathbb{R}^{d_i}$ for each vertex in $H - G_i$, and taking the Cartesian product of these k extended 1-local box representations, we obtain an intersection representation of H of the desired kind.

Similarly, consider any intersection representation $\{B_1(v) \times \cdots \times B_k(v) \mid v \in V(H)\}$ of H, where for every $v \in V(H)$ and every $i \in [k]$ the box $B_i(v)$ is d_i-dimensional and 1-local. Then by Lemma 11 the set $\{B_i(v) \mid v \in V(H)\}$ is an intersection representation of some graph G_i whose complement G_i^c is in $\overline{\mathcal{C}}$. Moreover, H^c is the union of these k graph $G_1^c, \ldots, G_k^c \in \overline{\mathcal{C}}$. This gives $\overline{\mathrm{box}}(H) = c_g^{\overline{\mathcal{C}}}(H^c) \leq k$, as desired.

(ii) For an example illustrating this case, see Fig. 3. If $\mathrm{box}_\ell(H) = c_\ell^{\mathcal{C}}(H^c) \leq k$, then there is a set $\{G_1, \ldots, G_t\}$ of t co-interval graphs such that $G_i \subseteq H^c$ for $i = 1, \ldots, t$, $E(H^c) = E(G_1) \cup \cdots \cup E(G_t)$ and every $v \in V(H^c)$ is contained in at most k such G_i, $i = 1, \ldots, t$. For each $i \in [t]$ consider an interval representation $\{I_i(v) \mid v \in V(G_i)\}$ of G_i^c. For $v \in H - G_i$ we

set $I_i(v) = \mathbb{R}$. Then $\{I_i(v) \mid v \in V(H)\}$ is an interval representation of the graph $\bar{G}_i^c$ obtained from G_i^c by adding vertices in H not in G_i as universal vertices.

Now for $v \in V(G)$ let $B(v) = I_1(v) \times \cdots \times I_t(v)$ be the Cartesian product of the t intervals associated with vertex v. As v is in G_i for at most k indices $i \in [t]$, $I_i(v) \neq \mathbb{R}$ for at most k indices $i \in [t]$. In other words, $B(v)$ is a k-local box. Finally, we claim that $\{B(v) \mid v \in V(H)\}$ is an intersection representation of H. Indeed, if $vw \notin E(H)$, then $vw \in E(H^c)$ and hence $vw \in E(G_i)$ for at least one $i \in [t]$. Then $I_i(v) \cap I_i(w) = \emptyset$ and thus $B(v) \cap B(w) = \emptyset$. And if $vw \in E(H)$, then $vw \notin E(H^c)$ and $vw \notin E(G_i')$ for every $i \in [t]$. Thus $I_i(v) \cap I_i(w) \neq \emptyset$ for every $i \in [t]$ and hence $B(v) \cap B(w) \neq \emptyset$.

This shows that if $\mathrm{box}_\ell(H) \leq k$, then H is the intersection graph of k-local boxes. On the other hand, if H admits an intersection representation with k-local t-dimensional boxes, then for each $i \in [t]$ projecting the boxes to coordinate i and considering the bounded intervals in this projection give an interval representation of some subgraph G_i of H^c. As before, we can check that $\{G_1, \ldots, G_t\}$ forms an injective k-local $\mathcal{C}$-cover of H^c, showing that $\mathrm{box}_\ell(H) = c_\ell^{\mathcal{C}}(H^c) \leq k$. $\square$

5. Conclusions

In this paper we have introduced the notions of the local boxicity $\mathrm{box}_\ell(H)$ and union boxicity $\overline{\mathrm{box}}(H)$ of a graph H. It holds that $\mathrm{box}_\ell(H) \leq \overline{\mathrm{box}}(H) \leq \mathrm{box}(H)$, where $\mathrm{box}(H)$ denotes the classical boxicity as introduced almost 50 years ago. Indeed, both new parameters are a better measure of the complexity of H. For example, if H is the complement of a matching on n edges, then $\mathrm{box}(H) = n$, simply because the n non-edges each have to be realized in a different dimension. On the other hand, we have $\mathrm{box}_\ell(H) = \overline{\mathrm{box}}(H) = 1$, and as these non-edges are vertex–disjoint, they also should be "counted only once". We have shown this phenomenon in a few more examples in the course of the paper. In fact, in many box representations from the literature many (if not all) dimensions are only used by few vertices. The resulting high boxicity may be misintepreted as the graph being very complex, which could be avoided by using local or union boxicity.

