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# ON THE STRUCTURE OF SEQUENTIALLY COHEN–MACAULAY BIGRADED MODULES

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ABSTRACT. Let $K$ be a field and $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be the standard bigraded polynomial ring over $K$. In this paper, we explicitly describe the structure of finitely generated bigraded "sequentially Cohen–Macaulay" $S$-modules with respect to $Q = (y_1, \dots, y_n)$. Next, we give a characterization of sequentially Cohen–Macaulay modules with respect to $Q$ in terms of local cohomology modules. Cohen–Macaulay modules that are sequentially Cohen–Macaulay with respect to $Q$ are considered.

## INTRODUCTION

Let $K$ be a field and $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be the standard bigraded $K$-algebra with $\deg x_i = (1, 0)$ and $\deg y_j = (0, 1)$ for all $i$ and $j$. We set the bigraded irrelevant ideals $P = (x_1, \dots, x_m)$ and $Q = (y_1, \dots, y_n)$. Let $M$ be a finitely generated bigraded $S$-module. The largest integer $k$ for which $H^k_Q(M) \neq 0$, is called the cohomological dimension of $M$ with respect to $Q$ and denoted by $\operatorname{cd}(Q, M)$. A finite filtration $\mathcal{D} : 0 = D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_r = M$ of bigraded submodules of $M$, is called the dimension filtration of $M$ with respect to $Q$ if $D_{i-1}$ is the largest bigraded submodule of $D_i$ for which $\operatorname{cd}(Q, D_{i-1}) < \operatorname{cd}(Q, D_i)$ for all $i = 1, \dots, r$, see [9]. In Section 1, we explicitly describe the structure of the submodules $D_i$ that extends [11, Proposition 2.2]. In fact, it is shown that $D_i = \bigcap_{\mathfrak{p}_j \notin B_{i,Q}} N_j$ for $i = 1, \dots, r-1$ where $0 = \bigcap_{j=1}^{s} N_j$ is a reduced primary decomposition of 0 in $M$ with $N_j$ is $\mathfrak{p}_j$-primary for $j = 1, \dots, s$ and

$$B_{i,Q} = \{\mathfrak{p} \in \operatorname{Ass}(M) : \operatorname{cd}(Q, S/\mathfrak{p}) \leq \operatorname{cd}(Q, D_i)\}.$$

In [10], we say $M$ is Cohen–Macaulay with respect to $Q$, if $\operatorname{grade}(Q, M) = \operatorname{cd}(Q, M)$. A finite filtration $\mathcal{F}$: $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ of $M$ by bigraded submodules $M$, is called a Cohen–Macaulay filtration with respect to $Q$ if each quotient $M_i/M_{i-1}$ is Cohen–Macaulay with respect to $Q$ and

$$0 \leq \operatorname{cd}(Q, M_1/M_0) < \operatorname{cd}(Q, M_2/M_1) < \cdots < \operatorname{cd}(Q, M_r/M_{r-1}).$$

If $M$ admits a Cohen–Macaulay filtration with respect to $Q$, then we say $M$ is sequentially Cohen–Macaulay with respect to $Q$, see [9]. Note that if $M$ is sequentially Cohen–Macaulay with respect to $Q$, then the filtration $\mathcal{F}$ is uniquely determined and it is just the dimension filtration of $M$ with respect to $Q$, that is, $\mathcal{F} = \mathcal{D}$. In

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Section 2, we give a characterization of sequentially Cohen–Macaulay modules with respect to $Q$ in terms of local cohomology modules which extends [4, Corollary 4.4] and [5, Corollary 3.10]. We apply this result and the description of the submodules $M_i$ mentioned earlier, showing that $S/I$ is sequentially Cohen–Macaulay with respect to $P$ and $Q$ where $I$ is the Stanley-Reisner ideal that corresponds to the natural triangulation of the projective plane $\mathbb{P}^2$. Here $S = K[x_1, x_2, x_3, y_1, y_2, y_3]$, $P = (x_1, x_2, x_3)$ and $Q = (y_1, y_2, y_3)$. Note that $S/I$ is Cohen–Macaulay of dimension 3, if $\operatorname{char} K \neq 2$.

In [10] we have shown that if $M$ is a finitely generated bigraded Cohen–Macaulay $S$-module which is Cohen–Macaulay with respect to $P$, then $M$ is Cohen–Macaulay with respect to $Q$. Inspired by this fact and the above example we have the following question: Let $I \subseteq S$ be a monomial ideal. Suppose $S/I$ is Cohen–Macaulay. If $S/I$ is sequentially Cohen–Macauly with respect to $P$, is $S/I$ sequentially Cohen–Macaulay with respect to $Q$? We do not know the answer of this question yet, however in the last section, we obtain some properties of a Cohen–Macaulay filtration with respect to $Q$ in general provided that the module itself is Cohen–Macaulay, see Propositions 3.3 and 3.4. Inspired by Proposition 3.4, we make the following question: Let $M$ be a finitely generated bigraded Cohen–Macaulay $S$-module such that $H^k_Q(M) \neq 0$ for all $\operatorname{grade}(Q, M) \leq k \leq \operatorname{cd}(Q, M)$. Is $H^s_P(M) \neq 0$ for all $\operatorname{grade}(P, M) \leq s \leq \operatorname{cd}(P, M)$? Of course the question has positive answer in the case that $M$ has only one(two) non-vanishing local cohomology with respect to $Q$. The projective plane $\mathbb{P}^2$ would also be the case as module with three non-vanishing local cohomology.

