Accelerated Information Dissemination on Networks with Local and Global Edges

Sarel Cohen[1], Philipp Fischbeck[2(✉)], Tobias Friedrich[2], Martin S. Krejca[3], and Thomas Sauerwald[4]

[1] School of Computer Science, The Academic College of Tel Aviv-Yaffo, Tel Aviv-Yaffo, Israel
sarelco@mta.ac.il
[2] Hasso Plattner Institute, University of Potsdam, Potsdam, Germany
{philipp.fischbeck,tobias.friedrich}@hpi.de
[3] Sorbonne University, CNRS, LIP6, Paris, France
martin.krejca@lip6.fr
[4] Department of Computer Science and Technology, University of Cambridge, Cambridge, UK
thomas.sauerwald@cl.cam.ac.uk

Abstract. Bootstrap percolation is a classical model for the spread of information in a network. In the round-based version, nodes of an undirected graph become active once at least r neighbors were active in the previous round. We propose the *perturbed* percolation process: a superposition of two percolation processes on the same node set. One process acts on a local graph with activation threshold 1, the other acts on a global graph with threshold r – representing local and global edges, respectively. We consider grid-like local graphs and expanders as global graphs on n nodes.

For the extreme case $r = 1$, all nodes are active after $O(\log n)$ rounds, while the process spreads only polynomially fast for the other extreme case $r \geq n$. For a range of suitable values of r, we prove that the process exhibits both phases of the above extremes: It starts with a polynomial growth and eventually transitions from at most cn to n active nodes, for some constant $c \in (0, 1)$, in $O(\log n)$ rounds. We observe this behavior also empirically, considering additional global-graph models.

Keywords: Bootstrap percolation · Random graphs · Expanders · Rumor spreading

1 Introduction

Information spreads very fast in networks (see, e.g., [23]). Several practical and theoretical studies concern n agents (nodes) interacting within a network and exchanging information via incident edges. These works have demonstrated that if each agent, once informed, informs all its agents in the neighborhood, the entire network is typically informed in a time that is at most logarithmic in the

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number of agents. This behavior even holds if each agent chooses only *one random* neighbor at each iteration (and a slightly faster dissemination is possible if an agent does not choose the same agent twice in a row [22]). A similar behavior occurs in the *bootstrap percolation* model [17], in which agents are informed once the number of informed neighbors reaches a certain threshold. This model has been extensively analyzed on a range of graph models, including hypercubes [9], grids [10], Erdős–Rényi graphs [31], preferential attachment graphs [4], random regular graphs [11], random geometric graphs [14], hyperbolic random graphs [15], inhomogeneous random graphs [1,5], geometric inhomogeneous random graphs [32], Kleinberg's small world model [24,27], as well as superpositions of Erdős–Rényi graphs and other graphs [37].

In the bootstrap percolation model, the process usually either reaches almost all agents quickly or terminates without having reached most of the agents [9, 31]. This speed is often attributed to the logarithmic diameter of the network, as well as to the existence of high-degree nodes, which are both prevalent in many real-world graphs as well as in their mathematical models. However, these models assume that information spreads the same way among all edges. If this is not the case, e.g., because agents need to be convinced of some information by more than one agent, the resulting behavior can be fundamentally different [12,16,26,28,35].

Typically, the edges of a graph describe the closeness of agents, i.e., two agents connected via an edge are close, while non-edges represent separation. This is particularly true for graph models that utilize an underlying geometry for determining the edge set. However, another perspective, found in epidemics, is that every pair of agents has an activation probability defined, e.g., by splitting the agents into groups [30]. In bootstrap percolation, one can set different activation thresholds based on the groups [13]. Further, one can model the closeness via different graphs on the same agents, namely via *local* and *global* edges, which are assigned different activation probabilities [7,8]. The underlying graphs represent different interactions, e.g., contacts within and across households [6].

We aim at understanding the effect of edge types on the speed of information dissemination. To this end, we analyze graphs that have two types of edges: one representing short edges, and another one representing long edges. The graph induced by the short edges (the *local* graph) models the local neighborhood of agents. These model whether two agents are close, e.g., people an agent is exposed to more often, such as colleagues, relatives or neighbours. The graph induced by the long edges (*global* graph) models non-local (global) contacts. This represents people who the agent has not that much contact with, e.g., people who live further away or celebrities from social media who the agent may never meet personally but is influenced by.

We employ the classic *bootstrap percolation* model as a foundation for the spread of information in networks as described above. In this model, each agent either has a certain piece of information (it is *active*) or it has not (*inactive*). Given a parameter $r \in \mathbb{N}^+$ (the *activation threshold*) and a set of initially active nodes, iteratively, at each round $t \in \mathbb{N}^+$, a node becomes (and remains) active if it has at least r neighbors that were active in round $t-1$.

Model. We propose the *perturbed percolation model*, which is the superposition of two bootstrap percolation processes on the same node set but with two different edge sets. One process acts on the *local* graph with an activation threshold of 1. The other process acts on the *global* graph with an activation threshold of r. This is similar to the above-mentioned models where nodes have local and global contacts with varying activation probabilities [8]. Note that a perturbed percolation process always percolates completely if the local graph is connected. However, the overall speed is majorly influenced by the global graph via r.

Theoretically and empirically, we analyze *how quickly* nodes become active in this model. We are interested in the two following activation rates: a *polynomial* rate, i.e., the number of active nodes in round t is a polynomial in t, and a *rapid* rate, i.e., the number of inactive nodes reduces from at least $(1-c)n$, for some constant $c \in (0,1)$, to none in $O(\log n)$ rounds.

Results. For our theoretical results, we analyze the activation rate of the perturbed percolation model on local graphs that we refer to as *polynomial-neighborhood graphs* (PNGs) with n nodes, characterized by having a polynomially expanding neighborhood w.r.t. the hop distance, including grid graphs, cycles, and, asymptotically almost surely (a.a.s.), random geometric graphs with expected polylogarithmic node degree. We prove the following landscape of perturbed percolation w.r.t. the activation threshold r, using PNGs as local and expanders as global graph:

- For the extreme case $r \geq n$, the process has a **polynomial** rate (Theorem 1).
- For the other extreme case $r = 1$, the process has a **rapid** rate (Corollary 1), i.e., adding global edges changes the rate immediately from polynomial to rapid.
- Our **main result** is that the process with suitable values of r between the extreme cases above, including $r = 2$, has a **polynomial-to-rapid** rate (Corollary 2), i.e., the process has a polynomial rate for a polynomial number of rounds (w.r.t. n) and then ends with a rapid rate. This result highlights that while the edges from the global graph speed up the overall process, it takes some (long) time for the process to actually switch to a rapid rate.

