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# Cliques in Hyperbolic Random Graphs

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**Abstract** Most complex real world networks display scale-free features. This characteristic motivated the study of numerous random graph models with a power-law degree distribution. There is, however, no established and simple model which also has a high clustering of vertices as typically observed in real data. Hyperbolic random graphs bridge this gap. This natural model has recently been introduced by Krioukov, Papadopoulos, Kitsak, Vahdat, and Boguñá [17] and has shown theoretically and empirically to fulfill all typical properties of real world networks, including power-law degree distribution and high clustering.

We study cliques in hyperbolic random graphs $G$ and present new results on the expected number of $k$-cliques $\mathbb{E}[K_k]$ and the size of the largest clique $\omega(G)$. We observe that there is a phase transition at power-law exponent $\beta = 3$. More precisely, for $\beta \in (2, 3)$ we prove $\mathbb{E}[K_k] = n^{k(3-\beta)/2}\Theta(k)^{-k}$ and $\omega(G) = \Theta(n^{(3-\beta)/2})$, while for $\beta \geqslant 3$ we prove $\mathbb{E}[K_k] = n\,\Theta(k)^{-k}$ and $\omega(G) = \Theta(\log(n)/\log\log n)$.

Furthermore, we show that for $\beta \geqslant 3$, cliques in hyperbolic random graphs can be computed in time $\mathcal{O}(n)$. If the underlying geometry is known, cliques can be found with worst-case runtime $\mathcal{O}(m \cdot n^{2.5})$ for all values of $\beta$.

**Keywords** hyperbolic random graphs, random graphs, scale-free networks, social networks, cliques

## 1 Introduction

Scale-free networks are ubiquitous in nature and society. They appear as a large array of real world graphs that (mostly) have been formed by au-

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tonomous agents. Popular examples include social networks, protein-protein interactions, sexual networks, electricity circuits, the WWW, the internet, and many more [22]. Even though the term "scale-free network" has never been well-defined [19], it has been observed that all of these graphs share similar characteristics. They have hub nodes (nodes that interconnect the graph), community structures (subgraphs with high edge density), very low diameter (longest shortest path), a giant component (a connected component containing almost all vertices) and—probably most importantly—their degree distribution follows a power law: $P(k) \sim k^{-\beta}$, where $P(k)$ is the fraction of nodes having degree $k$.

Over the course of the last decade, research has been striving to produce generative models for these types of networks that are able to accurately predict properties of real world graphs. Popular models include the preferential attachment graphs [2] and variants of inhomogeneous random graphs [28]. The latter generalizes the Erdős-Rényi random graphs $G_{n,p}$ by using non-uniform edge probabilities. These models excel at modeling the power-law degree distribution; they also have a giant component, hub nodes, and low diameter. Due to their independent edge probabilities, they are accessible to rigorous studies. Independent edge probabilities also imply, however, that the graphs have low clustering, meaning that no community structures exist.

In contrast, most real world graphs do have high clustering. In the case of social networks this is easy to envision: two people are much more likely to be connected if they already have a friend in common. A number of fixes to the above models have been proposed to incorporate that intuition [20, 21, 29] (e.g., first construct a random graph, and then replace all nodes with $k$-cliques). Often, however, these fixes seem artificial and introduce structural artifacts that are unlikely to appear as such in nature.

Krioukov et al. [17] took a different approach by assuming an underlying hyperbolic geometry to the network. Similarly to the well-known geometric random graphs in Euclidean space [25], they introduced *hyperbolic random graphs* in which all nodes are placed in the hyperbolic plane, and two nodes are connected whenever they have a small distance from each other. Clustering then naturally emerges from the geometric interpretation. When two nodes are close to a third node, it is likely that they are also close to each other and the network thus attains a constant clustering coefficient [5, 13, 17]. Furthermore, the hyperbolic geometry enforces a power-law degree distribution and the presence of hub nodes.