## 1. The dimension filtration with respect to $Q$

Let $K$ be a field and $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ the standard bigraded polynomial ring over $K$. In other words, $\deg x_i = (1, 0)$ and $\deg y_j = (0, 1)$ for all $i$ and $j$. We set the bigraded irrelevant ideals $P = (x_1, \dots, x_m)$ and $Q = (y_1, \dots, y_n)$, and let $M$ be a finitely generated bigraded $S$-module. We denote by $\operatorname{cd}(Q, M)$ the cohomological dimension of $M$ with respect to $Q$ which is the largest integer $i$ for which $H^i_Q(M) \neq 0$. Notice that $0 \leq \operatorname{cd}(Q, M) \leq n$.

We recall the following facts which will be used in the sequel.

**Fact 1.1.**
$$\operatorname{grade}(P, M) \leq \dim M - \operatorname{cd}(Q, M),$$
and the equality holds if $M$ is Cohen–Macaulay, see [10, Formula 5].

Let $q \in \mathbb{Z}$. In [10], we say $M$ is relative Cohen–Macaulay with respect to $Q$ if $H^i_Q(M) = 0$ for all $i \neq q$. In other words, $\operatorname{grade}(Q, M) = \operatorname{cd}(Q, M) = q$. From now on, we omit the word "relative" for simplicity and say $M$ is Cohen–Macaulay with respect to $Q$.

**Fact 1.2.** If $M$ is Cohen–Macaulay with respect to $Q$ with $|K| = \infty$, then
$$\operatorname{cd}(P, M) + \operatorname{cd}(Q, M) = \dim M,$$
see [10, Theorem 3.6].

**Fact 1.3.** The exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated bigraded $S$-modules yields

$$\operatorname{cd}(Q, M) = \max\{\operatorname{cd}(Q, M'), \operatorname{cd}(Q, M'')\},$$

see the general version of [2, Proposition 4.4].

**Fact 1.4.**

$$\operatorname{cd}(Q, M) = \max\{\operatorname{cd}(Q, S/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Ass}(M)\},$$

see the general version of [2, Corollary 4.6].

For a finitely generated bigraded $S$-module $M$, there is a unique largest bigraded submodule $N$ of $M$ for which $\operatorname{cd}(Q, N) < \operatorname{cd}(Q, M)$, see [9, Lemma 1.9]. We recall the following definition from [9].

**Definition 1.5.** We call a filtration $\mathcal{D}$: $0 = D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_r = M$ of bigraded submodules of $M$ the dimension filtration of $M$ with respect to $Q$ if $D_{i-1}$ is the largest bigraded submodule of $D_i$ for which $\operatorname{cd}(Q, D_{i-1}) < \operatorname{cd}(Q, D_i)$ for all $i = 1, \ldots, r$.

**Remark 1.6.** Let $\mathcal{D}$ be the dimension filtration of $M$ with respect to $Q$. For all $i$, the exact sequence $0 \to D_{i-1} \to D_i \to D_i/D_{i-1} \to 0$ by using Fact 1.3 yields

$$\operatorname{cd}(Q, D_i) = \max\{\operatorname{cd}(Q, D_{i-1}), \operatorname{cd}(Q, D_i/D_{i-1})\} = \operatorname{cd}(Q, D_i/D_{i-1}).$$

Thus, $\operatorname{cd}(Q, D_{i-1}/D_{i-2}) < \operatorname{cd}(Q, D_i/D_{i-1})$ for all $i$.

Let $\mathcal{D}$ be the dimension filtration of $M$ with respect to $Q$. We set

$$B_{i,Q} = \{\mathfrak{p} \in \operatorname{Ass}(M) : \operatorname{cd}(Q, S/\mathfrak{p}) \leq \operatorname{cd}(Q, D_i)\}, \;\; I_{i,Q} = \prod_{\mathfrak{p} \in B_{i,Q}} \mathfrak{p}$$

and

$$A_{i,Q} = \{\mathfrak{p} \in \operatorname{Ass}(M) : \mathfrak{p} \in V(I_{i,Q})\} \quad \text{for} \quad i = 1, \ldots, r.$$

