

# Upper Domination: Complexity and Approximation

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**Abstract.** We consider UPPER DOMINATION, the problem of finding a maximum cardinality *minimal* dominating set in a graph. We show that this problem does not admit an  $n^{1-\epsilon}$  approximation for any  $\epsilon > 0$ , making it significantly harder than DOMINATING SET, while it remains hard even on severely restricted special cases, such as cubic graphs (APX-hard), and planar subcubic graphs (NP-hard). We complement our negative results by showing that the problem admits an  $O(\Delta)$  approximation on graphs of maximum degree  $\Delta$ , as well as an EPTAS on planar graphs. Along the way, we also derive essentially tight  $n^{1-\frac{1}{d}}$  upper and lower bounds on the approximability of the related problem MAXIMUM MINIMAL HITTING SET on  $d$ -uniform hypergraphs, generalising known results for MAXIMUM MINIMAL VERTEX COVER.

## 1 Introduction

A dominating set of an undirected graph  $G = (V, E)$  is a set of vertices  $S \subseteq V$  such that all vertices outside of  $S$  have a neighbour in  $S$ . The problem of finding the smallest dominating set of a given graph is one of the most widely studied problems in computational complexity. In this paper, we focus on a related problem that “flips” the optimisation objective. In UPPER DOMINATION we are given a graph and we are asked to find a *maximum cardinality* dominating set that is still minimal. A dominating set is minimal if any proper subset of it is no longer dominating, that is, if it does not contain obviously redundant vertices.

Considering a MaxMin or MinMax version of a problem by “flipping” the objective is not a new idea; in fact, such questions have been posed before for

many classical optimisation problems. Some of the most well-known examples include the MINIMUM MAXIMAL INDEPENDENT SET problem [9, 10, 14, 19] (also known as MINIMUM INDEPENDENT DOMINATING SET), the MAXIMUM MINIMAL VERTEX COVER problem [7, 26] and the LAZY BUREAUCRAT problem [2, 4], which is a MinMax version of KNAPSACK. The initial motivation for this type of question was rather straightforward: most classical optimisation problems admit an easy, naive heuristic algorithm which starts with a trivial solution and then gradually tries to improve it in an obvious way until it gets stuck. For example, one can produce a (maximal) independent set of a graph by starting with a single vertex and then adding vertices to the current solution while maintaining an independent set. What can we say about the worst-case performance of such a basic algorithm? Motivated by this initial question the study of MaxMin and MinMax versions of standard optimisation problems has gradually grown into a sub-field with its own interest, often revealing new insights on the structure of the original problems. UPPER DOMINATION is a natural example of this family of problems, on which somewhat fewer results are currently known. A typical pattern that often shows up in this line of research is that MaxMin versions of classical problems turn out to be much harder than the originals, especially when one considers approximation. For example, MAXIMUM MINIMAL VERTEX COVER does not admit any  $n^{\frac{1}{2}-\epsilon}$  approximation, while VERTEX COVER admits a 2-approximation [7]; LAZY BUREAUCRAT is APX-hard while KNAPSACK admits a PTAS [2]; and though MINIMUM MAXIMAL INDEPENDENT SET and INDEPENDENT SET share the same (inapproximable) status, the proof of inapproximability of the MinMax version is considerably simpler, and was known long before the corresponding hardness results for INDEPENDENT SET [14].

Our first contribution is to show that this pattern also holds for UPPER DOMINATION: while DOMINATING SET admits a greedy  $\ln n$  approximation, UPPER DOMINATION does not admit an  $n^{1-\epsilon}$  approximation for any  $\epsilon > 0$ , unless  $P=NP$ . We establish this by considering the related MAXIMUM MINIMAL HITTING SET problem: given a  $d$ -uniform hypergraph, find the largest minimal set of vertices that intersects all hyperedges. Observe that the previously studied MAXIMUM MINIMAL VERTEX COVER problem is a special case of this problem for  $d = 2$ . We show, for any  $d$ , an approximation algorithm with ratio  $n^{1-\frac{1}{d}}$ , for MAXIMUM MINIMAL HITTING SET on  $d$ -uniform hypergraphs, as well as a tight  $n^{1-\frac{1}{d}-\epsilon}$  inapproximability bound, exactly matching, and subsuming, the corresponding tight  $\sqrt{n}$  approximation results for MAXIMUM MINIMAL VERTEX COVER given in [7]. We then obtain the inapproximability of UPPER DOMINATION by performing a reduction from an instance with sufficiently large  $d$ . We also show that UPPER DOMINATION remains hard on two restricted cases: the problem is still APX-hard on cubic graphs, and NP-hard on planar subcubic graphs. Since the problem is easy on graphs of maximum degree 2, our results completely characterise the complexity of the problem in terms of maximum degree (the best previously known result was NP-hardness for planar graphs of maximum degree 6 [1]). Given the general behavior of this type of problem, and the above results on UPPER DOMINATION in particular, the questions remains *why* are such problems typically so much harder than their original versions. Consider the following