In future research, established boxicity results should be revisited to see whether one can improve the upper bounds using local or union boxicity. For example, it is known that if H is a planar graph, then $\mathrm{box}(H) \leq 3$ [19]. Moreover, the octahedral graph O is planar and has boxicity 3, because its complement O^c is the matching on three edges (c.f. the proof of Theorem 6(iii) and Fig. 2). By (2) we have that $\mathrm{box}_\ell(H) \leq \overline{\mathrm{box}}(H) \leq 3$ whenever H is planar. However, $\mathrm{box}_\ell(O) = \overline{\mathrm{box}}(O) = 1$, because O^c is the vertex–disjoint union of co-interval graphs, i.e., $O^c \in \bar{\mathcal{C}}$. Hence it is natural to ask the following.

Question 12. *Is there a planar graph H with $\mathrm{box}_\ell(H) = 3$?*

For general graphs H we proved that the local boxicity $\mathrm{box}_\ell(H)$ and the union boxicity $\overline{\mathrm{box}}(H)$ can be arbitrarily far from the classical boxicity $\mathrm{box}(H)$. But we do not know whether if $\mathrm{box}(H)$ is large, then $\mathrm{box}_\ell(H)$ and $\overline{\mathrm{box}}(H)$ can be very *close* to $\mathrm{box}(H)$. We construct graphs in the proof of Theorem 6(i) with large local boxicity, but one can show that these have even larger boxicity.

Question 13. *Is there for every $k \in \mathbb{N}$ a graph H_k such that $\mathrm{box}_\ell(H_k) = \overline{\mathrm{box}}(H_k) = \mathrm{box}(H) = k$?*

Another interesting research direction concerns the computational complexity. It is known that for every $k \geq 2$ deciding whether a given graph H satisfies $\mathrm{box}(H) \leq k$ is NP-complete [3,11]. When $k = 1$, one can decide if $\mathrm{box}(H) \leq k$ and $\overline{\mathrm{box}}(H) \leq k$ in polynomial time via interval graph recognition [2], because $\mathrm{box}(H) \leq 1$ if and only if H is an interval graph and $\overline{\mathrm{box}}(H) \leq 1$ (equivalently $\mathrm{box}_\ell(H) \leq 1$, as already mentioned in the introduction) if and only if $H^c \in \bar{\mathcal{C}}$, i.e., the complement of H is the vertex–disjoint union of co-interval graphs.

Question 14. *For $k \geq 2$, is it NP-complete to decide whether $\mathrm{box}_\ell(H) \leq k$ (or $\overline{\mathrm{box}}(H) \leq k$) for a given graph H?*

Let us remark that for general covering numbers the computational complexity of computing $c_g^{\mathcal{G}}(H)$ tends to be harder than that of $c_\ell^{\mathcal{G}}(H)$, which in turn tends to be harder than for $c_f^{\mathcal{G}}(H)$. For example, for $\mathcal{G}$ being the class of star forests, computing $c_g^{\mathcal{G}}(H)$ is NP-complete [8,12], while computing $c_\ell^{\mathcal{G}}(H)$ and $c_f^{\mathcal{G}}(H)$ is polynomial-time solvable [10]. The same holds when $\mathcal{G}$ is the class of all matchings as discussed in [10]. And for $\mathcal{G}$ being the class of bipartite graphs, computing $c_g^{\mathcal{G}}(H)$ and $c_\ell^{\mathcal{G}}(H)$ is NP-complete [7], while computing $c_f^{\mathcal{G}}(H)$ is polynomial-time solvable since $c_f^{\mathcal{G}}(H) = 1$ if H is bipartite and $c_f^{\mathcal{G}}(H) = 2$ otherwise.

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