The model achieved remarkable results for greedy forwarding. Embedding the internet graph in the hyperbolic plane, an autonomous system can route packets using only the hyperbolic location of the destination and its own neighbors [4, 24]. This approach both eliminates the need for the currently used routing tables, and it is nearly optimal: on average, path lengths using greedy routing are just 10% longer than optimal routing paths. This result suggests that there is an underlying hyperbolic metric to at least the internet graph and that hyperbolic geometry might be what unites most scale-free networks.

| | | Hyperbolic Random Graphs | |
|---|---|---|---|
| | | $\frac{1}{2} < \alpha < 1$ | $\alpha \geqslant 1$ |
| $\mathbb{E}[K_k]$ | $\leqslant$ | $\frac{n^{(1-\alpha)k}}{k^k \exp(k(\alpha \frac{C}{2} - 1))} \left(\frac{\alpha k c_1^{k-1}}{(1-\alpha)k+1} + 1\right)$ | $nk^{-k} \frac{\alpha k e (c_1 e^{-C/2+1})^{k-1}}{(\alpha-1)k+1} (1 + o(1))$ |
| | $\geqslant$ | $\left(\frac{e^{-\alpha C/2} n^{1-\alpha}(1-o(1))}{k}\right)^k$ | $nk^{-k} \left(\frac{e^{-C/2}}{\pi}\right)^{k-1} (1 + o(1))$ |
| | $=$ | $n^{(1-\alpha)k} \Theta(k)^{-k}$ | $n \cdot \Theta(k)^{-k}$ |
| $\omega(G)$ | $\leqslant$ | $c_1 e^{-\alpha \frac{C}{2} + 1} n^{1-\alpha} (1 + o(1))$ | $\frac{\log n}{\log \log n} (1 + o(1))$ |
| | $\geqslant$ | $e^{-\alpha \frac{C}{2}} n^{1-\alpha} (1 - o(1))$ | $\frac{\log n}{\log \log n} (1 - o(1))$ |
| | $=$ | $\Theta(n^{1-\alpha})$ | $\frac{\log n}{\log \log n} (1 \pm o(1))$ |

**Table 1** New results on the expected number of $k$-cliques $\mathbb{E}[K_k]$ and the size of the largest clique $\omega(G)$ in hyperbolic random graphs drawn from the step model. Sections 4 and 5 prove the upper and lower bounds on $\mathbb{E}[K_k]$. Section 6 proves the bounds on $\omega(G)$.

## 2 Our Contribution

Closely related to clustering and community structures, we analyze the emergence of cliques in hyperbolic random graphs (we note that we focus on the so-called step model, see Section 3 for more details). In particular, we present bounds on the expected number of k-cliques and the size of the largest clique. The results are summarized in Table 1. We observe a phase transition at power-law exponent $\beta = 3$, with smaller exponents yielding polynomial-size cliques and larger exponents yielding logarithmic-size cliques. While Clique is NP- and W[1]-complete for general graphs, we show that the largest clique of hyperbolic random graphs (in the step model) can be found in linear time, if $\beta \geqslant 3$, and in polynomial time $\mathcal{O}(m \cdot n^{2.5})$ if the node coordinates in the hyperbolic space are known. These findings stand in contrast to previous results on similar models like Chung-Lu [11], which need exponential time for a power-law of $2 < \beta < 3$.

*Comparison with other scale-free models* Using the results of Janson, Łuczak, and Norros [15], we compare the *clique numbers* (i.e., the size of the largest clique) of some popular scale-free network models to hyperbolic random graphs in Table 2.

We notice that the (asymptotic) clique number is nearly the same for Chung-Lu [1], Norros-Reittu [23] and hyperbolic random graphs, in the case where the power law exponent is $2 < \beta < 3$. An intuitive explanation for this phenomenon is that all these models have a tightly connected *core*—a subgraph of polynomial size in which the edge probability is $1 - o(1)$ or even 1. Large cliques emerge as a consequence of this core.

But even when such a core does not exist in the graph (which is the case for $\beta \geqslant 3$), one would expect to have small communities and therefore cliques in the graph. In particular, due to the large clustering coefficient it is likely that a node's neighbors (or a subset of the neighbors) form a clique. Consequently,