We complement our theoretical results by empirical analyses (Figs. 1 and 2). Next to Erdős–Rényi graphs as global graphs, we also include Barabási–Albert and hyperbolic random graphs, which are not covered by our theoretical analysis. For all cases, we observe a clear distinction between the polynomial and the rapid rate.

Framework (Informal Description). Our main result follows from our more general result (Theorem 3) based on proving the following three independent properties, assuming a graph with n nodes:

1. Any *bootstrap* percolation process on the local graph, for any initial active set of size 1, has polynomial rate.

2. For the perturbed percolation process, a.a.s. for an initial number of rounds polynomial in n, no inactive node has at least r global edges to active nodes.
3. Asymptotically a.s., any *bootstrap* percolation process on the global graph, for any initial active set of linear size, percolates completely in a number of rounds logarithmic in n.

Combining all three properties shows a polynomial-to-rapid rate. We note that Property 3 considers the classic bootstrap percolation setting but requires to *first* fix the random graph and *then* the initial set (even adversarially). Typically, this order is reversed. Thus, we believe our results proving this property (Theorem 6 and 8) to be of independent interest. In addition, in Theorem 8 we provide an improved bound of $r \cdot n/\ln n$ for the size of the initial set in Property 3 for the special case of Erdős–Rényi graphs.

Outline. In Sect. 2, we introduce our notation as well as our model and the graph classes we consider. Sections 3 and 4 contain our theoretical results. The former considers the extreme cases of the activation threshold r, the latter suitable intermediate values. Our main result of these sections is Corollary 2. In Sect. 5, we discuss our empirical results, and we provide an outlook in Sect. 6.

2 Preliminaries

Let $\mathbb{N}$ denote the set of natural numbers, including 0. For $m, n \in \mathbb{N}$, let $[m..n] := [m, n] \cap \mathbb{N}$, and let $[m] := [1..m]$. We consider undirected, finite graphs. Given such a graph G, let $V(G)$ denote its set of nodes and $E(G)$ its set of edges. We denote the minimum and maximum node degree of G by $d_{\min}(G)$ and $d_{\max}(G)$, respectively, dropping G if it is clear from context.

We use big-O notation only in combination with a graph G. The asymptotics of the notation are then with respect to $|V(G)|$ (which we usually call n). Additionally, the notation $\widetilde{O}$ allows for factors polylogarithmic in $|V(G)|$. In the same context, a *constant* is a value $\Theta(1)$, that is, a value bounded independently of $|V(G)|$.

An event A occurs *asymptotically almost surely (a.a.s.)* if and only if $\Pr[A] = 1 - o(1)$.

2.1 Percolation Processes

We introduce the perturbed percolation process, which is a superposition of two classical bootstrap percolation processes, using different edges and thresholds.

Bootstrap Percolation. Let G be a graph with n nodes, $r \in \mathbb{N}_{>0}$, and $I \subseteq V(G)$. The *bootstrap percolation process P on G with threshold r and initial active set I* is a deterministic discrete-time process on $V(G)$ in which each node is either *active* or *inactive*. In each round, each node adjacent to at least r active

nodes becomes active. Let $(A_t)_{t\in\mathbb{N}}$ denote the sequence of sets of active nodes over time. Note that $A_0 = I$ and that, for all $t \in \mathbb{N}$ with $t \geq n-1$, $A_{t+1} = A_t$. We say that P *percolates completely* if and only if $|A_{n-1}| = n$.

Let $t_1, t_2 \in \mathbb{N}$, and let $T = \min\{t \in \mathbb{N} \mid A_t = A_{n-1}\}$. We say that P has a *polynomial activation rate for* $[t_1..t_2]$ if and only if there is a constant $c > 0$ such that for all $t \in [t_1..t_2]$ it holds that $|A_t| = \widetilde{O}(t^c + 1)$. Further, P has a *rapid activation rate for* $[t_1..t_2]$ if and only if $t_2 = t_1 + O(\log n)$, there is a constant $c \in (0,1)$ such that $|A_{t_1}| \leq cn$, and $|A_{t_2}| = n$. We say P has a *polynomial* (resp. *rapid*) *activation rate* if and only if it has a polynomial (resp. rapid) activation rate for $[0..T]$. Last, we say that P has a *polynomial-to-rapid activation rate* if it has both a *polynomial activation rate* and *rapid activation rate*. Note that this is equivalent to the existence of $t_1, t_2 \in \mathbb{N}$ and a constant $c > 0$ such that $t_1 \in \Omega(n^c)$ and that P has a polynomial activation rate for $[0..t_1]$ and a rapid activation rate for $[t_2..T]$.

Perturbed Percolation. Let $G = (V, E)$ be a graph decomposable into a *local graph* $G_\ell = (V, E_\ell)$ and a *global graph* $G_\mathrm{g} = (V, E_\mathrm{g})$ (each possibly random), i.e., $E = E_\ell \cup E_\mathrm{g}$. Further, let $r \in \mathbb{N}_{>0}$ and $I \subseteq V$. The *perturbed percolation process* P *on* G *with threshold* r *and initial active set* I is the union of the bootstrap percolation process on G_ℓ with threshold 1 and the one on G_g with threshold r, both with initial active set I. That is, in each round, each node with an active neighbor in G_ℓ or at least r active neighbors in G_g becomes active. The notion of polynomial/rapid activation rate from bootstrap percolation naturally extends to P.

We introduce *randomization* into the connections via a random permutation of the nodes. To this end, we assume w.l.o.g. that there exists a bijective labeling $\ell\colon V(G) \to [1..n]$. Let σ be a permutation over $[1..n]$, chosen uniformly at random, independently of any other potential random choices, and let G'_g be identical to G_g. Then $E_\mathrm{g} = \big\{\{\sigma(\ell(u)), \sigma(\ell(v))\} \in V(G)^2 \mid \{u, v\} \in E(G'_\mathrm{g})\big\}$. Technically, G_g is random (due to σ), and G'_g represents a (possibly deterministic) isomorphic representation of G_g. However, throughout the paper, we refer to both graphs as the global graph. When talking about the graph itself, we refer to G'_g, which can be deterministic. In contrast, if we refer to its edges, we refer to the set $E(G_\mathrm{g})$, which is random. Without randomization, there always exist perturbed percolation processes with (solely) rapid activation rates, due to possible dependencies between G_ℓ and G_g. In particular, there are graphs G_ℓ and G_g in the graph classes below such that the perturbed percolation process ends within $O(\log n)$ rounds. Randomization eliminates such cases. In case that G_ℓ and G_g are independent, randomization does not change anything. In particular, it is not required for our results concerning random graphs.