*extension* problem: Given a graph  $G = (V, E)$  and a set  $S \subseteq V$ , does there exist a minimal dominating set of *any size* that contains  $S$ ? Even though questions of this type are typically trivial for problems such as INDEPENDENT SET and LAZY BUREAUCRAT, it can be shown by a more or less easy modification of the proof of analogous results in [8, 22] that in the case of UPPER DOMINATION, deciding the existence of such a minimal dominating set is NP-hard in general graphs. This helps explain the added difficulty of this problem, and more generally of problems of this type, since any natural algorithm that gradually builds a solution would have to contend with (some version of) this extension problem. In this paper we show that the extension problem for UPPER DOMINATION remains hard even for planar cubic graphs.

We complement the above negative results by giving some approximation algorithms for the problem in restricted cases. Specifically, we show that the problem admits an  $O(\Delta)$ -approximation on graphs with maximum degree  $\Delta$ , as well as an EPTAS on planar graphs.

*Previous results.* It has long been known that UPPER DOMINATION is NP-complete in general [11], and even for graphs of maximum degree 6 [1]. Some polynomial-time solvable graph classes are also known. This is mainly due to the fact that on certain graph classes (like bipartite graphs) the independence number and upper domination number coincide and for those graph classes, the independence number can be computed in polynomial-time. We refer to the textbook on domination [16] for further details. We mention that the problem is polynomial for bipartite graphs [12], chordal graphs [20], generalised series-parallel graphs [15] and graphs with bounded clique-width [13]. Recently, the complexity of UPPER DOMINATION in monogenic classes of graphs defined by a single forbidden induced subgraph has led to a complexity dichotomy: if the unique forbidden induced subgraph is a  $P_4$  or a  $2K_2$  (or an induced subgraph of these), then UPPER DOMINATION is polynomial; otherwise, it is NP-complete [1].

## 2 Preliminaries and Combinatorial Bounds on $\Gamma(G)$

We only deal with undirected simple connected graphs  $G = (V, E)$ . The number of vertices  $n = |V|$  is known as the order of  $G$ . As usual,  $N(v)$  denotes the open neighbourhood of  $v$ , and  $N[v]$  is the closed neighbourhood of  $v$ , i.e.,  $N[v] = N(v) \cup \{v\}$ , which easily extends to vertex sets  $X$ , i.e.,  $N(X) = \bigcup_{x \in X} N(x)$  and  $N[X] = N(X) \cup X$ . The cardinality of  $N(v)$  is known as the degree of  $v$ , denoted as  $\deg(v)$ . The maximum degree in a graph is written as  $\Delta$ . A graph of maximum degree three is called subcubic, and if all degrees equal three, it is called cubic.

Given a graph  $G = (V, E)$ , a subset  $S$  of  $V$  is a *dominating set* if every vertex  $v \in V \setminus S$  has at least one neighbour in  $S$ , i.e., if  $N[S] = V$ . A dominating set is minimal if no proper subset of it is a dominating set. Likewise, a vertex set  $I$  is *independent* if  $N(I) \cap I = \emptyset$ . An independent set is maximal if no proper superset is independent. In the following we use classical notations:  $\gamma(G)$  and

$\Gamma(G)$  are the minimum and maximum cardinalities over all minimal dominating sets in  $G$ ,  $\alpha(G)$  and  $i(G)$  are the maximum and minimum cardinalities over all maximal independent sets, and  $\tau(G)$  is the size of a minimum vertex cover, which equals  $|V| - \alpha(G)$  by Gallai's identity. A minimal dominating set  $D$  of  $G$  with  $|D| = \Gamma(G)$  is also known as an *upper dominating set* of  $G$ .