| | Power-Law Exponent | | |
|---|---|---|---|
| **Random Graph Model** | $2 < \beta < 3$ | $\beta = 3$ | $\beta > 3$ |
| **Hyperbolic (new results)** | $\Theta(n^{\frac{3-\beta}{2}})$ | $\Theta\left(\frac{\log n}{\log\log n}\right)$ | $\Theta\left(\frac{\log n}{\log\log n}\right)$ |
| **Chung-Lu** | $\Theta(n^{\frac{3-\beta}{2}})$ | $\Theta(1)$ | 3 |
| **Norros-Reittu** | $\Theta(n^{\frac{3-\beta}{2}} \log^{-\frac{\beta-1}{2}} n)$ | $\Theta(1)$ | 3 |
| **Generalized RG** | $\Omega(n^{\frac{3-\beta}{1+\beta}}), \mathcal{O}(n^{\frac{3-\beta}{1+\beta}} \log^{\frac{\beta-1}{\beta+1}} n)$ | $\Theta(1)$ | 3 |
| **Pref. Attachment** | $\Theta(1)$ | $\Theta(1)$ | $\Theta(1)$ |

**Table 2** Comparison of our new results on the clique number $\omega(G)$ of hyperbolic random graphs to known results by Janson et al. [15] for other scale-free random graph models. All bounds hold with high probability.

the hyperbolic random graph has in this case a largest clique of size $\Theta(\frac{\log n}{\log\log n})$. Previous scale-free network models with independent edge probabilities predict in this case a largest clique of size $\leqslant 3$ asymptotically almost surely (i.e., with probability $\geqslant 1 - o(1)$), which is typically not true for real-world networks. For instance, the networks Gnutella31, cit-Patents and com-Amazon from [18] have, respectively, estimated power-law exponents of $4.8, 4.02$ and $3.58$ but clique numbers of $4, 11$ and $7$.

*Organization* The remainder of this paper is organized as follows. Section 3 contains a brief introduction to the hyperbolic random graph model. Section 4 shows the upper bounds on $\mathbb{E}[K_k]$ in Table 1, while Section 5 shows the lower bounds on $\mathbb{E}[K_k]$. By applying these results, in Section 6 we obtain bounds on the size of the largest clique $\omega(G)$ in the hyperbolic random graph. Finally, in Section 7 we present efficient algorithms for finding the largest clique in hyperbolic random graphs. Section 8 contains concluding remarks.

## 3 Preliminaries

In this section, we briefly describe the hyperbolic graph generation. For a more in-depth introduction to the topic of hyperbolic random graphs, we refer the reader to [13, 17].

Following convention, we use the *native* representation of the hyperbolic plane $\mathbb{H}_2$. Here, a point $x$ is identified by a radial and an angular coordinate $(r_x, \varphi_x)$, where the radial coordinate denotes the hyperbolic distance from the coordinate origin. The hyperbolic space is also typically equipped with some negative curvature $K < 0$. In our case, however, it has been shown that there exists a coupling between random hyperbolic graphs on different curvatures [3]. Therefore, using different curvatures is equivalent to rescaling other model parameters—which is why we simply set $K = -1$.

To obtain a graph $G$ with $n$ nodes, let $D_{R_n}$ be a disc in $\mathbb{H}^2$ of radius $R_n = 2\ln n + C$, where $C$ adjusts the average degree of $G$. The disc is centered in the point of origin. Afterwards, $n$ points are sampled in $D_{R_n}$ as follows. Let $\alpha > \frac{1}{2}$ be some constant. The probability density for the radial coordinate $r$ of a point $p = (r, \varphi)$ is given by

$$\rho(r) := \alpha \frac{\sinh(\alpha r)}{\cosh(\alpha R_n) - 1},$$

and the angular coordinate $\varphi$ is sampled uniformly from $[0, 2\pi]$. We write $r_u$ and $\varphi_u$ to refer to the polar coordinates of a point $u$. For the sake of brevity, we omit the dependence on $n$ and simply write $R$ throughout the rest of the paper.