Throughout the paper, we assume the following order of events: 1. Fix G_ℓ and G_g in some order. 2. Randomize G_g as described above. 3. Fix an initial active set of nodes. Note that this implies that the initial active set can be chosen adversarially w.r.t. the realizations of the resulting graph of the perturbed percolation process.

2.2 Graph Classes

As local graphs, we consider graphs with polynomially expanding neighborhoods. As global graphs, we consider expanders, especially random regular graphs and Erdős–Rényi graphs.

Polynomial-Neighborhood Graphs. For a *connected* graph $G = (V, E)$, let $d_G \colon V^2 \to \mathbb{N}$ denote the distance between all pairs of nodes in G. That is, for all $u, v \in V$, the value $d_G(u, v)$ is the length of a shortest path from u to v. Further, for all $u \in V$ and all $h \in [0..|V| - 1]$, let $B_h(u) = \{v \in V \mid d_G(u, v) \leq h\}$ denote the ball of distance at most h around u.

Let $c > 0$ be a constant. We say that G is a *polynomial-neighborhood graph (PNG) of growth* c if and only if for all $u \in V$ and all $h \in [0..|V| - 1]$ it holds that $|B_h(u)| = \widetilde{O}(h^c + 1)$.

Examples of PNGs include grid graphs (with and without looping boundaries), cycles, and, a.a.s., random geometric graphs with expected node degree polylogarithmic in n.

Expanders. We call a graph an *expander* if and only if its spectral expansion λ is bounded away from 1 from above and below (see Sect. 4.2 for more details). We note that expanders can be deterministic or random. It is well-known that both Erdős–Rényi Graphs [19] and random d-regular graphs are expanders [25] (see Theorems 4 and 5).

Random Regular Graphs. Let $n \in \mathbb{N}_{>0}$, let $d \in [3..n-1]$, and let $\mathcal{G}_{n,d}$ denote the class of all (deterministic) d-regular graphs with n nodes. Each uniform sample G from $\mathcal{G}_{n,d}$ is a *random d-regular graph with n nodes*, denoted as $G_{n,d}$.

Erdős–Rényi Graphs. Let $n \in \mathbb{N}_{>0}$ and $p \in [0, 1]$. A graph G is an *Erdős–Rényi graph with n nodes and edge probability p*, denoted as $G_{n,p}$, if and only if $|V(G)| = n$ and each $e \in V^2 \setminus \{(v, v) \mid v \in V\}$ is in $E(G)$ with probability p, independent of all other choices.

3 Extreme Thresholds

We consider perturbed percolation on PNGs with n nodes as local graphs for the extreme cases of $r \geq n$ and $r = 1$, where r is the threshold of the global graph.

Case $r \geq n$. This case is equivalent to *bootstrap* percolation on PNGs with a threshold of 1. We show that regardless of the (bootstrap) threshold, the rate of the process on PNGs is polynomial if the initially active set is constant. We note that the perturbed percolation process percolates completely if and only if the local graph is connected.

Theorem 1. *Let $c > 0$ be a constant, and let G be a PNG of growth c. Further, let $I \subseteq V(G)$ such that $|I| = \Theta(1)$, and let $r' \in [n-1]$. Then the* bootstrap *percolation process on G with threshold r' and initial active set I has a polynomial activation rate.*

Proof. Let $t \in \mathbb{N}$, and recall that A_t is the set of active nodes at the end of round t. From each $u \in I$, the bootstrap percolation process reaches at most $B_t(u)$, that is, it holds that $|A_t| \leq \sum_{u \in I} |B_t(u)|$. Since G is a PNG of growth c and since $|I| = \Theta(1)$, it follows that $|A_t| = \widetilde{O}\big(|I| \cdot (t^c + 1)\big) = \widetilde{O}(t^c + 1)$, which concludes the proof.

Case $r = 1$. It follows from the literature that the rate is rapid from the start (Corollary 1) if the global graph is an Erdős–Rényi graph, as the diameter of the graph is logarithmic.

Theorem 2 [34, **Theorem 4**]. *Let $n \in \mathbb{N}_{>0}$, $\varepsilon > 0$ be a constant, and let G be a graph with n nodes that is decomposable into a connected local graph and into a $G_{n,\varepsilon/n}$ as a global graph. Then a.a.s., G has a diameter of $O(\log n)$.*

For d-regular expanders, it is well-known that the diameter is $O(\log n)$ [29, page 455].

The following statement immediately follows (as it only requires that the diameter is $O(\log n)$), noting that the diameter of a $G_{n,p}$ does not increase when p increases.

Corollary 1. *Let $G = (V, E)$ be a graph with $n \in \mathbb{N}_{>0}$ nodes that is decomposable into a connected local graph and into a global graph G_{g}. Further, let $c \in (0, 1)$ be a constant, and let $I \subseteq V(G)$ such that $I \neq \emptyset$ and $|I| \leq cn$.*

1. *Let G_g be $G_{n,p}$ with $p \in [\Omega(1/n), 1]$. Then a.a.s., the perturbed percolation process on G with threshold 1 and initial active set I has a rapid activation rate.*
2. *For $d \in [3..n-1]$, let G_g be a d-regular expander with n nodes. Then a.a.s., the perturbed percolation process on G with threshold 1 and initial active set I has a rapid activation rate.*

4 Polynomial-to-Rapid Activation Rate

We prove the emergence of a polynomial-to-rapid activation rate for suitable values of r between the extreme cases considered above. Our main result is the following.