For any subset  $S \subseteq V$  and  $v \in S$  we define the private neighbourhood of  $v$  with respect to  $S$  as  $pn(v, S) := N[v] \setminus N[S \setminus \{v\}]$ . Any  $w \in pn(v, S)$  is called a *private neighbour of  $v$  with respect to  $S$* .  $S$  is called *irredundant* if every vertex in  $S$  has at least one private neighbour, i.e., if  $|pn(v, S)| > 0$  for every  $v \in S$ .  $IR(G)$  denotes the cardinality of the largest irredundant set in  $G$ , while  $ir(G)$  is the cardinality of the smallest maximal irredundant set in  $G$ . We can now observe the validity of the well-known domination chain.

$$ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G)$$

The domination chain is largely due to the following two combinatorial properties: (1) Every maximal independent set is a minimal dominating set. (2) A dominating set  $S \subseteq V$  is minimal if and only if  $|pn(v, S)| > 0$  for every  $v \in S$ . Observe that  $v$  can be a private neighbour of itself, i.e., a dominating set is minimal if and only if it is also an irredundant set. Actually, every minimal dominating set is also a maximal irredundant set.

Any minimal dominating set  $D$  for a graph  $G = (V, E)$  can be associated with a partition of  $V$  into four sets  $F, I, P, O$  given by:  $I := \{v \in D : v \in pn(v, D)\}$ ,  $F := D \setminus I$ ,  $P \in \{B \subseteq N(F) \setminus D : |pn(v, D) \cap B| = 1 \text{ for all } v \in F\}$  with  $|F| = |P|$ ,  $O = V \setminus (D \cup P)$ . This representation is not necessarily unique since there might be different choices for  $P$  and  $O$ , but for every partition of this kind, the following properties hold: **(1)** Every vertex  $v \in F$  has at least one neighbour in  $F$ , called a **friend**. **(2)** The set  $I$  is an **independent set** in  $G$ . **(3)** The subgraph induced by  $F \cup P$  has an edge cut set separating  $F$  and  $P$  that is also a perfect matching; hence,  $P$  is a set of **private neighbours** for  $F$ . **(4)** The neighbourhood of a vertex in  $I$  is always a subset of  $O$ , which are otherwise the **outsiders**. This partition is also related to a different characterisation of  $\Gamma(G)$  in terms of so-called upper perfect neighbourhoods [16].

**Lemma 1.** *For any connected graph  $G$  with  $n > 0$  vertices we have:*

$$\alpha(G) \leq \Gamma(G) \leq \max \left\{ \alpha(G), \frac{n}{2} + \frac{\alpha(G)}{2} - 1 \right\} \tag{1}$$

**Lemma 2.** *For any connected graph  $G$  with  $n > 0$  vertices, minimum degree  $\delta$  and maximum degree  $\Delta$ , we have:*

$$\alpha(G) \leq \Gamma(G) \leq \max \left\{ \alpha(G), \frac{n}{2} + \frac{\alpha(G)(\Delta - \delta)}{2\Delta} - \frac{\Delta - \delta}{\Delta} \right\} \tag{2}$$

Note that Lemma 2 generalises the earlier result of Henning and Slater on upper bounds on  $IR(G)$  (and hence on  $\Gamma(G)$ ) for  $\Delta$ -regular graphs  $G$  [17].

### 3 Hardness Results for Upper Domination

In this section we demonstrate several results that indicate that UPPER DOMINATION is a rather hard problem: it does not admit any non-trivial approximation in polynomial time, and it remains hard even in quite restricted cases.

#### 3.1 Hardness of Approximation on General Graphs

We show that UPPER DOMINATION is hard to approximate in two steps: first, we show that a related natural problem, MAXIMUM MINIMAL HITTING SET, is hard to approximate, and then we show that this problem is essentially equivalent to UPPER DOMINATION.

The MAXIMUM MINIMAL HITTING SET problem is the following: we are given a hypergraph, that is, a base set  $V$  and a collection  $F$  of subsets of  $V$ . We wish to find a set  $H \subseteq V$  such that: (1) For all  $e \in F$  we have  $e \cap H \neq \emptyset$  (i.e.,  $H$  is a hitting set) (2) For all  $v \in H$  there exists  $e \in F$  such that  $e \cap H = \{v\}$  (i.e.,  $H$  is minimal) (3)  $H$  is as large as possible. This problem generalises UPPER DOMINATION: given a graph  $G = (V, E)$ , we can produce a hypergraph by keeping the same set of vertices and creating a hyperedge for each closed neighbourhood  $N[v]$  of  $G$ . An upper dominating set of the original graph is now exactly a minimal hitting set of the constructed hypergraph. We will also show that MAXIMUM MINIMAL HITTING SET can be reduced to UPPER DOMINATION.