In the most general model, the probability that two nodes $u, v$ with relative angle $\Delta\theta$ connect is $p(u,v) := (\exp(\frac{1}{2T}(\mathrm{d}(u,v) - R)) + 1)^{-1}$, where $T$ is the temperature of the model, and

$$\cosh(\mathrm{d}(u,v)) := \cosh r_u \cosh r_v - \sinh r_u \sinh r_v \cos \Delta\theta \tag{1}$$

defines the distance $\mathrm{d}(u,v)$ between two points $u, v$ in $\mathbb{H}^2$. This produces a power-law graph with exponent $\beta = 2\alpha + 1$ if $\alpha \geqslant \frac{1}{2}$, and $\beta = 2$ otherwise [10, 13, 17]. We assume that $\alpha > \frac{1}{2}$, i.e., $\beta > 2$. Gugelmann, Panagiotou and Peter [13, 26] have shown that the average degree is then

$$\delta = (1 + o(1)) \frac{2\alpha^2 e^{-C/2}}{(\alpha - 1/2)^2} \left( \lim_{t \to T} \frac{t}{\sin(\pi t)} \right).$$

Observe that when $T \to 0$, $p(u,v)$ becomes a step function that connects two nodes if and only if they have hyperbolic distance at most $R$ from each other. We call this case the *step model* and the case $T > 0$ the *binomial model.* In this paper, we focus on the step model. Then, for two nodes with given radial coordinates $r, y$, the maximal angle such that they are connected by an edge is by Equation (1)

$$\theta(r,y) = \max_\varphi \{\mathrm{d}((r,0),(y,\varphi)) \leqslant R\} = \arccos\left(\frac{\cosh(y)\cosh(r) - \cosh(R)}{\sinh(y)\sinh(r)}\right). \tag{2}$$

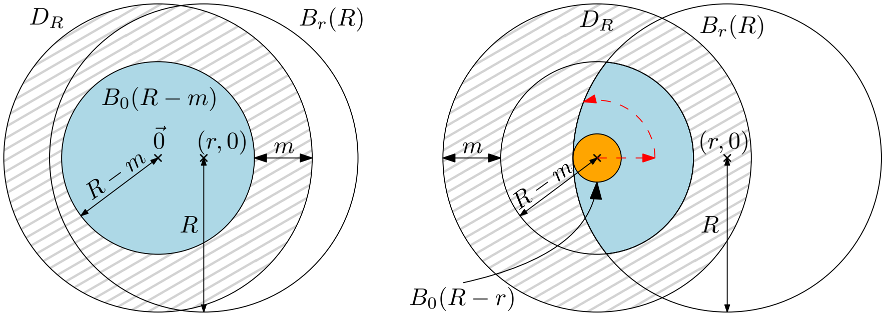
This complicated expression is closely approximated by the following Lemma from [13]. Notice that the second condition in the statement is required as otherwise $r + y < R$ and the two corresponding nodes are always connected by the triangle inequality.

**Lemma 1 ([13])** *Let $0 \leqslant r, y \leqslant R$ and $y + r \geqslant R$. Then,*

$$\theta(r,y) = \theta(y,r) = 2e^{\frac{R-r-y}{2}}(1 \pm \Theta(e^{R-r-y})).$$

For most computations on hyperbolic random graphs, one needs close approximations of the probability that a sampled point falls in a certain area. To this end, Gugelmann et al. [13] define the probability measure of a set $S \subseteq D_R$ as

$$\mu(S) := \int_S f(y)\,\mathrm{d}y,$$




(a) First case in Equation (4). Here, the disk $B_r(R)$ fully encloses $B_0(R-m)$, so their intersection is determined by the smaller disk $B_0(R-m)$.

(b) Second case in Equation (4). The two disks $B_r(R)$ and $B_0(R-m)$ intersect. The orange circle $B_0(R-r)$ is fully enclosed in the intersection. For the remainder, we integrate over the area as indicated by the red arrows.

**Figure 1** Proof illustration for Lemma 3. The gray area denotes the disc $D_R$. Note that these illustrations should only be understood schematically, as the behavior of circles in the hyperbolic plane is different to the classical Euclidean geometry.

where $f(r)$ is the probability mass of a single point $p = (r, \theta)$, which is $f(r) := \frac{\alpha \sinh(\alpha r)}{2\pi(\cosh(\alpha R)-1)}$. To simplify computation, we normally use the following approximation of $f(r)$.