Corollary 2. *Let G be a graph with $n \in \mathbb{N}_{\geq 3}$ nodes that is decomposable into a PNG as local graph and into a graph with spectral expansion $\lambda \in \mathbb{R}_{>0}$ and $d_{\max} = O(d_{\min})$ as global graph. Let $d = 2|E(G)|/n$, and let $r \in [2..(1-\lambda)d_{\min}^2/(4d)]$. Then a.a.s., there exists a $V' \subseteq V(G)$ with $|V'| = n - n^{3/4}$ such that for all $v \in V'$, the perturbed percolation process on G with threshold r and initial active set $\{v\}$ has a polynomial-to-rapid rate.*

We prove this result by applying a general framework for proving that a perturbed percolation process P has a transition from polynomial to rapid rate on a graph $G = (V, E)$ with $|V| = n$. To this end, let G_ℓ denote the local graph that P acts on, and let G_g denote the global graph. Further, let P_ℓ and P_g denote the bootstrap percolation processes on G_ℓ and G_g, respectively. Last, let $(A_t)_{t\in[0..n-1]}$ denote the set of active nodes of P_ℓ after each round, and for all $v \in V$ and $U \subseteq V$, let $\Gamma_\mathrm{g}(v, U) = \{u \in U \mid \{u, v\} \in E(G_\mathrm{g})\}$.

Framework. The framework comprises the following three independent properties:

1. For all $v \in V(G)$, the process P_ℓ with initial active set $\{v\}$ has a polynomial activation rate and percolates completely.
2. There are constants $c_1, c_2 \in (0, 1)$ and a set $|V'| \geq n - n^{1-c_1}$ such that for all $v \in V'$, having initial active set $\{v\}$ implies that for all $u \in V$, $|\Gamma_\mathrm{g}(u, A_{n^{c_2}})| < r$.
3. There exists a constant $c_3 > 1$ such that for all $I \subseteq V$ with $|I| \geq n/c_3$, the process P_g with initial active set I has a rapid activation rate.

Properties 1 and 3 consider exclusively P_ℓ and P_g, respectively, whereas Property 2 connects P_ℓ with the global graph. Our framework yields the following general theorem.

Theorem 3 (Polynomial-to-rapid rate). *Let $n \in \mathbb{N}_{\geq 3}$, let $r \in [2..n-1]$, and let G be a graph with n nodes, decomposable into a local graph and into a global graph. Assume that P is a perturbed percolation process on G with threshold r and some initial active set such that Properties 1 to 3 are all satisfied. Then P has a polynomial-to-rapid rate.*

Proof. By Property 2, there exists a $c_2 > 0$ such that during the initial n^{c_2} rounds of P, all activations are exclusively due to the local graph. By Property 1, it follows that P has a polynomial activation rate for $[0..n^{c_2}]$.

Now consider the first round t^* such that the number of active nodes is at least n/c_3, where c_3 is from Property 3. Note that such a t^* exists, as the number of active nodes strictly increases each round until complete percolation, since the process on the local graph percolates completely. Further note that, due to Property 3, the number of active nodes in round $t^* - 1$ is less than n/c_3. By Property 3, for any set of active nodes in round t^*, the process P percolates completely in $O(\log n)$ rounds. Thus, the process P has a rapid activation rate, starting from round $t^* - 1$, which concludes the proof.

In the following, we prove the properties of our framework separately. As Theorem 1 already proves Property 1, we are left to consider Properties 2 and 3.

4.1 Polynomial Rate

We show that Property 2 is satisfied for PNGs as local graph and for global graphs with a bounded maximum degree, which includes expanders and, a.a.s., Erdős–Rényi graphs.

Lemma 1. *Let $n \in \mathbb{N}_{\geq 3}$, $r \in [2..n-1]$, and $c_1, c_2 \in \mathbb{R}_{>0}$ with $c_2 < 1/3$ be constants. Further, let G be a graph with n nodes, decomposable into a PNG with growth c_1 as local graph and into a global graph G_g with $d_{\max}(G_g) \leq n^{c_2}$. Then with probability at least $1-n^{-1/12}$, there exists a $V' \subseteq V(G)$ with $|V'| = n-n^{3/4}$ such that for all $v \in V'$, the perturbed percolation process on G with threshold r and initial active set $\{v\}$ has a polynomial activation rate for $[0..n^{(1/3-c_2)/c_1}]$.*

Proof. By monotonicity, it suffices to consider the case $r = 2$. Pick any node $v \in V$ as the initially active node. Let B_v be all nodes that get activated in the local graph G_ℓ after $O(n^{(1/3-c_2)/c_1})$ rounds. Hence, $|B_v| = O(n^{1/3-c_2})$. Note that within the graph G_g, due to the random labeling of the nodes, we can regard the subset B_v in G_g as a random set of size $|B_v|$. In particular, the events of any two nodes x, y being in B_v are negatively correlated. Now let $Z_v \subseteq V$ be the set of nodes in $V(G_g)$ that have at least 2 neighbors in B_v. Then,

$$\mathrm{E}[|Z_v|] \leq n \cdot \binom{d_{\max}(G_g)}{2} \cdot \left(\frac{|B_v|}{n}\right)^2 \leq \frac{n^{2c_2} \cdot n^{2/3-2c_2}}{n} = n^{-1/3}.$$

Hence by Markov's inequality, the probability of any activation occurring via global edges is $\Pr[|Z_v| \geq 1] \leq n^{-1/3}$.

Now define $Y := \{v \in V \mid |Z_v| \geq 1\}$. Then $\mathrm{E}[|Y|] \leq n^{2/3}$, and by another application of Markov's inequality, $\Pr[|Y| \geq n^{3/4}] \leq n^{-1/12}$.

4.2 Rapid Rate on the Global Graph

We show that expander graphs satisfy Property 3 (Theorem 6). For the special case of Erdős-Rényi graphs, we prove an even stronger bound, showing complete percolation in $O\big(\log(n)/\log\log n\big)$ rounds (Theorem 8). We note that due to our assumption that the random graphs are revealed *before* the initial active set is chosen, our theorems show that a.a.s. the global graphs have immediately a rapid activation rate for *arbitrary* sufficiently large initial active sets. This includes cases where the initial set is chosen adversarially w.r.t. the global graph. In contrast, classic results typically fix the global graph *after* or independent of the initial set [11,31], thus not allowing for adversarially chosen initial sets.