**Lemma 1.7.** *Let the notation be as above. Then the following statements hold*

$$A_{i,Q} = B_{i,Q} = \operatorname{Ass}(D_i) \quad \textit{for} \quad i = 1, \ldots, r.$$

*Consequently,*

$$\operatorname{Supp}(D_i) \subseteq V(I_{i,Q}) \quad \textit{for} \quad i = 1, \ldots, r.$$

*Proof.* In order to show the first equality, we note that $B_{i,Q} \subseteq A_{i,Q}$ for $i = 1, \ldots, r$. Now let $\mathfrak{p} \in A_{i,Q}$. Then $\mathfrak{p} \in \operatorname{Ass}(M)$ with $I_{i,Q} \subseteq \mathfrak{p}$. Hence $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{q} \in \operatorname{Ass}(M)$ with $\operatorname{cd}(Q, S/\mathfrak{q}) \leq \operatorname{cd}(Q, D_i)$. The canonical epimorphism $S/\mathfrak{q} \to S/\mathfrak{p}$ yields $\operatorname{cd}(Q, S/\mathfrak{p}) \leq \operatorname{cd}(Q, S/\mathfrak{q})$ by Fact 1.3. It follows that $\mathfrak{p} \in B_{i,Q}$ and hence $A_{i,Q} \subseteq B_{i,Q}$.

To show the second equality, let $\mathfrak{p} \in B_{i,Q}$. Then there is a submodule $N \subseteq M$ such that $N \cong S/\mathfrak{p}$ and $\operatorname{cd}(Q, S/\mathfrak{p}) \leq \operatorname{cd}(Q, D_i)$. Using Fact 1.3 we have

$$\operatorname{cd}(Q, N + D_i) = \max\{\operatorname{cd}(Q, D_i), \operatorname{cd}(Q, N/(N \cap D_i))\} = \operatorname{cd}(Q, D_i),$$

and hence $N \subseteq D_i$. This shows $\mathfrak{p} \in \operatorname{Ass}(D_i)$ and therefore $B_{i,Q} \subseteq \operatorname{Ass}(D_i)$. Now let $\mathfrak{p} \in \operatorname{Ass}(D_i)$. Then $\mathfrak{p} \in \operatorname{Ass}(M)$ and $\operatorname{cd}(Q, S/\mathfrak{p}) \leq \operatorname{cd}(Q, D_i)$ by Fact 1.4. This shows $\mathfrak{p} \in B_{i,Q}$ and hence $\operatorname{Ass}(D_i) \subseteq B_{i,Q}$.

In the following we describe the structure of the submodules $D_i$ in the dimension filtration of $\mathcal{D}$ with respect to $Q$ which extends [11, Proposition 2.2].

**Proposition 1.8.** *Let $\mathcal{D}$ be the dimension filtration of $M$ with respect to $Q$. Then*

$$D_i = H^0_{I_{i,Q}}(M) = \bigcap_{\mathfrak{p}_j \notin B_{i,Q}} N_j$$

*for $i = 1, \ldots, r-1$ where $0 = \bigcap_{j=1}^s N_j$ is a reduced primary decomposition of $0$ in $M$ with $N_j$ is $\mathfrak{p}_j$-primary for $j = 1, \ldots, s$.*

*Proof.* In order to prove the first equality, we have $V(\mathrm{Ann}(D_i)) = \mathrm{Supp}(D_i) \subseteq V(I_{i,Q})$ for $i = 1, \ldots, r-1$ by Lemma 1.7. Since $I_{i,Q}$ is finitely generated, it follows that $I_{i,Q}^{k_i} \subseteq \mathrm{Ann}(D_i)$ for some integer $k_i$ and hence $I_{i,Q}^{k_i} D_i = 0$ for some $k_i$. Thus $D_i = H^0_{I_{i,Q}}(D_i) \subseteq H^0_{I_{i,Q}}(M)$ for $i = 1, \ldots, r-1$.

Now we prove the equality by decreasing induction on $i$. For $i = r-1$, we assume that $D_{r-1} \subsetneq H^0_{I_{r-1,Q}}(M) \subseteq D_r = M$. It follows from the definition dimension filtration that $\mathrm{cd}(Q, H^0_{I_{r-1,Q}}(M)) = \mathrm{cd}(Q, M)$. Note that

$$\mathrm{Ass}\, H^0_{I_{i,Q}}(M) = A_{i,Q} = \mathrm{Ass}(D_i) \quad \text{for} \quad i = 1, \ldots, r-1$$