Let us note that MAXIMUM MINIMAL HITTING SET, as defined here, also generalises MAXIMUM MINIMAL VERTEX COVER, which corresponds to instances where the input hypergraph is actually a graph. We recall that for this problem there exists a  $n^{1/2}$ -approximation algorithm, while it is known to be  $n^{1/2-\epsilon}$ -inapproximable [7]. Here, we generalise this result to arbitrary hypergraphs, taking into account the sizes of the hyperedges allowed.

**Theorem 1.** *For all  $\epsilon > 0, d \geq 2$ , if there exists a polynomial-time approximation algorithm for MAXIMUM MINIMAL HITTING SET which on hypergraphs  $G = (V, F)$  where hyperedges have size exactly  $d$  has approximation ratio  $n^{\frac{d-1}{d}-\epsilon}$ , where  $|V| = n$ , then  $P=NP$ . This is still true for hypergraphs where  $|F| \in O(|V|)$ .*

*Proof.* Fix some constant hyperedge size  $d$ . We will present a reduction from MAXIMUM INDEPENDENT SET, which is known to be inapproximable [18]. Specifically, for all  $\epsilon > 0$ , it is known to be NP-hard to distinguish for an  $n$ -vertex graph  $G$  if  $\alpha(G) > n^{1-\epsilon}$  or  $\alpha(G) < n^\epsilon$ .

Take an instance  $G = (V, E)$  of MAXIMUM INDEPENDENT SET. If  $d > 2$  we begin by turning  $G$  into a  $d$ -uniform hypergraph  $G' = (V, H)$  such that any (non-trivial) hitting set of  $G'$  is a vertex cover of  $G$  and vice-versa (for  $d = 2$  we simply set  $G' = G$ ). We proceed as follows: for every edge  $e \in E$  and every  $S \subseteq V \setminus e$  with  $|S| = d - 2$  we construct in  $H$  the hyperedge  $e \cup S$  (with size exactly  $d$ ). Thus,  $|H| = O(n^d)$ . Any vertex cover of  $G$  is also a hitting set of  $G'$ . For the converse, we only want to prove that any hitting set of  $G'$  of size at most  $n - d$  is also a vertex cover of  $G$  (this is without loss of generality, since  $d$

is a constant, so we will assume  $\alpha(G) > d$ . Take a hitting set  $C$  of  $G'$  with at most  $n - d$  vertices; take any edge  $e \in E$  and a set  $S$  with  $S \subseteq V \setminus (C \cup e)$  and  $|S| = d - 2$  (such a set  $S$  exists since  $|V \setminus C| \geq d$ ). Now,  $(e \cup S) \in H$ , therefore  $C$  must contain a vertex of  $e$ . We thus conclude that the maximum size of  $V \setminus C$ , where  $C$  is a hitting set of  $G'$  is either at least  $n^{1-\varepsilon}$  or at most  $n^\varepsilon$ , that is, the maximum size of  $V \setminus C$  is  $\alpha(G)$ .

We now add some vertices and hyperedges to  $G'$  to obtain a hypergraph  $G''$ . For every set  $S \subseteq V$  such that  $|S| = d - 1$  and  $V \setminus S$  is a hitting set of  $G'$ , we add to  $G''$   $n$  new vertices, call them  $u_{S,i}$ ,  $1 \leq i \leq n$ . Also, for each such vertex  $u_{S,i}$  we add to  $G''$  the hyperedge  $S \cup \{u_{S,i}\}$ ,  $1 \leq i \leq n$ . This completes the construction. It is not hard to see that  $G''$  has hyperedges of size exactly  $d$ , and its vertex and hyperedge set are both of size  $O(n^d)$ .

Let us analyse the approximability gap of this reduction. First, suppose that there is a minimal hitting set  $C$  of  $G'$  with  $|V \setminus C| > n^{1-\varepsilon}$ . Then, there exists a minimal hitting set of  $G''$  with size at least  $n^{d-O(d\varepsilon)}$ . To see this, consider the set  $C \cup \{u_{S,i} \mid S \subseteq V \setminus C, 1 \leq i \leq n\}$ . This set is a hitting set, since  $C$  hits all the hyperedges of  $G'$ , and for every new hyperedge of  $G''$  that is not covered by  $C$  we select  $u_{S,i}$ . It is also minimal, because  $C$  is a minimal hitting set of  $G'$ , and each  $u_{S,i}$  selected has a private hyperedge. To calculate its size, observe that for each  $S \subseteq V \setminus C$  with  $|S| = d - 1$  we have  $n$  vertices. There are at least  $\binom{n^{1-\varepsilon}}{d-1}$  such sets.