Expanders. For any graph G, for all $v \in V(G)$, let $\deg(v)$ be the degree of v, let $d = 2|E(G)|/n$ denote the average degree, and, for all $S \subseteq V(G)$, let $\mathrm{vol}(S) := \sum_{u \in S} \deg(u)$. We define the *normalized Laplacian matrix of* G by

$$\boldsymbol{L}_{u,v} = \begin{cases} 1 & \text{if } u = v, \\ -\frac{1}{\sqrt{\deg(u)\cdot\deg(v)}} & \text{if } \{v,w\} \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$ the n eigenvalues of $\boldsymbol{L}$. Further, $\lambda := \max_{i \geq 2} |1-\lambda_i|$ denotes the *spectral expansion*. A graph is called an *expander*

if $\lambda \leq 1-c$ for some constant $c > 0$ (in other words, all eigenvalues are sufficiently far away from 0 and 2).

The following result shows that Erdős-Rényi graphs are expanders.

Theorem 4 [19, **Theorem 1.2**]. *Let $G = G_{n,p}$ be an Erdős-Rényi graph with expected degree $p(n-1) \geq c_1 \cdot \ln(n)$ for a sufficiently large constant $c_1 > 0$. Then a.a.s., the spectral expansion of $\boldsymbol{L}$ satisfies $\lambda(G) = O((p(n-1))^{-1/2})$.*

A similar result was shown by Friedman [25] for random regular graphs (for simplicity, we only state a slightly weaker version of his main result, which suffices for our purposes).

Theorem 5 [25, **Theorem A**]. *Let G be a $G(n, 2d)$ random $2d$-regular graph. Then for all $d = O(1)$, a.a.s., the spectral expansion of $\boldsymbol{L}$ satisfies $\lambda(G) = O(d^{-1/2})$.*

Our main result of this section is the rapid activation rate of expanders.

Theorem 6. *Let $n \in \mathbb{N}_{\geq 3}$, and let $G = (V, E)$ with $|V| = n$, with spectral expansion $\lambda > 0$, and with $d_{\max} = O(d_{\min})$. Further, let $d = 2|E(G)|/n$, let $r \in [2..(1-\lambda)d_{\min}^2/(4d)]$, and let $I \subseteq V$ with $|I| \geq 4\frac{r-1}{(1-\lambda)\cdot d_{\min}^2/d} \cdot n$. Then the bootstrap percolation process on G with threshold r and initial active set I percolates completely after $O(\frac{\log n}{1-\lambda})$ rounds.*

In case of Erdős-Rényi graphs with $p = \Omega(\log n/n)$ or random $2d$-regular graphs, $1-\lambda$ is bounded below by a positive constant, and thus the process percolates rapidly. We remark that the result and proof of Theorem 6 share some ideas with the work by [20], who investigate the size of *smallest* contagious sets in various classes of expander graphs. However, one key difference is that Theorem 6 provides a guarantee so that *all* sets of a certain size percolate, and it additionally establishes a bound on the number of steps until complete percolation.

We use the following version of the expander mixing-lemma to show Theorem 6.

Lemma 2 (Non-regular-expander mixing-lemma). *For all $S \subseteq V$ of a graph with spectral expansion λ, denoting with $e(S, V \setminus S)$ the number of edges between S and $V \setminus S$, we have*

$$\left| e(S, V \setminus S) - \frac{\mathrm{vol}(S) \cdot \mathrm{vol}(V \setminus S)}{\mathrm{vol}(G)} \right| \leq \lambda \cdot \frac{\mathrm{vol}(S) \cdot \mathrm{vol}(V \setminus S)}{\mathrm{vol}(G)}.$$

Proof (Proof of Theorem 6). We establish the result in two stages, depending on whether $|S|$ is greater or smaller than $n/2$. In the first stage, we show that whenever the set of active nodes S with $|S| = \varepsilon \cdot n$ satisfies $4\frac{r-1}{(1-\lambda)\cdot d_{\min}^2/d} \leq \varepsilon \leq 1/2$, then the number of active nodes increases by a factor of $1 + \Omega(1-\lambda)$. Applying Lemma 2 with S yields

$$e(S, V \setminus S) \geq (1-\lambda) \cdot \frac{\mathrm{vol}(S) \cdot \mathrm{vol}(V \setminus S)}{\mathrm{vol}(G)} \geq (1-\lambda) \cdot \frac{d_{\min}^2 \varepsilon n(1-\varepsilon)}{d}.$$

Now define $N := \{v \in V \setminus S \mid \deg_S(v) \geq r\} \subseteq V \setminus S$, which are the nodes that get activated by S in the next round. By decomposing $e(S, V \setminus S) = e(S, N) + e(S, (V \setminus S) \setminus N)$,

$$e(S, V \setminus S) \leq |N| \cdot d_{\max} + (|V \setminus S| - |N|) \cdot (r-1),$$

and rearranging gives

$$|N| \geq \frac{e(S, V \setminus S) - (n - |S|) \cdot (r-1)}{d_{\max}} \geq \frac{(1-\lambda) \cdot \frac{d_{\min}^2}{d} \varepsilon(1-\varepsilon)n - n \cdot (r-1)}{d_{\max}}. \tag{1}$$

Hence, if $1/2 \geq \varepsilon \geq 4\frac{r-1}{(1-\lambda) \cdot d_{\min}^2/d}$, we conclude that

$$|N| \geq \frac{\left(\frac{1-\lambda}{2}\frac{d_{\min}^2}{d}\varepsilon - \frac{1-\lambda}{4}\frac{d_{\min}^2}{d}\varepsilon\right) \cdot n}{d_{\max}} \geq \frac{\frac{1-\lambda}{4}\frac{d_{\min}^2}{d}\varepsilon}{d_{\max}} \cdot n = \frac{1-\lambda}{4} \cdot \frac{d_{\min}^2}{d_{\max} \cdot d} \cdot |S|.$$

Recall that we assumed $d_{\max} = O(d_{\min})$. Thus in the next step, we can replace S by $S \cup I$ and obtain an at least exponential growth (with factor $\Theta(1-\lambda)$) in the number of active nodes until $|S| > n/2$.