by [7, Proposition 3.13](c) and Lemma 1.7. It follows that $\mathrm{cd}(Q, H^0_{I_{r-1,Q}}(M)) = \mathrm{cd}(Q, D_{r-1,Q})$, and hence $\mathrm{cd}(Q, D_{r-1,Q}) = \mathrm{cd}(Q, M)$, a contradiction. Thus $D_{r-1,Q} = H^0_{I_{r-1,Q}}(M)$. Now let $1 < i < r-1$, and assume that $D_i = H^0_{I_{i,Q}}(M)$. We show $D_{i-1} = H^0_{I_{i-1,Q}}(M)$. Assume $D_{i-1} \subsetneq H^0_{I_{i-1,Q}}(M)$. As $H^0_{I_{i-1,Q}}(M) \subseteq H^0_{I_{i,Q}}(M) = D_i$, we have $\mathrm{cd}(Q, H^0_{I_{i-1,Q}}(M)) \geq \mathrm{cd}(Q, D_i)$. Since $\mathrm{Ass}\, H^0_{I_{i-1,Q}}(M) = \mathrm{Ass}(D_{i-1})$, it follows that $\mathrm{cd}(Q, D_{i-1}) = \mathrm{cd}(Q, H^0_{I_{i-1,Q}}(M)) \geq \mathrm{cd}(Q, D_i)$, a contradiction. Therefore, $D_{i-1} = H^0_{I_{i-1,Q}}(M)$. The second equality follows from Lemma 1.7 and [7, Proposition 3.13](a). □

**Remark 1.9.** Let $\mathcal{D}$ be the dimension filtration of $M$ with respect to $Q$ with $\mathrm{cd}(Q, M) = q$. We call the submodule

$$D_{r-1} = \bigcap_{\mathfrak{p}_j \notin B_{r-1,Q}} N_j = \bigcap_{\mathrm{cd}(Q, S/\mathfrak{p}_j) = q} N_j,$$

the *unmixed component of $M$ with respect to $Q$* and denote it by $u_{Q,M}(0)$. Notice that $u_{\mathfrak{m},M}(0) = u_M(0)$ introduced by Schenzel in [11]. If $M$ is relatively unmixed with respect to $Q$, that is, $\mathrm{cd}(Q, M) = \mathrm{cd}(Q, S/\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$, then by Proposition 1.8 we have

$$D_i = \bigcap_{\mathfrak{p}_j \notin B_{i,Q}} N_j = \bigcap_{j=1}^{s} N_j = 0 \quad \text{for all} \quad i < r.$$

**Corollary 1.10.** *Let $\mathcal{D}$ be the dimension filtration of $M$ with respect to $Q$. Then for $i = 1, \ldots, r$ we have*

$$\mathrm{Ass}(M/D_i) = \mathrm{Ass}(M) - \mathrm{Ass}(D_i).$$

*Proof.* The assertion follows from Proposition 1.8, Lemma 1.7 and the fact that $\operatorname{Ass} M/H^0_{I_{i,Q}}(M) = \operatorname{Ass}(M) - A_{i,Q}$, see [7, Proposition 3.13](c). □

## 2. Sequentially Cohen–Macaulay with respect to $Q$

We recall the following definition from [9].

**Definition 2.1.** Let $M$ be a finitely generated bigraded $S$-module. We call a finite filtration $\mathcal{F}$: $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ of $M$ by bigraded submodules $M$ a Cohen–Macaulay filtration with respect to $Q$ if

(a) Each quotient $M_i/M_{i-1}$ is Cohen–Macaulay with respect to $Q$;
(b) $0 \leq \operatorname{cd}(Q, M_1/M_0) < \operatorname{cd}(Q, M_2/M_1) < \cdots < \operatorname{cd}(Q, M_r/M_{r-1})$.

We call $M$ to be *sequentially Cohen–Macaulay with respect to* $Q$ if $M$ admits a Cohen–Macaulay filtration with respect to $Q$.

Note that if $M$ is sequentially Cohen–Macaulay with respect to $Q$, then the filtration $\mathcal{F}$ in the definition above is uniquely determined and it is just the dimension filtration of $M$ with respect to $Q$ defined in Definition 1.5, see [9, Proposision 1.12].

We have the following characterization of sequentially Cohen–Macaulay modules with respect to $Q$ in terms of local cohomology modules which extends [4, Corollary 4.4] and [5, Corollary 3.10].

**Proposition 2.2.** *Let $\mathcal{D}$: $0 = D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_r = M$ be the dimension filtration of $M$ with respect to $Q$. Then the following statements are equivalent:*

(a) *$M$ is sequentially Cohen–Macaulay with respect to $Q$;*
(b) *$H^k_Q(M/D_{i-1}) = 0$ for $i = 1, \ldots, r$ and $k < \operatorname{cd}(Q, D_i)$;*
(c) *$\operatorname{grade}(Q, M/D_{i-1}) = \operatorname{cd}(Q, D_i)$ for $i = 1, \ldots, r$.*