**Lemma 2** *The density function $f(r)$ is approximated by*

$$f(r) = \tfrac{\alpha}{2\pi} e^{\alpha(r-R)} \cdot (1 + \Theta(e^{-\alpha R} - e^{-2\alpha r})).$$

*Proof* By applying $\frac{1}{1+x} = 1 - \Theta(x)$ for $x = o(1)$, we obtain

$$\begin{aligned} f(r) &= \frac{1}{2\pi} \cdot \frac{\alpha \sinh(\alpha r)}{\cosh(\alpha R) - 1} \\ &= \tfrac{\alpha}{2\pi} \cdot \frac{e^{\alpha r} - e^{-\alpha r}}{e^{\alpha R} + e^{-\alpha R} - 2} \\ &= \tfrac{\alpha}{2\pi} \cdot \frac{e^{\alpha r}(1 - e^{-2\alpha r})}{e^{\alpha R}(1 - \Theta(e^{-\alpha R}))} \\ &= \tfrac{\alpha}{2\pi} e^{\alpha(r-R)}(1 + \Theta(e^{-\alpha R} - e^{-2\alpha r})). \end{aligned}$$

□

We define the *ball* with radius $x$ around a certain point $(r, \theta)$ as

$$B_{r,\theta}(x) := \{(r', \theta') \mid \mathrm{d}((r', \theta'), (r, \theta)) \leqslant x\}.$$

We write $B_r(x)$ for $B_{r,0}(x)$. Using these definitions, we can formulate the following Lemma which was partially proven by Gugelmann et al. [13].

**Lemma 3** *For any $0 \leqslant r, m \leqslant R$ we have*

$$\mu(B_0(r)) = e^{-\alpha(R-r)}(1 - \Theta(e^{-\alpha r})) \tag{3}$$

$$\mu(B_r(R) \cap B_0(R-m)) = \begin{cases} \mu(B_0(R-m)) & \text{if } r \leqslant m, \\ \frac{4\alpha}{\pi(2\alpha-1)} e^{\frac{m-r}{2} - \alpha m} \cdot \mathcal{E} & \text{if } r > m, \end{cases} \tag{4}$$

*with error term $\mathcal{E} = 1 \pm \mathcal{O}(e^{(m-r)(\alpha-\frac{1}{2})})$ if $\alpha \neq \frac{3}{2}$ and $\mathcal{E} = 1 \pm \mathcal{O}(e^{m-r}(r-m))$ otherwise.*

*Proof* The proof of Equation (3) can be found in [13]. In the following, we show Equation (4).

Consider first the case $r \leqslant m$, for which Figure 1a contains an illustration. In that case, the ball $B_r(R)$ fully encloses $B_0(R-m)$, as all points in $B_0(R-m)$ have at most distance $R-m$ to 0; and by the triangle inequality at most distance $R-m+r \leqslant R$ to the center of $B_r(R)$. Therefore, the intersection of those two areas has the probability measure $\mu(B_0(R-m)) = e^{-\alpha m}(1 \pm o(1))$, proving the first case.

We now assume $r \geqslant m$. Figure 1b contains an illustration for this case. Then, we can write

$$\mu(B_r(R) \cap B_0(R-m)) = \mu(B_0(R-r)) + 2 \int_{R-r}^{R-m} \int_0^{\theta(r,y)} f(y) \,\mathrm{d}\theta \,\mathrm{d}y,$$

where $\theta(r,y) = \arccos\left(\frac{\cosh(r)\cosh(y) - \cosh(R)}{\sinh(r)\sinh(y)}\right)$ is given by the definition of the distance function, see Equation (2). The first part of the sum vanishes in the error term $\mathcal{E}$, since it simplifies to $(1 \pm o(1))e^{-\alpha r} = e^{\frac{m-r}{2} - \alpha m} \cdot \mathcal{O}(e^{(m-r)(\alpha - \frac{1}{2})})$.

For the second part of the sum, we have

$$2 \int_{R-r}^{R-m} \int_0^{\theta(r,y)} f(y) \,\mathrm{d}\theta \,\mathrm{d}y = 2 \int_{R-r}^{R-m} \theta(r,y) \cdot f(y) \,\mathrm{d}y.$$

By simplifying $\theta(r,y)$ using Lemma 1 and $f(y)$ using Lemma 2, this term can be transformed to obtain

$$\frac{2\alpha}{\pi}(1 + \mathcal{O}(e^{-\alpha R})) \int_{R-r}^{R-m} e^{\frac{R-r-y}{2} + \alpha y - \alpha R}(1 + \mathcal{O}(\pm e^{R-r-y} - e^{-2\alpha y})) \,\mathrm{d}y. \tag{5}$$

Observe that the dominant error term within the integral is $\mathcal{O}(\pm e^{R-r-y})$. This holds since $-2\alpha y < R-r-y$ follows from $(1-2\alpha)y < 0 < R-r$ and thereby $\mathcal{O}(\pm e^{R-r-y} - e^{-2\alpha y}) = \mathcal{O}(\pm e^{R-r-y})$.