For the converse direction, we want to show that if  $|V \setminus C| < n^\varepsilon$  for all hitting sets  $C$  of  $G'$ , then any minimal hitting set of  $G''$  has size at most  $n^{1+O(d\varepsilon)}$ . Consider a hitting set  $C'$  of  $G''$ . Then,  $C' \cap V$  is a hitting set of  $G'$ . Let  $S \subset V$  be a set of vertices such that  $S \cap C' \neq \emptyset$ . Then  $u_{S,i} \notin C'$  for all  $i$ , because the (unique) hyperedge that contains  $u_{S,i}$  also contains some other vertex of  $C'$ , contradicting minimality. Now, because  $V \cap C'$  is a hitting set of  $G'$  we have  $|V \setminus C'| \leq n^\varepsilon$ . Thus, the maximum number of different sets  $S \subseteq V$  such that some  $u_{S,i} \in C'$  is  $\binom{n^\varepsilon}{d-1}$  and the total size of  $C'$  is at most  $|C' \cap V| + n^{\varepsilon(d-1)+1} \leq n^{1+O(d\varepsilon)}$ .  $\square$

**Corollary 1.** *For any  $\varepsilon > 0$  MAXIMUM MINIMAL HITTING SET is not  $n^{1-\varepsilon}$ -approximable, where  $n$  is the number of vertices of the input hypergraph, unless  $P=NP$ . This is still true for hypergraphs  $G = (V, F)$  where  $|F| \in O(|V|)$ .*

A graph is called co-bipartite if its complement is bipartite. Using Corollary 1 and the reduction of [21] from MINIMUM HITTING SET to MINIMUM DOMINATING SET, the following holds.

**Theorem 2.** *For any  $\varepsilon > 0$  UPPER DOMINATION, even restricted to co-bipartite graphs, is not  $n^{1-\varepsilon}$ -approximable, where  $n$  is the number of vertices of the input graph, unless  $P=NP$ .*

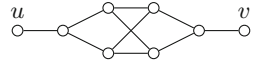
Note that, in fact, the inapproximability bound given in Theorem 1 is tight, for every fixed  $d$ , a fact that we believe may be of independent interest. This is shown in the following theorem, which also generalises results on MAXIMUM MINIMAL VERTEX COVER [7].

**Theorem 3.** For all  $d \geq 1$ , there exists a polynomial-time algorithm which, given a hypergraph  $G = (V, F)$  such that all hyperedges have size at most  $d$ , produces a minimal hitting set  $H$  of  $G$  with size  $\Omega(n^{1/d})$ . This shows an  $O(n^{\frac{d-1}{d}})$ -approximation for MAXIMUM MINIMAL HITTING SET on such hypergraphs.

### 3.2 Hardness on Cubic and Subcubic Planar Graphs

UPPER DOMINATION is known to be NP-hard on planar graphs of maximum degree six [1]. We strengthen this result in two ways: first, we show that even for cubic graphs the problem is APX-hard; second, the problem remains NP-hard for planar subcubic graphs. We complement this hardness with an EPTAS on planar graphs.

**Theorem 4.** UPPER DOMINATION is APX-hard on cubic graphs.



*Proof.* (Sketch) We present a reduction from MAXIMUM INDEPENDENT SET on cubic graphs, which is APX-hard [25]. Let  $G = (V, E)$  be the cubic input graph. Build  $G'$  from  $G$  by replacing every  $(u, v) \in E$  by a structure of six new vertices, as shown on the right. Any  $IS \subset V$  is an independent set for  $G$  if and only if  $G'$  contains an upper dominating set of cardinality  $|IS| + 3|E|$ .  $\square$

**Theorem 5.** UPPER DOMINATION is NP-hard on planar subcubic graphs.

### 3.3 On MINIMAL DOMINATING SET EXTENSION

Algorithms working on combinatorial graph problems often try to look at local parts of the graph and then extend some part of the (final) solution that was found and fixed so far. For many maximisation problems, like UPPER IRREDUNDANCE or MAXIMUM INDEPENDENT SET, it is trivial to obtain a feasible solution that extends a given vertex set by some greedy strategy, or to know that no such extension exists. This is not true for UPPER DOMINATION, as we show next. Let us first define the problem formally.