*Proof.* $(a) \Rightarrow (b)$: We proceed by decreasing induction on $i$. As $D_i/D_{i-1}$ is Cohen–Macaulay with respect to $Q$ for all $i$, thus for $i = r$ we have $H^k_Q(M/D_{r-1}) = 0$ for $k < \operatorname{cd}(Q, M)$. Now let $1 < i < r$, and assume that $H^k_Q(M/D_{i-1}) = 0$ for $k < \operatorname{cd}(Q, D_i)$. The exact sequence

$$0 \to D_{i-1}/D_{i-2} \to M/D_{i-2} \to M/D_{i-1} \to 0,$$

induces the following long exact sequence

$$\cdots \to H^k_Q(D_{i-1}/D_{i-2}) \to H^k_Q(M/D_{i-2}) \to H^k_Q(M/D_{i-1}) \to \cdots . \tag{1}$$

As $D_{i-1}/D_{i-2}$ is Cohen–Macaulay with respect to $Q$, we have $H^k_Q(D_{i-1}/D_{i-2}) = 0$ for $k < \operatorname{cd}(Q, D_{i-1})$. By Remark 1.6, we have $\operatorname{cd}(Q, D_{i-1}) = \operatorname{cd}(Q, D_{i-1}/D_{i-2}) < \operatorname{cd}(Q, D_i)$. Hence by using (1) and the induction hypothesis, we have $H^k_Q(M/D_{i-2}) = 0$ for $k < \operatorname{cd}(Q, D_{i-1})$, as desired.

$(b) \Rightarrow (a)$: By Remark 1.6 we have $\operatorname{cd}(Q, D_i/D_{i-1}) < \operatorname{cd}(Q, D_{i+1}/D_i)$ for all $i$. Thus it suffices to show that $D_i/D_{i-1}$ is Cohen–Macaulay with respect to $Q$ for all $i$. We prove this statement by decreasing induction on $i$. In condition (b), we first assume $i = r$. It follows that $M/D_{r-1}$ is Cohen–Macaulay with respect to $Q$. Now let $1 < i < r$, and assume that $D_i/D_{i-1}$ is Cohen–Macaulay with respect to $Q$. The exact sequence

$$0 \to D_i/D_{i-1} \to M/D_{i-1} \to M/D_i \to 0,$$

induces the following long exact sequence

$$(2) \qquad \cdots \to H_Q^{k-1}(D_i/D_{i-1}) \to H_Q^{k-1}(M/D_{i-1}) \to H_Q^{k-1}(M/D_i) \to \cdots .$$

Suppose $k < \operatorname{cd}(Q, D_{i-1})$. Induction hypothesis and our assumption say that $H_Q^{k-1}(D_i/D_{i-1}) = H_Q^{k-1}(M/D_i) = 0$. Hence $H_Q^{k-1}(M/D_{i-1}) = 0$ by (2). We have $H_Q^k(M/D_{i-2}) = 0$ for $k < \operatorname{cd}(Q, D_{i-1})$ because of our assumption again. Thus $H_Q^k(D_{i-1}/D_{i-2}) = 0$ for $k < \operatorname{cd}(Q, D_{i-1})$ by (1). Therefore $D_{i-1}/D_{i-2}$ is Cohen–Macaulay with respect to $Q$, as desired.

$(b) \Rightarrow (c)$: We set $\operatorname{cd}(Q, D_i) = \operatorname{cd}(Q, D_i/D_{i-1}) = q_i$ for $i = 1, \ldots, r$. Our assumption says that $\operatorname{grade}(Q, M/D_{i-1}) \geq q_i$ for $i = 1, \ldots, r$. We only need to know $H_Q^{q_i}(M/D_{i-1}) \neq 0$. Consider the long exact sequence

$$(3) \qquad \cdots \to H_Q^{q_i-1}(M/D_i) \to H_Q^{q_i}(D_i/D_{i-1}) \to H_Q^{q_i}(M/D_{i-1}) \to \cdots .$$

Since $q_i - 1 < q_i < q_{i+1}$, it follows from our assumption that $H_Q^{q_i-1}(M/D_i) = 0$. If $H_Q^{q_i}(M/D_{i-1}) = 0$, then by (3) we have $H_Q^{q_i}(D_i/D_{i-1}) = 0$, a contradiction. The implication $(c) \Rightarrow (b)$ is obvious. $\square$

As an application of Proposition 1.8 and Proposition 2.2 we have

**Example 2.3.** Let $I$ be the Stanley-Reisner ideal that corresponds to the natural triangulation of the projective plane $\mathbb{P}^2$. Then

$$I = (x_1x_2x_3, x_1x_2y_1, x_1x_3y_2, x_1y_1y_3, x_1y_2y_3, x_2x_3y_3, x_2y_1y_2, x_2y_2y_3, x_3y_1y_2, x_3y_1y_3).$$