We now first compute the integral without the error term and later add the error term. We obtain

$$\begin{aligned}
&\frac{2\alpha}{\pi}(1+\mathcal{O}(e^{-\alpha R}))\int_{R-r}^{R-m} e^{\frac{R-r-y}{2}+\alpha y-\alpha R}\,\mathrm{d}y \\
&= \frac{4\alpha}{\pi(2\alpha-1)}(1+\mathcal{O}(e^{-\alpha R}))\left[e^{\frac{R-r-y}{2}+\alpha y-\alpha R}\right]_{R-r}^{R-m} \\
&= \frac{4\alpha}{\pi(2\alpha-1)}(1+\mathcal{O}(e^{-\alpha R}))\left(e^{\frac{m-r}{2}-\alpha m}-e^{-\alpha r}\right) \\
&= \frac{4\alpha}{\pi(2\alpha-1)}e^{\frac{m-r}{2}-\alpha m}(1+\mathcal{O}(e^{-\alpha R}-e^{(m-r)(\alpha-\frac{1}{2})})) \\
&= \frac{4\alpha}{\pi(2\alpha-1)}e^{\frac{m-r}{2}-\alpha m}(1+\mathcal{O}(e^{(m-r)(\alpha-\frac{1}{2})})),
\end{aligned}$$

since again the dominating error term is $e^{(m-r)(\alpha-\frac{1}{2})} > e^{-R(\alpha-\frac{1}{2})} > e^{-\alpha R}$.

It is left to bound the error term in Equation (5). To this end, we compute

$$\begin{aligned}
&\int_{R-r}^{R-m} \mathcal{O}(e^{\frac{3}{2}(R-r-y)+\alpha y-\alpha R})\,\mathrm{d}y \\
&= e^{\frac{m-r}{2}-\alpha m}\cdot\begin{cases}
\mathcal{O}(e^{m-r}) \leqslant \mathcal{O}(e^{(m-r)(\alpha-\frac{1}{2})}), & \text{if } \alpha > \frac{3}{2}, \\
\mathcal{O}(e^{m-r}(r-m)), & \text{if } \alpha = \frac{3}{2}, \\
\mathcal{O}(e^{(m-r)(\alpha-\frac{1}{2})}), & \text{if } \alpha < \frac{3}{2}.
\end{cases}
\end{aligned}$$

Plugging everything together, we obtain Equation (4). □

Finally, let us also restate a useful result from [3]. Consider two vertices $u, v$ in the hyperbolic random graph. Moving one vertex closer to 0—i. e., decreasing $r_u$—typically does not result in a monotone behavior of $\mathrm{d}(u, v)$. In particular, $u$ can first move closer to $v$; and then farther away again. However, if $u$, $v$ had distance at most $x$ to each other and to the origin 0, this fact remains true even when $u$ is moved closer to the center. In this sense, a node's neighborhood is monotone in its radial coordinate: $B_u(R) \cap D_R \subset B_{u'}(R) \cap D_R$ for $u' < u$. The next lemma formalizes this intuition.

**Lemma 4 ([3])** *Consider two nodes $u = (r_u, \varphi_u), v = (r_v, \varphi_v)$ in the hyperbolic random graph. If $\mathrm{d}(u, v) \leqslant x$ and $r_u, r_v \leqslant x$, then it holds*

$$\mathrm{d}(u', v') \leqslant x,$$

*where $u' = (r'_u, \varphi_u), v' = (r'_v, \varphi_v)$ with $r'_u \leqslant r_u$ and $r'_v \leqslant r_v$.*

Using these results, we compute an upper bound on the expected number of $k$-cliques in the hyperbolic random graph.

## 4 Proof of the Upper Bound

The goal of this section is to show the upper bounds for $\mathbb{E}[K_k]$ stated in Table 1. The following theorem summarizes these results in their asymptotic form.