MINIMAL DOMINATING SET EXTENSION

**Input:** A graph  $G = (V, E)$ , a set  $S \subseteq V$ .

**Question:** Does  $G$  have a minimal dominating set  $S'$  with  $S' \supseteq S$ ?

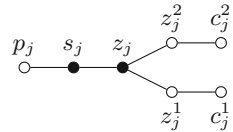
Notice that this problem is trivial on some input with  $S = \emptyset$  by using a greedy approach. If  $S$  is an independent set in  $G$ , it is also always possible to extend  $S$  to a minimal dominating set, simply by greedily extending it to a maximal independent set. If  $S$  however contains two adjacent vertices, we arrive at the problem of fixing at least one private neighbour for these vertices. This problem of preserving irredundance of the vertices in  $S$  while extending  $S$  to dominate the whole graph turns out to be a quite difficult task.

In [8] it is shown that this kind of extension of partial solutions is NP-hard for the problem of computing prime implicants of the dual of a Boolean function; a problem which can also be seen as the problem of finding a minimal hitting set for the set of prime implicants of the input function. Interpreted in this way, the proof from [8] yields NP-hardness for the minimal extension problem for 3-HITTING SET. The standard reduction from HITTING SET to DOMINATING SET however does not transfer this result to MINIMAL DOMINATING SET EXTENSION; observe that if we represent the hitting-set input-hypergraph  $H = (V, F)$  with partial solution  $S \subset V$  (w.l.o.g. irredundant) by  $G = (V \cup F, E)$  with  $E = \{(v, f) : v \in V, f \in F, v \in f\} \cup (V \times V)$ , the set  $S$  can always be extended to a minimal dominating set by simply adding all edge-vertices which are not dominated by  $S$ . One can repair this by adjusting this construction to forbid the edge-vertices in minimal solutions in the following way: for each edge-vertex  $f$ , add three new  $a_f, b_f, c_f$  with edges  $(f, a_f), (a_f, b_f), (b_f, c_f)$  and include  $a_f$  and  $b_f$  in  $S$ . This way,  $f$  is the only choice for a private neighbour for  $a_f$ .

We will show that MINIMAL DOMINATING SET EXTENSION remains hard even for very restricted cases. Our proof is based on a reduction from the NP-complete 4-BOUNDED PLANAR 3-CONNECTED SAT problem (4P3C3SAT for short) [23], the restriction of 3-satisfiability to clauses in  $C$  over variables in  $V$ , where each variable occurs in at most four clauses and the associated bipartite graph  $(C \cup X, \{(c, x) \in C \times X : (x \in c) \vee (\neg x \in c)\})$  is planar.

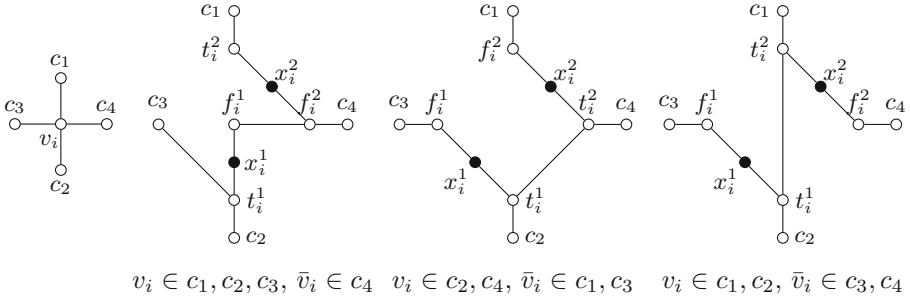
**Theorem 6.** MINIMAL DOMINATING SET EXTENSION is NP-complete, even when restricted to planar cubic graphs.

*Proof.* (Sketch) Consider an instance of 4P3C3SAT with clauses  $c_1, \dots, c_m$  and variables  $v_1, \dots, v_n$ . By definition, the graph  $G = (V, E)$  with  $V = \{c_1, \dots, c_m\} \cup \{v_1, \dots, v_n\}$  and  $E = \{(c_j, v_i) : v_i \text{ or } \bar{v}_i \text{ is literal of } c_j\}$  is planar. Replace every vertex  $v_i$  by six new vertices  $f_i^1, x_i^1, t_i^1, t_i^2, x_i^2, f_i^2$  with edges  $(f_i^j, x_i^j), (t_i^j, x_i^j)$  for  $j = 1, 2$ .