Consider now the second stage, where we assume $|S| > n/2$ (thus $\varepsilon > 1/2$). As before, we infer in the same way $e(S, V \setminus S) \geq (1-\lambda) \cdot d_{\min}^2 \varepsilon n(1-\varepsilon)/d$. Recalling that $N = \{v \in V \setminus S \mid \deg_S(v) \geq r\}$, we obtain the following refined version of (1), using that $\varepsilon \geq 1/2$,

$$\begin{aligned} |N| &\geq \frac{(1-\lambda)\frac{d_{\min}^2}{d}\varepsilon(1-\varepsilon)n - (|V \setminus S| - |N|) \cdot (r-1)}{d_{\max}} \\ &\geq \frac{\frac{1-\lambda}{2}\frac{d_{\min}^2}{d}(1-\varepsilon)n - (1-\varepsilon) \cdot n \cdot (r-1)}{d_{\max}}. \end{aligned}$$

Hence, if $r - 1 \leq \frac{(1-\lambda)d_{\min}^2}{4d}$, we conclude that

$$|N| \geq \frac{\frac{1-\lambda}{4}\frac{d_{\min}^2}{d}(1-\varepsilon)n}{d_{\max}} = \frac{1-\lambda}{4} \cdot \frac{d_{\min}^2}{d_{\max} \cdot d} \cdot |V \setminus S|.$$

Thus, if $|S| > n/2$, the set of inactive nodes decreases exponentially in each round.

Erdős-Rényi Graphs. We first prove an upper bound for the time until complete percolation for bootstrap processes on Erdős-Rényi graphs, showing Property 3, which is better than the one following from Theorem 6. Then, we show that there exists an initial active set such that the time needed for complete percolation matches this bound. We make use of the well-known Chernoff bounds.

Theorem 7 (Chernoff bounds [3, Theorems A.1.12 and A.1.13]). *Let $n \in \mathbb{N}_{>0}$, $p \in [0,1]$, and $X \sim \mathrm{Bin}(n,p)$. Then*

1. *for all $\beta > 1$, it holds that $\Pr[X \geq \beta np] \leq (e^{\beta-1}\beta^{-\beta})^{np}$, and*
2. *for all $a \in (0, np]$, it holds that $\Pr[X < np - a] < \exp\big(-a^2/(2np)\big)$.*

The following bound shows a rapid activation rate for sufficiently large initial active sets.

Theorem 8. *Let $n \in \mathbb{N}_{\geq 3}$, $p \geq 20\ln(n)/n$, $r \in [2..\ln n]$. Further, let $I \subseteq V(G_{n,p})$ with $|I| = r \cdot n/\ln n$. Then a.a.s., the bootstrap percolation process on $G_{n,p}$ with threshold r and initial active set I percolates completely in at most $(1+o(1))\ln(n)/\ln\ln n$ rounds.*

Proof. We prove several claims about $G = G_{n,p}$, which ultimately show Theorem 8.

Claim (8.1). The minimum degree of a node of G is a.a.s. at least $13\ln n$.

Proof. The degree of each node v is a binomial random variable with parameters $n-1$ and p. By assumption $(n-1)p \geq (1-o(1))20\ln n$ and, by Theorem 7, Item 2, the probability that it is smaller than $13\ln n$ is at most

$$e^{-(1+o(1))(49/40)\ln n} = \frac{1}{n^{49/40-o(1)}}.$$

The assertion of the claim thus follows from the union bound.

Claim (8.2). Asymptotically almost surely, for every two disjoint sets C and B in G, with $|C| = n/2$ and $|B| = rn/\ln n$, there is a node c in C that has at least r neighbors in B.

Proof. Fix two disjoint sets B and C as above. Clearly it suffices to prove the claim for $p = 20\ln(n)/n$. For every node $v \in C$, the expected number of neighbors of v in B is $p|B| = 20r$. By Theorem 7, Item 2, the probability it has less than r neighbors in B is at most (with room to spare) $e^{-19^2r^2/(40r)} < \frac{1}{100}$. These events for distinct nodes $v \in C$ are pairwise independent, hence the probability that there is no node $v \in C$ as above is at most $(1/100)^{n/2}$. As there are less than 4^n pairs of sets B, C as above, the result follows by the union bound, since $4^n/100^{n/2} = o(1)$.

Claim (8.3). Asymptotically almost surely, for any two disjoint sets of nodes B and C, where $|B| \geq n/2$, $n - |B| \geq 12\ln n$ and $|C| = (n-|B|)/2$, there is a node in C that has at least r neighbors in B.

Proof. As before, fix two disjoint sets B, C as above, and note that we may assume that $p = 20\ln(n)/n$. For every fixed $v \in C$ the expected number of neighbors of v in B is $p|B| \geq 10\ln n$. As $r \leq \ln n$, the probability that v has less than r neighbors in B is at most $e^{-81\ln(n)/20} < n^{-4}$, by Theorem 7, Item 2. Therefore the probability that this is the case for every $v \in C$ is smaller than $(1/n^4)^{|C|}$. The number of possible pairs of sets B and C as above is smaller than $n^{3|C|}$ (as the number of choices for the complement of B is $\binom{n}{2|C|} \leq n^{2|C|}$), and the claim follows by the union bound.

Claim (8.4). Asymptotically almost surely, for every $y \leq \frac{n}{100 \ln n}$, no set of y nodes of G spans more than $y \ln n$ edges.

Proof. Fix a set Y of y nodes. The expected number of edges in it is $\binom{y}{2}p \leq \frac{y^2 10 \ln n}{n}$. By Theorem 7, Item 1, with

$$\beta = \frac{y \ln n}{(y^2 10 \ln n)/n} = \frac{n}{10y} \ (> 10 \ln n)$$

the probability that Y spans at least $y \ln n$ edges is at most

$$(e^{\beta-1}/\beta^{\beta})^{(y^2 10 \ln n)/n} \leq \beta^{-0.9\beta(y^2 10 \ln n)/n} = (10y/n)^{0.9y \ln n} < e^{-2y \ln n}.$$

The number of sets of size y is $\binom{n}{y} \leq e^{y \ln n}$. We conclude by noting that the probability that there is a set Y spanning $y \ln n$ edges for any $y \leq \frac{n}{100 \ln n}$ is at most

$$\sum_{y=1}^{n/(100 \ln n)} e^{y \ln n} \cdot e^{-2y \ln n} = o(1).$$

Claim (8.5). Asymptotically almost surely, for every set B of nodes of size $n-x$, where $12 \ln n \leq x \leq n/1000$, the number of nodes outside B that do not have at least $\ln n$ $(\geq r)$ neighbors in B is smaller than $10x/\ln n$.