**Theorem 1** *In a hyperbolic random graph, the expected number of k-cliques is at most* $n^{(1-\alpha)k}\Theta(k)^{-k}$ *and* $n \cdot \Theta(k)^{-k}$ *if* $\frac{1}{2} < \alpha < 1$ *and* $\alpha \geqslant 1$*, respectively.*

In a clique, each pair of nodes is connected. To compute an upper bound on the probability that $k$ nodes form a clique, we examine a relaxed condition; namely, that all nodes connect to one specific node $v$.

For a set $U$ of $k$ independently sampled points, let $v \in U$ be the node with the largest radial coordinate, i. e., $r_v = \max_{u \in U}\{r_u\}$. We begin by computing the probability density function of $r_v$ which we call $\rho_v(r)$. By the definition of the cumulative distribution function, we have

$$\begin{aligned}\Pr[r_v \leqslant x] = \Pr[\forall u \in U \colon r_u \leqslant x] &= \left(\int_0^x \frac{\alpha \sinh(\alpha r)}{\cosh(\alpha R) - 1}\,\mathrm{d}r\right)^k \\ &= \left(\frac{\cosh(\alpha x) - 1}{\cosh(\alpha R) - 1}\right)^k.\end{aligned}$$

The resulting probability density function is given by

$$\begin{aligned}\rho_v(r) &= \frac{\partial}{\partial r}\left(\frac{\cosh(\alpha r) - 1}{\cosh(\alpha R) - 1}\right)^k \\ &= \alpha k \sinh(\alpha r)\frac{(\cosh(\alpha r) - 1)^{k-1}}{(\cosh(\alpha R) - 1)^k} \\ &= \alpha k e^{\alpha k(r-R)}(1 - e^{-2\alpha r}) \cdot \frac{(1 + e^{-2\alpha r} - 2e^{-\alpha r})^{k-1}}{(1 + e^{-2\alpha R} - 2e^{-\alpha R})^k} \\ &= \alpha k e^{\alpha k(r-R)}(1 - \mathcal{O}(e^{-\alpha r}))^k \\ &\leqslant \alpha k e^{\alpha k(r-R)},\end{aligned}$$

where we used the fact $\frac{1}{1+x} = 1 - \Theta(x)$ for $x = o(1)$ to bound the error term. Following the explanation above, the probability that a set $U$ of $k$ independently sampled nodes forms a clique is upper bounded by the probability that all nodes are connected to $v$. Formally,

$$\begin{aligned}\Pr[U \text{ is clique}] &\leqslant \Pr[\forall u \in U \colon \mathrm{d}(u,v) \leqslant R] \\ &= \int_0^R \rho_v(r) \cdot \Pr[\forall u \in U \colon u \in B_r(R) \mid r_v = r]\,\mathrm{d}r \\ &= \int_0^R \rho_v(r) \cdot \left(\frac{\mu(B_r(R) \cap B_0(r))}{\mu(B_0(r))}\right)^{k-1} \mathrm{d}r\end{aligned}$$

For the last equality, observe that we condition on the fact that the largest radial coordinate among the nodes in $U$ is $r$, i. e., all other radial coordinates are $\leqslant r$. Hence, the probability that a node $u$ is connected to $v$ is the probability that $u \in B_r(R) \cap B_0(R)$, conditioned on the fact that $r_u \leqslant r$, that is, $u \in B_0(r)$.

We split the integral in two parts. If $r < R/2$, then by the triangle inequality it follows that all $k$ nodes are connected. This agrees with Lemma 3 since $r < R/2$ implies $r < m$, and we obtain

$$\int_0^{R/2} \rho_v(r) \cdot \left(\frac{\mu(B_r(R) \cap B_0(r))}{\mu(B_0(r))}\right)^{k-1} \mathrm{d}r$$
$$= \int_0^{R/2} \rho_v(r)\,\mathrm{d}r \leqslant \left(\frac{\cosh(\alpha R/2) - 1}{\cosh(\alpha R) - 1}\right)^k \leqslant e^{-\alpha k \frac{R}{2}}. \tag{6}$$