Depending on whether  $v_i$  appears negated or non-negated in these clauses, we differentiate between the three cases depicted in Fig. 1. Observe that all other cases are rotations of these three cases and/or invert the roles of  $v_i$  and  $\bar{v}_i$  and that the maximum degree of the vertices which replace  $v_i$  is three. Next, replace each clause-vertex  $c_j$  by the subgraph on the right. The vertices  $c_j^1, c_j^2$  somehow take the role of the old vertex  $c_j$  regarding its neighbours:  $c_j^1$  is adjacent to two of the literals of  $c_j$  and  $c_j^2$  is adjacent to the remaining literal. This way, all vertices have degree at most three and the choices of literals to connect to  $c_j^1$  and  $c_j^2$  can be made such that planarity is preserved. □





**Fig. 1.** Construction of Theorem 6: A variable  $v_i$  appearing in four clauses  $c_1, \dots, c_4$ , of the original instance is transformed to one of the subgraphs on the right, depending on which clauses it appears positive in. Black vertices denote elements of  $S$ .

## 4 Approximation Algorithms

### 4.1 Bounded-Degree Graphs

Unlike the general case, UPPER DOMINATION admits a simple constant factor approximation when restricted to graphs of maximum degree  $\Delta$ . This follows by the fact that any dominating set in such a graph has size at least  $\frac{n}{\Delta+1}$ . We show that this can be improved.

**Theorem 7.** Consider some graph-class  $\mathcal{G}(p, \rho)$  with the following properties:

- One can properly colour every  $G \in \mathcal{G}(p, \rho)$  with  $p$  colours in polynomial time.
- For any  $G \in \mathcal{G}(p, \rho)$ , MAXIMUM INDEPENDENT SET is  $\rho$ -approximable in polynomial time.

Then, for every  $G \in \mathcal{G}(p, \rho)$ , UPPER DOMINATION is approximable in polynomial time within ratio at most  $\max \left\{ \rho, \frac{\Delta \rho p + \Delta - 1}{2\rho\Delta} \right\}$ .

The proof idea uses Eq. (2) and the fact that any maximal independent set is a minimal dominating set. We distinguish two cases, and run a different MAXIMUM INDEPENDENT SET algorithm for each case. We output the best among the computed solutions.

Any connected graph of maximum degree  $\Delta$ , except a complete graph or an odd cycle, can be coloured with at most  $\Delta$  colours [24]; also, MAXIMUM INDEPENDENT SET is approximable within ratio  $(\Delta+3)/5$  in graphs of maximum degree  $\Delta$  [5]. So, the class  $\mathcal{G}(\Delta, (\Delta+3)/5)$  contains all graphs of maximum degree  $\Delta$ .

**Corollary 2.** UPPER DOMINATION is approximable in polynomial time within a ratio of  $(6\Delta^2 + 2\Delta - 3)/10\Delta$  in general graphs.

Theorem 7 can be improved for regular graphs where  $\Gamma(G) \leq \frac{n}{2}$  [17].

**Corollary 3.** UPPER DOMINATION in regular graphs is approximable in polynomial time within ratio  $\Delta/2$ .

## 4.2 Planar Graphs

In this section we present an EPTAS (a PTAS with running time  $f(\frac{1}{\epsilon}) \cdot poly(|I|)$ ) for UPPER DOMINATION on planar graphs. We use techniques based on the ideas of Baker [3]. As we shall see, some complications arise in applying these techniques, because of the hardness of extending solutions to this problem.

We use the notion of outerplanar graphs. An *outerplanar* (or 1-outerplanar) graph  $G$  is a graph such that there is a planar embedding of  $G$ , where all vertices are incident to the outer face of  $G$ . For  $k > 1$ , graph  $G$  is a  $k$ -outerplanar graph if there is a planar embedding of  $G$ , such that when all vertices, incident to the outer face are removed,  $G$  is a  $(k - 1)$ -outerplanar graph. Removing stepwise the vertices that are incident to the outer face, the vertices of  $G$  can be partitioned into levels  $L_1, \dots, L_k$ . We write  $|L_i|$  for the number of vertices in level  $L_i$  (if  $i < 1$  or  $i > k$  we write  $|L_i| = 0$ ). Bodlaender [6] proved that every  $k$ -outerplanar graph has treewidth of at most  $3k - 1$ . This implies the following corollary:

**Corollary 4.** *The maximum minimal dominating set  $\Gamma(G)$  of a  $k$ -outerplanar graph  $G$  can be computed in time  $f(k)n$ .*

To obtain the EPTAS, we use the fact that every planar graph is  $k$ -outerplanar for some  $k$ . By removing some of the levels  $L_i$  we split the graph  $G$  into several  $\ell$ -outerplanar subgraphs  $G_i$  of some small  $\ell < k$ . The maximum minimal dominating set  $\Gamma(G_i)$  can be computed using the above corollary. Finally the partial solutions of  $G_i$  are merged to obtain a minimal dominating set for  $G$ . In the following theorem we analyse how the maximum of the subgraphs  $\Gamma(G_i)$  correlates to the maximum  $\Gamma(G)$  of the graph  $G$ .