Proof. Fix a set B as above and a subset C of $10x/\ln n$ nodes in its complement. We bound the probability that no node of C has at least $\ln n$ neighbors in B as follows. By Sect. 4.2, a.a.s. each node in the graph has degree at least $13 \ln n$. Assume this is the case. Then every node of C has at least $12 \ln n$ neighbors in the complement B' of B (as it has at most $\ln n$ neighbors in B). By Sect. 4.2, a.a.s., the number of edges spanned by the set C is at most $|C| \ln n$. Thus the number of edges between C and $B' \setminus C$ has to be at least $10|C| \ln n = 100x$. The expected number of edges is

$$|C|(|B'| - |C|)p \leq \frac{10x}{\ln n} x \frac{20 \ln n}{n} = \frac{200x^2}{n}.$$

Applying Theorem 7, Item 1, with

$$\beta = \frac{100x}{200x^2/n} = \frac{n}{2x} \geq 500 \ (> e^5)$$

we conclude that the probability of having that many edges is at most

$$(e^{\beta-1}\beta^{-\beta})^{200x^2/n} \leq \beta^{-0.8\beta 200x^2/n} = (2x/n)^{80x}.$$

The number of choices for the sets B' and C is smaller than $\binom{n}{x}^2 \leq (en/x)^{2x}$. Thus, by the union bound the probability that there are sets B, C violating the claim is at most

$$\sum_{x \geq 12 \ln n} (en/x)^{2x} (2x/n)^{80x}.$$

Since $x \leq n/1000$, $2x/n \leq 1/500$ and hence $(2x/n)^{80x} \leq \big(x/(250n)\big)^{40x} < \big(x/(250n)\big)^{2x}$ showing that the sum above is at most $\sum_{x \geq 12 \ln n} (e/250)^{2x} = o(1)$.

We now prove that the number of rounds until complete percolation is a.a.s. $(1 + o(1)) \ln(n)/\ln\ln n$. Assuming that all claims hold, starting with any set A of $rn/\ln n$ nodes, by Claim 8.2, in one round at least $n/2$ nodes become active. By Claim 8.3, in 9 additional rounds the number of inactive nodes drops to at most $n/2^{10} < n/1000$. By Claim 8.5, in each round from now on, the number of inactive nodes drops by a factor of at least $\ln n/10$, as long as this number is above $12 \ln n$. Once below $12 \ln n$, one final step activates all remaining nodes, as the minimum degree is at least $13 \ln n$, by Claim 8.1. This completes the proof.

Note that the bound in Theorem 8 is optimal for $p = \Theta(\log(n)/n)$ in the sense that there is an initial active set A such that the process takes, for some $\varepsilon \in (0, 1]$, at least $(1 - \varepsilon) \ln(n)/\ln\ln n$ rounds. This is the case since a.a.s. $d_{\max}(G) = O(\log n)$ (similar to Sect. 4.2). Assuming this is the case, for every node v the number of nodes within distance at most t is at most $\big(O(\log n)\big)^t$. For $t = (1-\varepsilon)\ln(n)/\ln\ln n$, this number is smaller than $n/2$. Hence there is a set A of $n/2 > rn/\ln n$ nodes so that the distance between A and v exceeds t. Thus, when starting with A of active nodes, t rounds do not suffice to activate v.

The following remark implies this is the same number of rounds the perturbed percolation with $r = 1$ for $p = \Theta(\log(n)/n)$ takes when starting from an active set of constant size.

Remark 1 [18, *Theorem 4*]. Let $n \in \mathbb{N}_{>0}$ and $p = \Theta(\log(n)/n)$. Then a.a.s., $G_{n,p}$ has a diameter of $\Theta(\log(n)/\log\log n)$.

5 Experimental Results

In this section, we provide empirical results on the polynomial-to-rapid activation rate both on the graphs analyzed above, and on further global-graph models. Our findings are consistent with our theoretical results as well as the expected behavior of the perturbed percolation process on such graph models. The Python implementation uses the libraries NetworKit [36] and igraph [21], collections of tools for generating and analyzing graphs. In particular, they provide implementations for several random graph models. All experiments were run on a machine with 4 Intel i7-7500U cores and 8 GB RAM. However, note that we are not concerned with wall clock times, and all experiments were finished within minutes.

5.1 Erdős–Rényi Graphs

Corollary 2 shows that a.a.s. there is a polynomial-to-rapid activation rate for a PNG local graph combined with a $G_{n,p}$, for some parameter range. In Fig. 1, we consider such configurations satisfying these conditions, in particular, with a two-dimensional torus on $n = 10^6$ nodes as local graph. All runs are on the same random $G_{n,p}$ with $p = 20 \ln(n)/n$, as it is generated once for consistent comparison. One can see the linear increase of the number of activations per round on the local graph. After 500 rounds, the number of new active nodes per round starts decreasing, as the set of active nodes wraps around the torus.

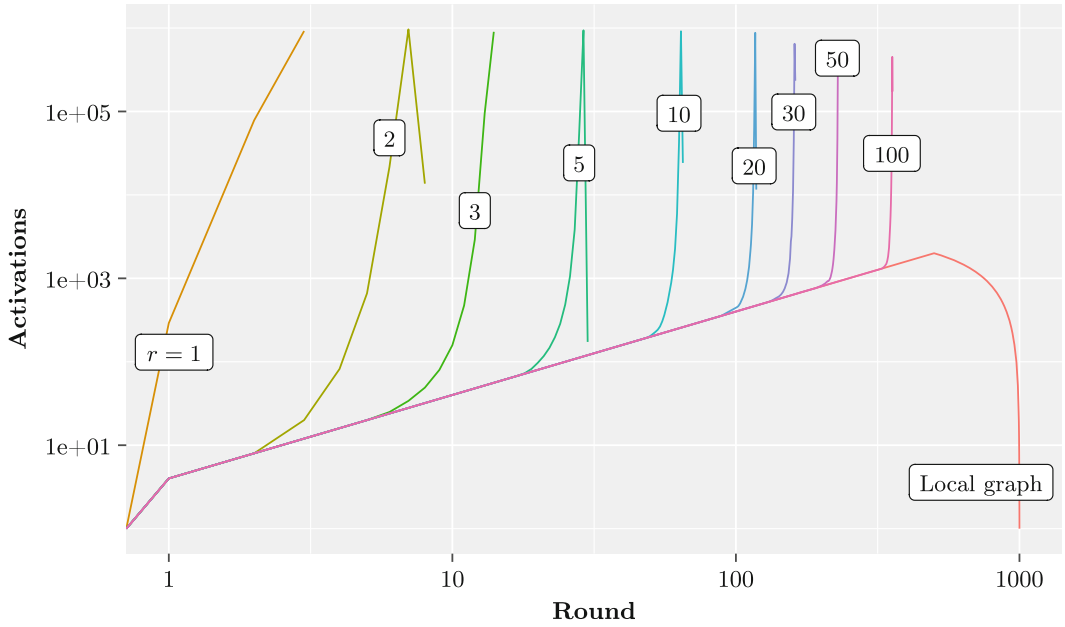

Fig. 1. The number of new active nodes in every round for different configurations. All runs are on $n = 10^6$ nodes with one initially active node. In one run ("Local graph"), we only consider the 2-dimensional torus on n nodes, while all other runs are on both the torus and a $G_{n,p}$ with $p = 20\ln(n)/n$ and a threshold r on the $G_{n,p}$. Note that both axes are logarithmic.