We set $R = S/I$ where $S = K[x_1, x_2, x_3, y_1, y_2, y_3]$, $P = (x_1, x_2, x_3)$ and $Q = (y_1, y_2, y_3)$. Our aim is to show that $R$ is sequentially Cohen–Macaulay with respect to $P$ and $Q$. Note that $R$ is Cohen–Macaulay of dimension 3 if $\operatorname{char} K \neq 2$. The ideal $I$ has the minimal primary decomposition $I = \bigcap_{i=1}^{10} \mathfrak{p}_i$ where $\mathfrak{p}_1 = (x_3, y_1, y_3), \mathfrak{p}_2 = (x_1, y_1, y_3), \mathfrak{p}_3 = (x_2, y_1, y_2), \mathfrak{p}_4 = (x_3, y_1, y_2), \mathfrak{p}_5 = (x_1, y_2, y_3), \mathfrak{p}_6 = (x_2, y_2, y_3), \mathfrak{p}_7 = (x_2, x_3, y_3), \mathfrak{p}_8 = (x_1, x_2, y_1), \mathfrak{p}_9 = (x_1, x_3, y_2), \mathfrak{p}_{10} = (x_1, x_2, x_3)$. As $P = \mathfrak{p}_{10} \in \operatorname{Ass}(R)$, we have $\operatorname{grade}(P, R) = 0$. By Fact 1.4 we have $\operatorname{cd}(P, R) = 2$ and $\operatorname{cd}(Q, R) = 3$. Since $R$ is Cohen–Macaulay, it follows from Fact 1.1 that $\operatorname{grade}(Q, R) = 1$. We first show that $R$ is sequentially Cohen–Macaulay with respect to $P$. By Proposition 1.8, $R$ has the dimension filtration

$$0 = R_0 \subsetneq R_1 \subsetneq R_2 \subsetneq R_3 = R,$$

with respect to $P$ where

$$R_1 = \bigcap_{i=1}^{9} \mathfrak{p}_i/I \quad \text{and} \quad R_2 = \bigcap_{i=1}^{6} \mathfrak{p}_i/I.$$

By Corollary 1.10 we have

$$\operatorname{Ass}(R_1) = \operatorname{Ass}(R) - \operatorname{Ass}(R/R_1) = \{\mathfrak{p}_{10}\}.$$

and

$$\mathrm{Ass}(R_2) = \mathrm{Ass}(R) - \mathrm{Ass}(R/R_2) = \{\mathfrak{p}_7, \mathfrak{p}_8, \mathfrak{p}_9, \mathfrak{p}_{10}\}.$$

It follows that $\mathrm{cd}(P, R_1) = 0$ and $\mathrm{cd}(P, R_2) = 1$. We set $I_1 = \bigcap_{i=1}^{9} \mathfrak{p}_i$ and $I_2 = \bigcap_{i=1}^{6} \mathfrak{p}_i$. In view of Proposition 2.2, we need to show

$$\mathrm{grade}(P, R_3/R_0) = \mathrm{grade}(P, R) = \mathrm{cd}(P, R_1) = 0,$$

$$\mathrm{grade}(P, R_3/R_1) = \mathrm{grade}(P, S/I_1) = \mathrm{cd}(P, R_2) = 1$$

and

$$\mathrm{grade}(P, R_3/R_2) = \mathrm{grade}(P, S/I_2) = \mathrm{cd}(P, R) = 2.$$

The first equality is obvious. As $P \not\subseteq \mathfrak{p}_i$ for $i = 1, \ldots, 9$, we have $\mathrm{grade}(P, S/I_1) \geq 1$. On the other hand, $\mathrm{grade}(P, S/I_1) \leq \dim S/I_1 - \mathrm{cd}(Q, S/I_1) = 3 - 2 = 1$. Thus the second equality holds. In order to show the third equality, we note that $S/I_2$ has dimension 3 and by using CoCoA [3] depth 2. Thus Fact 1.1 can not be used to compute $\mathrm{grade}(P, S/I_2)$. We set $\mathfrak{q}_1 = \mathfrak{p}_1 \cap \mathfrak{p}_2 = (x_1x_3, y_1, y_3)$, $\mathfrak{q}_2 = \mathfrak{p}_3 \cap \mathfrak{p}_4 = (x_2x_3, y_1, y_2)$ and $\mathfrak{q}_3 = \mathfrak{p}_5 \cap \mathfrak{p}_6 = (x_1x_2, y_2, y_3)$. Consider the exact sequence

$$0 \to S/\mathfrak{q}_1 \cap \mathfrak{q}_2 \to S/\mathfrak{q}_1 \oplus S/\mathfrak{q}_2 \to S/(\mathfrak{q}_1 + \mathfrak{q}_2) \to 0.$$