**Theorem 8.** *Let  $G = (V, E)$  be a  $k$ -outerplanar graph with levels  $L_1, \dots, L_k \subseteq V$ . For some  $i \leq k$ , let  $G_1$  be the subgraph which is induced by levels  $L_1, \dots, L_{i-1}$  and let  $G_2$  be the subgraph induced by levels  $L_{i+1}, \dots, L_k$ . Then,  $\Gamma(G_1) + \Gamma(G_2) \geq \Gamma(G) - \sum_{j=i-3}^{i+3} |L_j|$ .*

Using the above theorem iteratively for several levels  $L_{i_1}, \dots, L_{i_s-1}$  yields the following

**Corollary 5.** *Let  $G = (V, E)$  be a  $k$ -outerplanar graph with levels  $L_1, \dots, L_k \subseteq V$ . For indices  $0 = i_0 < i_1 < \dots \leq i_s = k$ , let  $G_j$  be the subgraph which is induced by levels  $L_{i_j}, \dots, L_{i_{j+1}}$ . Then,  $\sum_{j=0}^{s-1} \Gamma(G_j) \geq \Gamma(G) - \sum_{k=0}^s \sum_{j=i_k-3}^{i_k+3} |L_j|$ .*

The following algorithm shows how partial solutions of subgraphs can be used to obtain a minimal dominating set for the whole graph  $G$ .

**Algorithm 1.** *Input: A minimal dominating set of subgraphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  of  $G = (V, E)$ , which are separated by level  $L_i$  such that  $V_1 \cup L_i \cup V_2 = V$ .*

1. Repeat the following steps until all vertices are covered by the dominating set.
2. Add vertex  $v \in L_i$  which is not covered by the dominating set.
3. Remove vertices in  $N[N[v]]$  from the dominating set until the dominating set is minimal.

**Theorem 9.** *Let  $G = (V, E)$  be a  $k$ -outerplanar graph with levels  $L_1, \dots, L_k \subseteq V$ . For some  $i \leq k$ , let  $G_1$  be the subgraph which is induced by levels  $L_1, \dots, L_{i-1}$  and let  $G_2$  be the subgraph induced by levels  $L_{i+1}, \dots, L_k$ . Let  $S_1$  and  $S_2$  be a minimal dominating set of  $G_1$  and  $G_2$ , respectively. Then Algorithm 1 returns a minimal dominating set  $S$  with  $|S| \geq |S_1| + |S_2| - |L_{i-1}| - |L_{i+1}|$ .*

We now state our final algorithm: An EPTAS for planar UPPER DOMINATION.

**Algorithm 2.** *Input: A  $k$ -outerplanar graph  $G = (V, E)$  for some  $k \in \mathbb{N}$  and parameter  $\epsilon$ .*

1. Let  $\mu = \lceil \frac{36}{\epsilon} \rceil$ .
2. Choose  $x$  such that  $0 \leq x < \mu$  and such that the following term is minimised

$$\sum_{j \in \mathbb{N}} \left( \sum_{i=-3}^3 |L_{j\mu+x+i}| \right) + |L_{j\mu+x-1}| + |L_{j\mu+x+1}|$$

3. Let  $G_i$  be the graph induced by levels  $L_{(i-1)\mu+x+1}, \dots, L_{i\mu+x-1}$  (note that  $L_i$  with  $i < 1$  or  $i > k$  are empty sets) and let  $H_i$  be the graph induced by levels  $L_1, \dots, L_{i\mu+x-1}$ .
4. Use Corollary 4 to compute the maximum minimal dominating set and its value  $\Gamma(G_i)$  for each graph  $G_i$  with  $0 \leq i \leq \lceil \frac{k}{\mu} \rceil$ .
5. Apply Algorithm 1 iteratively to graph  $H_i$  and  $G_{i+1}$  with separating level  $L_{i\mu+x}$  for all  $0 \leq i \leq \lceil \frac{k}{\mu} \rceil$  (starting from  $H_0 = G_0$ ) to obtain a minimal dominating set for  $H_{i+1}$ .
6. Return the minimal dominating set for  $(H_{\lceil \frac{k}{\mu} \rceil}) = G$ .

**Theorem 10.** *Algorithm 2 returns a minimal dominating set  $S$  of size  $|S| \geq (1 - \epsilon)\Gamma(G)$  in time bounded by  $f(\frac{1}{\epsilon})n + O(n^2)$ .*

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