On Weakly and Strongly Popular Rankings

Extended Abstract

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ABSTRACT

Van Zuylen et al. [26] introduced the notion of a popular ranking in a voting context, where each voter submits a strictly-ordered list of all candidates. A popular ranking π of the candidates is at least as good as any other ranking σ in the following sense: if we compare π to σ, at least half of all voters will always weakly prefer π. Whether a voter prefers one ranking to another is calculated based on the Kendall distance.

A more traditional definition of popularity—as applied to popular matchings, a well-established topic in computational social choice—is stricter, because it requires at least half of the voters *who are not indifferent between* π *and* σ to prefer π. In this paper, we derive structural and algorithmic results in both settings, also improving upon the results in [26]. We also point out connections to the famous open problem of finding a Kemeny consensus with 3 voters.

KEYWORDS

majority rule; Kemeny consensus; complexity; preference aggregation; popular matching

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1 INTRODUCTION

A fundamental question in preference aggregation is the following: given a number of voters who rank candidates, can we construct a ranking that expresses the preferences of the entire set of voters as a whole? A common way of evaluating how close the constructed ranking is to a voter's preferences is the Kendall distance, which measures the pairwise disagreements between two rankings. Among others, a well-known rank aggregation method is the Kemeny ranking method [20], in which the winning ranking minimises the sum of its Kendall distances to the voters' rankings.

For the preference aggregation problem, van Zuylen et al. [26] introduce a new rank aggregation method called popular ranking, which is also based on the Kendall distance. Each voter can compare two given rankings π and σ, and prefers the one that is closer to her submitted ranking in terms of the Kendall distance. Van Zuylen et al. define π to be a winning ranking in an instance if for any ranking σ, at least half of the voters prefer π to σ or are indifferent between them. This implies that there is no ranking σ such that switching to σ from π would benefit a majority of all voters.

According to the definition of popularity in [26], even in a situation where exactly half of the voters are indifferent between rankings π and σ, whilst the other half of the voters prefer σ to π, σ is not more popular than π. This example demonstrates how hard it is for the dissatisfied voters to find a ranking that overrules π—the definition requires them to find a profiting set of voters who build an *absolute majority*, that is, a majority of *all* voters for this endeavour.

A straightforward option would be to require only a *simple majority*, this is, a majority of the *non-abstaining* voters, to profit from switching to σ from π. Excluding the abstaining voters in a pairwise majority voting rule is common practice [12]. It is also analogous to the classical popularity notion in the matching literature [1, 9, 24]. In this paper, we propose an alternative definition of a popular ranking. We define π to be a *strongly popular ranking* if for every ranking σ, at least half of the non-abstaining voters prefer π to σ. This means that switching from π to σ would harm at least as many voters as it would benefit.

Related literature. The most common approach to aggregating voters' preferences is to search for a ranking that minimises the sum of the distances to the voters' rankings. If the Kendall distance [21] is used as the metric on rankings, then this optimality concept corresponds to the Kemeny consensus [20, 23]. Deciding whether a given ranking is a Kemeny consensus is NP-complete, and calculating a Kemeny consensus is NP-hard [6] even if there are only 4 voters [7, 13], or at least 6 of them [2]. Interestingly, the complexity of the problem is open for 3 and 5 voters [2, 7].

Majority voting rules offer another natural way of aggregating voters' preferences. The earliest reference for this might be from Condorcet [8], who uses pairwise comparisons to calculate the winning candidate, establishing his famous paradox on the smallest voting instance not admitting any majority winner. The absolute and simple majority voting rules have both been extensively discussed in the setting where the goal is to choose the winning candidate [4, 5].

The concept of majority voting readily translates to other scenarios, where voters submit preference lists. One such field is the area of matchings under preferences, where popular matchings [17, 1, 24, Chapter 7, 9] serve as a voting-based alternative concept to the well-known notion of stable matchings [3, 16] in two-sided markets. Besides two-sided matchings, majority voting has also been defined for the house allocation problem [1, 25], the roommates problem [14, 18], spanning trees [10], permutations [26], the ordinal group activity selection problem [11], and very recently, to

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branchings [19]. The notion of popularity is aligned with simple majority in all these papers, with one exception, namely [26], which defines popularity based on the absolute majority rule.

A part of this work revisits the paper from van Zuylen et al. [26]. They show that a popular ranking—according to their definition of popularity—need not necessarily exist. More precisely, they show that the acyclicity of a structure known as the majority graph is a necessary, but not sufficient condition for the existence of a popular ranking. They also prove that if the majority graph is acyclic, then one can efficiently compute a ranking, which may or may not be popular, but for which the voters have to solve an NP-hard problem to compute a ranking that a majority of them prefer.

Our contribution. We study both the weaker notion of popularity from [26] and the stronger notion of popularity analogous to the one in the matching literature, which excludes abstaining voters. In the full version of the paper [22], we present the following results.

(1) For at most 5 voters, the two notions are equivalent, but with 6 voters this does not hold anymore.
(2) We give a sufficient condition for the two notions to be equivalent for a given ranking π.
(3) In the case of 2 or 3 voters, one can find a popular ranking of either kind and verify the weak or strong popularity of a given ranking in polynomial time.
(4) The problem of verifying the weak or strong popularity of a given ranking for 4 voters is polynomial-time solvable if and only if it is polynomial-time solvable for 5 voters.
(5) If finding a ranking that is more popular in either sense than a given ranking in an instance with 4 (or 5) voters were polynomial-time solvable, then the famously open Kemeny consensus problem for 3 voters would also be polynomial-time solvable.

