Marriages are made in calculation

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Abstract

Butterflies in the stomach, racing heart and sweaty hands... is he really the one? The mathematician simply sits down to the computer and calculates it quickly. On the side she makes important discoveries that come handy in various other fields of life, such as job applications.

In our everyday life we are surrounded by mathematics. It plays a role in planning logistics, scheduling buses, organizing soccer tournaments, establishing evacuation routes, only to name a few. But is it possible to model 'partner finding' as well with the help of numbers and forms? In my PhD thesis, I studied the 'stable marriage problem', which has been in the spotlight of research for over 50 years. In 2012, Alvin Roth and Lloyd Shapley were awarded the Nobel Memorial Prize in Economic Sciences for their achievements in the field. What is this mathematical model that is so often used in the industry and can be illustrated through romantic bonds?

Imagine a group of men and a group of women, all of whom are assumed to be heterosexual. Every person draws up a list. The first on this list is his or her true love, the second is their second best match and so on. Our goal now is to match these people so that their marriages stand the test of time. Instability is induced by a man and a woman who are not married to each other, yet they prefer each other to their respective partners or to being single. In a stable matching no such pair occurs.

David Gale and Lloyd Shapley proved that such a stable matching always exists, moreover, it can be found via a simple procedure. This procedure –also often referred to as 'algorithm' by us mathematicians– works through a series of old-fashioned proposals. Each man asks out the woman of his first-choice. Every woman who received at least one request takes the best one very cautiously only temporarily and she rejects the rest of them. Thus we have some couples now, but they are not married yet. In the second round, each single man asks out the best woman on his list who has not rejected him yet. Then the women contemplate whether they stay with their current partner or accept one of the new offers. If a woman decides to take a new offer, then her previous partner becomes single again. The algorithm runs in such a manner until every man is either in a relationship or he has been rejected by every woman on his list. The matching derived by this procedure is not only stable, but it is also the best stable matching for all men — and at the same time, it is the worst stable matching for all women. The reason for this is that while men were expressing their preferences, the women were forced to sit quietly and wait for offers. The takeaway message is thus: the more active you are during dating, the better partner you receive. And this advice is not coming from a motivational self-help book, but it is proved with mathematical rigor.

We all know that feelings cannot be modeled as ordered lists, moreover, probably nobody would like to automatize dating. Thus, stable matchings serve rather as a metaphor for the main concept of stability, and the statements in the topic marriage are not to be taken literally. However, the concept has been used for over 60 years in assigning medical residents to hospitals. In this application, the job openings play the role of women, while the prospective residents are the men. The goal is to find an assignment in which every resident is matched to the best hospital that could not fill its positions with better applicants. The wide range of possible applications contains admission decision at colleges and in schools, schedules at sports events, living donor kidney exchange programs, online auctions and distribution of dormitory places.

It is easy to see on these applications that the efficiency of the matching algorithm is absolutely necessary: it must run very fast even if there are a huge number of participants. For example, the residency matching program in the U.S. assigns over 40000 residents to hospitals each year. Besides, the algorithm must be adjusted to various extensions of the basic problem.

Many of these extensions are discussed in my thesis with one of them being the case of changing preferences. The above described algorithm is based on the assumption that no new person arrives to the scheme and that every participant sticks to their original preference list. Reality proves to be different though, since there will always be a resident who changes their preferences in the last minute. The smallest change in the allocation can induce a huge chaos in the system, in which one resident reassignment is followed by another one. When and how does this avalanche come to a halt? I have shown for complex and realistic systems –for example when hospitals advertise several open positions– that after altering their preference lists the participants can soon find a new stable matching on their own. The main point here is that the participants stabilize the matching without the interaction of any outside authority. This principle can be carried over to other problems as well, such as player transfer in soccer championships or allocating roommates in dormitories.

During a longer research stay in India I noticed that the Western view on arranged marriages is quite narrow. Several people who underwent the process, testified that arranged marriages are often success stories. As a researcher of stable matchings, I had to take a closer look at the problem, of course, from a scientific point of view. The original model can easily be extended to capture the wish of parents as well. The young men and women form the previously described system. Naturally, they all have their own preferences, but their parents' wishes might differ from this as day and night. Strict parents prescribe a forced partner, while the lenient ones only forbid some potential relationships. The goal is to find a matching that is stable with respect to the youngsters preferences, moreover it contains all forced marriages while it avoids all forbidden ones. In this way all parents and their children will be made content and on the top of this, there is no inclination for an affair among the youngsters. A forced marriage translates into the situation in the residency placement application where a hospital director is determined to hire a resident even if he or she belongs to the weaker candidates.

It is easy to see that the wishes of the parents can undermine the stability of the marriages. Any marriage scheme will be destabilized by a forbidden relationship in which the man and the woman both are each other's dream partners. Thus the next question rises naturally: How can one find a matching that is approved of by all parents and it has the fewest possible number of potential affairs? It turns out that computing such a matching is at least as hard as the solving a so-called NP-complete problem. These problems have been well studied by mathematicians and computer scientists for decades, and despite of a considerable amount of effort on their side, nobody has managed to come up with a fast algorithm for any NP-complete problem. A one-million-dollar prize is offered as a reward for the person who fist cracks the nut and answers the question whether such a fast algorithm can exist at all. It is therefore very probable that a good set of marriages under forced and forbidden pairs cannot be computed in reasonable time, even with the fastest computers of the world. This of course also holds in all applications: manipulation in the form of forced and forbidden pairs make the problem significantly hard to solve. What could frighten away backward-thinking parents from matchmaking if not this result?

The existing allocation models are not sufficient for several highly complex problems. For example, supply chains of large firms have a flow of goods through numerous vendors. Each of these vendors can specify their own preferences over their possible deals. I extended the above described proposal-acceptance algorithm for such networks. Moreover, I studied the case of market manipulation in the form of forced and forbidden deals. These results opened doors to novel areas of applications where stability has not been used yet.

The models developed by me can efficiently handle preference changes and they capture complex networks. Furthermore, I showed that manipulation in the form of forced and forbidden pairs is next to impossible to implement. The main strength of these results is the fact that they can be universally used in all fields of applications. Additionally, I also proved that a breakup is not the end of the world and that arranged marriage is rather a bad idea. In the end, one receives the same advice from mathematics: The right thing to do is to listen to the heart. Is there a soft kernel hiding in the ice cold mathematics, if one takes a closer look?