With the introduction of the $G_{n,p}$ with $r = 1$, the process completely percolates within three rounds, reflecting the rapid percolation. However, as r is increased, the effect of the global graph is withheld until some number of nodes are activated in the polynomial phase. Only then does the change to a rapid rate arise, and the process quickly percolates within few rounds, driven by the global edges. Even at $r = 100$, this effect is still observed.

5.2 Other Global-Graph Models

While our results only apply to the $G_{n,p}$ and expander graphs as global graph, we have strong reason to believe the same behavior can be observed for other global-graph models. We focus on two such models: (1) The Barabási-Albert (BA) model [2] uses a preferential-attachment approach, where nodes are iteratively added and connect to a fixed number of previous nodes proportional to their degree. (2) The hyperbolic random graph (HRG) model [33] randomly places nodes in a hyperbolic disk according to some probability distribution, and connects them if and only if they are close to each other. Both models exhibit small diameter and a power-law degree distribution, which should be beneficial for fast percolation. However, due to the underlying geometry, the HRG model has a large clustering coefficient, i.e., the neighbors of a node are likely to be neighbors of each other. We expect this feature to further accelerate the process, as this makes global edges more likely to hit the same nodes.

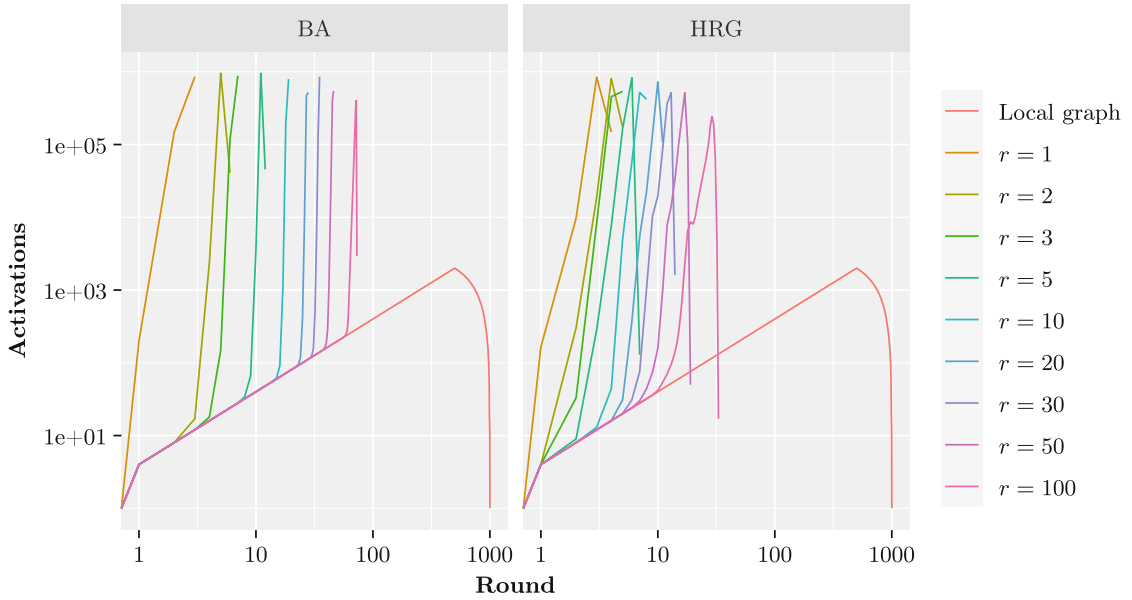

Fig. 2. The number of new active nodes in every round for different configurations. All runs are on $n = 10^6$ nodes with one initially active node. In one run ("Local graph"), we only consider the 2-dimensional torus on n nodes, while all other runs are on both the torus and a Barabási–Albert random graph (left), or hyperbolic random graph (right). Note that both axes are logarithmic.

The experiment setup is analogous to that described in Sect. 5.1, with the two-dimensional torus on n nodes as local graph. For the BA model, the number of attachments is chosen such that the expected average degree is $20 \ln n$. For the HRG model, we configure an expected power-law exponent of 3, and an expected average degree of $20 \ln n$. We consider the threshold model, i.e., a temperature of $T = 0$.

Our results of one run are depicted in Fig. 2. Again, the process for $r = 1$ reflects the rapid percolation, and for increasing r, the effect of the global edges is delayed until a threshold is reached. However, this threshold is reached earlier than in the $G_{n,p}$ version. For example, for $r = 100$, the $G_{n,p}$ has no activation by global edges until round 328, while with the BA model, this first happens in round 57. This can be explained by high-degree nodes in the BA model being more probable to reach the threshold r quickly.

For the HRG model, this effect is even stronger, with the first activation through global edges occurring in round 10 for $r = 100$. Even though both the BA and the HRG graph share the average degree and power-law exponent, the HRG graph has more high-degree nodes, in particular, those close to the center of the disk. Such nodes turn active very early, and then (through their high degree and high clustering) quickly activate the remaining nodes.

6 Outlook

With Lemma 1 and Theorems 6 and 8, we have shown bounds on the length of the initial (polynomial) and final (rapid) phase. It would be interesting to further analyze and tighten this gap. Our experiments (see Fig. 1) suggest that the transition is rather sharp once the first global activation occurs. Additionally, our experiments suggest that this behavior is very similar for other global-graph models, although we believe that the polynomial phase might be much shorter in the presence of a heavy-tailed degree distribution. Rigorously proving activation rates on such graph models would increase our understanding even further.

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