2 DEFINITIONS AND EXAMPLE

We are given a set of candidates and a set of voters, each of whom submits a ranking. A *ranking* π is a permutation of all candidates. The *rank* of candidate a in ranking π is the position it appears at in π, and it is denoted by $\text{rank}_\pi[a]$. The *Kendall distance* $K(\pi, \sigma)$ between two rankings π and σ is defined as the number of pairwise disagreements between π and σ, or, formally as

$$K(\pi, \sigma) = |\{(a, b) : \text{rank}_\pi[a] > \text{rank}_\pi[b] \text{ and } \text{rank}_\sigma[a] < \text{rank}_\sigma[b]\}| \\ + |\{(a, b) : \text{rank}_\pi[a] < \text{rank}_\pi[b] \text{ and } \text{rank}_\sigma[a] > \text{rank}_\sigma[b]\}|.$$

The Kendall distance is equivalent to the number of swaps that the bubble sort algorithm [15] executes when converting ranking π to ranking σ. Voters prefer rankings that are similar to their submitted ranking. More precisely, voter v *prefers* ranking σ to ranking π if $K(\sigma, \pi_v) < K(\pi, \pi_v)$. Analogously, voter v *abstains* between π and σ if $K(\sigma, \pi_v) = K(\pi, \pi_v)$.

In the instance depicted by Figure 1, let $\sigma_1 = [1, 2, 3], [5, 6, 4], [9, 7, 8]$ and $\sigma_2 = [1, 2, 3], [4, 5, 6], [7, 8, 9]$. Clearly v_4 prefers σ_2 to σ_1, since $\pi_{v_4} = \sigma_2$ and $\pi_{v_4} \neq \sigma_1$, that is, $K(\pi_{v_4}, \sigma_2) = 0 < K(\pi_{v_4}, \sigma_1)$. In the same instance, v_1 is an abstaining voter since $K(\pi_{v_1}, \sigma_2) = 4 = K(\pi_{v_1}, \sigma_1)$.

We now define the two different notions of popularity. The first notion of an weakly popular ranking corresponds to the popular ranking as defined in [26].

$$\begin{aligned}
\pi_{v_1} &= [1, 2, 3], [6, 4, 5], [8, 9, 7] \\
\pi_{v_2} &= [2, 3, 1], [4, 5, 6], [9, 7, 8] \\
\pi_{v_3} &= [3, 1, 2], [5, 6, 4], [7, 8, 9] \\
\pi_{v_4} &= [1, 2, 3], [4, 5, 6], [7, 8, 9] \\
\pi_{v_5} &= [1, 2, 3], [5, 4, 6], [9, 7, 8] \\
\pi_{v_6} &= [1, 2, 3], [5, 6, 4], [7, 9, 8]
\end{aligned}$$

Figure 1: A voting instance with 6 voters $v_1, v_2, \ldots v_6$ and 9 candidates $1, 2, \ldots, 9$.

Definition 2.1. *Ranking σ is* more popular *than ranking π in* the weak sense *if $K(\sigma, \pi_v) < K(\pi, \pi_v)$ for an absolute majority of all voters. Ranking π is* weakly popular *if no ranking σ is more popular than π in the weak sense.*

If we consider $\sigma_3 = [2, 1, 3], [4, 5, 6], [7, 8, 9]$, then in the instance in Figure 1, $\sigma_2 = [1, 2, 3], [4, 5, 6], [7, 8, 9]$ is more popular than σ_3 in the weak sense. Notice that σ_3 and σ_2 only differ in their ordering of the pair of candidates $\{1, 2\}$. So since five out of six voters prefer candidate 1 to candidate 2, they form an absolute majority of all voters who prefer σ_2 to σ_3.

This definition requires more than half of the n voters to prefer σ to π in order to declare σ to be more popular than π. Abstaining voters make it hard to beat π in such a pairwise comparison. However, if σ only needs to receive more votes than π among the voters not abstaining between these two rankings, then it can beat π. This leads to the notion of strong popularity. Let $V_{\text{abs}}(\pi, \sigma)$ be the set of voters who abstain in the vote between π and σ, that is, $v \in V_{\text{abs}}(\pi, \sigma)$ if and only if $K(\pi_v, \pi) = K(\pi_v, \sigma)$. In Figure 1, $V_{\text{abs}}(\sigma_1, \sigma_2) = \{v_1, v_2, v_3\}$ with σ_1 and σ_2 as before.

Definition 2.2. *Ranking σ is* more popular *than ranking π in* the strong sense *if $K(\sigma, \pi_v) < K(\pi, \pi_v)$ for a majority of the non-abstaining voters $V \setminus V_{abs}(\pi, \sigma)$. Ranking π is* strongly popular *if no ranking σ is more popular than π in the strong sense.*

It follows directly from the two definitions above that strongly popular rankings are weakly popular as well, but weakly popular rankings are not necessarily strongly popular. In the instance in Figure 1, $\sigma_1 = [1, 2, 3], [5, 6, 4], [9, 7, 8]$ is more popular than $\sigma_2 = [1, 2, 3], [4, 5, 6], [7, 8, 9]$ in the strong sense, since v_5 and v_6 prefer σ_1 to σ_2, while v_1, v_2, and v_3 abstain. Notice that σ_1 is not more popular than σ_2 in the weak sense, because two voters do not constitute an absolute majority of all 6 voters, only a majority of the non-abstaining 3 voters.

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