Since $\mathrm{grade}(P, S/\mathfrak{q}_1 \oplus S/\mathfrak{q}_2) = 2$ and $\mathrm{grade}(P, S/(\mathfrak{q}_1 + \mathfrak{q}_2)) = 1$, it follows that $\mathrm{grade}(P, S/(\mathfrak{q}_1 \cap \mathfrak{q}_2)) \geq 2$. As $\mathrm{cd}(P, S/(\mathfrak{q}_1 \cap \mathfrak{q}_2)) = 2$, we have $\mathrm{grade}(P, S/(\mathfrak{q}_1 \cap \mathfrak{q}_2)) = 2$. Consider the exact sequence

$$0 \to S/I_2 \to S/\mathfrak{q}_1 \cap \mathfrak{q}_2 \oplus S/\mathfrak{q}_3 \to S/(\mathfrak{q}_1 + \mathfrak{q}_3) \cap (\mathfrak{q}_2 + \mathfrak{q}_3) \to 0. \tag{4}$$

The exact sequence

$$0 \to S/(\mathfrak{q}_1 + \mathfrak{q}_3) \cap (\mathfrak{q}_2 + \mathfrak{q}_3) \to S/(\mathfrak{q}_1 + \mathfrak{q}_3) \oplus S/(\mathfrak{q}_2 + \mathfrak{q}_3) \to S/(\mathfrak{q}_1 + \mathfrak{q}_2 + \mathfrak{q}_3) \to 0$$

yields that $\mathrm{grade}(P, S/(\mathfrak{q}_1 + \mathfrak{q}_3) \cap (\mathfrak{q}_2 + \mathfrak{q}_3)) \geq 1$. Hence by (4) we have $\mathrm{grade}(P, S/I_2) \geq 2$. As $\mathrm{cd}(P, S/I_2) = 2$, we conclude that $\mathrm{grade}(P, S/I_2) = 2$, as desired.

Next, we show that $R$ is sequentially Cohen–Macaulay with respect to $Q$. By Proposition 1.8, $R$ has the dimension filtration $0 = R_0 \subsetneq R_1 \subsetneq R_2 \subsetneq R_3 = R$ with respect to $Q$ where $R_1 = \bigcap_{i=7}^{10} \mathfrak{p}_i/I$ and $R_2 = \mathfrak{p}_{10}/I$. By Corollary 1.10 we have $\mathrm{cd}(Q, R_1) = 1$ and $\mathrm{cd}(Q, R_2) = 2$. We set $J = \bigcap_{i=7}^{10} \mathfrak{p}_i$. In view of Proposition 2.2, we need to show

$$\mathrm{grade}(Q, R_3/R_0) = \mathrm{grade}(Q, R) = \mathrm{cd}(Q, R_1) = 1,$$

$$\mathrm{grade}(Q, R_3/R_1) = \mathrm{grade}(Q, S/J) = \mathrm{cd}(Q, R_2) = 2$$

and

$$\mathrm{grade}(Q, R_3/R_2) = \mathrm{grade}(Q, S/\mathfrak{p}_{10}) = \mathrm{cd}(Q, R) = 3.$$

The first and the third statements are obvious. In order to prove the second equality, consider the exact sequence

$$0 \to S/J \to S/\cap_{i=7}^{9} \mathfrak{p}_i \oplus S/\mathfrak{p}_{10} \to S/\cap_{i=7}^{9} (\mathfrak{p}_i + \mathfrak{p}_{10}) \to 0. \tag{5}$$

An exact sequence argument shows that

$$\mathrm{grade}(Q, S/\cap_{i=7}^{9} \mathfrak{p}_i) = \mathrm{grade}(Q, S/\cap_{i=7}^{9} (\mathfrak{p}_i + \mathfrak{p}_{10})) = 2.$$

Thus it follows from (5) that $\operatorname{grade}(Q, S/J) \geq 2$. On the other hand,

$$\operatorname{grade}(Q, S/J) \leq \dim S/J - \operatorname{cd}(P, S/J) = 3 - 1 = 2.$$

Therefore, $\operatorname{grade}(Q, S/J) = 2$, as desired.

## 3. Cohen–Macaulay modules that are sequentially Cohen–Macaulay with respect to $Q$

In [10] we have shown that if $M$ is a finitely generated bigraded Cohen–Macaulay $S$-module which is Cohen–Macaulay with respect to $P$, then $M$ is Cohen–Macaulay with respect to $Q$. Inspired by this fact and Example 2.3 we have the following question

**Question 3.1.** *Let $I \subseteq S$ be a monomial ideal. Suppose $S/I$ is Cohen–Macaulay. If $S/I$ is sequentially Cohen–Macauly with respect to $P$, is $S/I$ sequentially Cohen–Macaulay with respect to $Q$?*

We do not know the answer of this question yet, however in this section, we obtain some properties of a Cohen–Macaulay filtration with respect to $Q$ in general provided that the module itself is Cohen–Macaulay.

**Fact 3.2.** For a Cohen–Macaulay filtration $\mathcal{F}$ with respect to $Q$ we recall the following fact from [9, Corollary 1.8]

$$\operatorname{grade}(Q, M_i) = \operatorname{grade}(Q, M) \quad \text{for} \quad i = 1, \ldots, r.$$

**Proposition 3.3.** *Let $M$ be a finitely generated bigraded Cohen–Macaulay $S$-module with $|K| = \infty$. Suppose $M$ is sequentially Cohen–Macaulay with respect to $Q$ with the Cohen–Macaulay filtration $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ with respect to $Q$. Then*

(a) $\operatorname{cd}(P, M_i) = \operatorname{cd}(P, M)$ *for* $i = 1, \ldots, r$.
(b) $\operatorname{grade}(Q, M_i) + \operatorname{cd}(P, M_i) = \dim M_i$ *for* $i = 1, \ldots, r$.

*Proof.* In order to prove (a), since $M_1$ is Cohen–Macaulay with respect to $Q$, it follows from Fact 1.2 that $\operatorname{cd}(P, M_1) + \operatorname{cd}(Q, M_1) = \dim M_1$. By Fact 3.2 we have $\operatorname{cd}(Q, M_1) = \operatorname{grade}(Q, M_1) = \operatorname{grade}(Q, M)$. Since $M$ is Cohen–Macaulay, it follows from [9, Lemma 1.11] that $\dim M_1 = \dim M$ and $\operatorname{cd}(P, M) = \dim M - \operatorname{grade}(Q, M)$ by Fact 1.1. Thus we conclude that $\operatorname{cd}(P, M_1) = \operatorname{cd}(P, M)$. As by Fact 1.3 we have $\operatorname{cd}(P, M_{i-1}) \leq \operatorname{cd}(P, M_i)$ for all $i$, the first equality follows.

For the proof (b), by [9, Lemma 1.11] we have $\dim M_i = \dim M$ for $i = 1, \ldots, r$. Thus the second equalities follow from Fact 1.1, Fact 3.2 and part (a). □

**Proposition 3.4.** *Let the assumptions and the notation be as in Proposition 3.3. Then the following statements are equivalent:*

(a) $\operatorname{cd}(P, M) + \operatorname{cd}(Q, M) = \dim M + r - 1$;
(b) $H^s_Q(M) \neq 0$ *for all* $\operatorname{grade}(Q, M) \leq s \leq \operatorname{cd}(Q, M)$.

*Proof.* We first assume that $r = 1$. As $M$ is Cohen–Macaulay, by Fact 1.1 and Fact 1.2 we have $\operatorname{cd}(P, M) + \operatorname{cd}(Q, M) = \dim M$ if and only if $M$ is Cohen–Macaulay with respect to $Q$. Thus the claim holds in this case. Now let $r \geq 2$. By Fact 1.1 we have

$\operatorname{cd}(P,M)+\operatorname{cd}(Q,M)=\dim M+r-1$ if and only if $\operatorname{cd}(Q,M)-\operatorname{grade}(Q,M)=r-1$. This is equivalent to saying that $\operatorname{cd}(Q,M_{i+1})=\operatorname{cd}(Q,M_i)+1$ for $i=1,\ldots,r-1$ by Fact 3.2. By [9, Proposition 1.7] this is equivalent to saying that $H^s_Q(M)\neq 0$ for all $\operatorname{grade}(Q,M)\leq s\leq \operatorname{cd}(Q,M)$. $\square$

The following example shows that the condition that "$M$ is Cohen–Macaulay" is required for Proposition 3.4.

**Example 3.5.** We set $K[x]=K[x_1,\ldots,x_m]$ and $K[y]=K[y_1,\ldots,y_n]$. Let $L$ be a non-zero finitely generated graded $K[x]$-module of depth 0 and dimension 1, and $N$ a non-zero finitely generated graded $K[y]$-module of depth 0 and dimension 1. We set $M=L\otimes_K N$ and consider it as $S$-module. One has $\operatorname{depth} M=0$ and $\dim M=2$. Hence $M$ is not Cohen–Macaulay. On the other hand, $\operatorname{grade}(Q,M)=\operatorname{depth} N=0$ and $\operatorname{cd}(Q,M)=\dim N=1=\dim L=\operatorname{cd}(P,M)$. Hence $M$ is sequentially Cohen–Macaulay with respect to $Q$ which satisfies condition (b) in Proposition 3.4, while the equality (a) does not hold.

The following question is inspired by Proposition 3.4.

**Question 3.6.** Let $M$ be a finitely generated bigraded Cohen–Macaulay $S$-module such that $H^k_Q(M)\neq 0$ for all $\operatorname{grade}(Q,M)\leq k\leq \operatorname{cd}(Q,M)$. Is $H^s_P(M)\neq 0$ for all $\operatorname{grade}(P,M)\leq s\leq \operatorname{cd}(P,M)$?

**Remark 3.7.** Of course the question has positive answer in the following cases, namely, if $M$ has only one(two) non-vanishing local cohomology with respect to $Q$. This immediately follows by Fact 1.1. The projective plane $\mathbb{P}^2$ given in Example 2.3 is also the case as module with three non-vanishing local cohomology.

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