When $r \geqslant \frac{R}{2}$, we estimate again using Lemma 3

$$\frac{\mu(B_r(R) \cap B_0(r))}{\mu(B_0(r))} = \tfrac{4\alpha}{\pi(2\alpha-1)} e^{\frac{R}{2} - r - \alpha(R-r) + \alpha(R-r)} \cdot \mathcal{E}$$
$$= \tfrac{4\alpha}{\pi(2\alpha-1)} \cdot e^{\frac{R}{2} - r} \cdot \mathcal{E},$$

where $\mathcal{E} = (1 \pm \mathcal{O}(e^{(R-2r)(\alpha - \frac{1}{2})} + e^{-\alpha r}))$, if $\alpha \neq \frac{3}{2}$ and $\mathcal{E} = (1 \pm \mathcal{O}(e^{(R-2r)}(2r - R) + e^{-\alpha r}))$ otherwise. Observe that in both cases, since $r \geqslant \frac{R}{2}$, the error term is upper bounded by a constant. Thus, we write

$$\frac{\mu(B_r(R) \cap B_0(r))}{\mu(B_0(r))} \leqslant c_1 e^{\frac{R}{2} - r} \tag{7}$$

for some large enough constant $c_1 > 1$. Then, we compute for the second part of the integration

$$\int_{R/2}^{R} \rho_v(r) \cdot \left(\frac{\mu(B_r(R) \cap B_0(r))}{\mu(B_0(r))}\right)^{k-1} \mathrm{d}r$$
$$\leqslant \int_{R/2}^{R} \alpha k e^{\alpha k (r-R)} \left(c_1 e^{R/2 - r}\right)^{k-1} \mathrm{d}r \tag{8}$$
$$= \frac{\alpha k c_1^{k-1}}{(\alpha - 1)k + 1} \left[e^{\alpha k (r-R) + (k-1)(\frac{R}{2} - r)}\right]_{R/2}^{R} \tag{9}$$
$$= \frac{\alpha k c_1^{k-1}}{(\alpha - 1)k + 1} \left(e^{-\frac{R}{2}(k-1)} - e^{-\frac{R}{2}\alpha k}\right), \tag{10}$$

where Equations (9) and (10) hold if $\alpha \neq 1$ and $k \neq 1/(1 - \alpha)$. In the following, we consider all possible combinations of $\alpha$ and $k$. Whenever possible, we continue computing with Equation (10), otherwise we use Equation (8). We distinguish the following cases:

(a) $\alpha = 1$. In this case, Equation (8) evaluates to

$$(8) = k c_1^{k-1} [e^{-\frac{R}{2}(k+1) + r}]_{R/2}^{R}$$
$$\leqslant k c_1^{k-1} e^{-\frac{R}{2}(k-1)}.$$

(b) $\alpha > 1$. Then, $0 > \frac{1}{1-\alpha} \neq k$ and thus we may use Equation (10):

$$(10) \leqslant \frac{\alpha k c_1^{k-1}}{(\alpha-1)k+1} e^{-\frac{R}{2}(k-1)}.$$

(c) $\frac{1}{2} \leqslant \alpha < 1$. In this case, the sign in front of the antiderivative depends on $k$:

(c.i) $k < \frac{1}{1-\alpha}$. In that case, $(\alpha-1)k > -1$, and Equation (10) is again upper bounded by

$$(10) \leqslant \frac{\alpha k c_1^{k-1}}{(\alpha-1)k+1} e^{-\frac{R}{2}(k-1)}.$$

(c.ii) $k = \frac{1}{1-\alpha}$. Then, we substitute $\alpha = \frac{k-1}{k}$ in Equation (8):

$$\begin{aligned}(8) &= \int_{R/2}^{R} \alpha k c_1^{k-1} e^{-\frac{R}{2}(k-1)} \,\mathrm{d}r \\ &= \alpha k c_1^{k-1} \frac{R}{2} e^{-\frac{R}{2}(k-1)}\end{aligned}$$

(c.iii) $k > \frac{1}{1-\alpha}$. Here, the sign of the antiderivative is negative, and we obtain

$$(10) \leqslant \frac{\alpha k c_1^{k-1}}{(1-\alpha)k+1} e^{-\alpha k \frac{R}{2}}.$$

Recall that we split the integral into two parts and thus have to add $e^{-\alpha k \frac{R}{2}}$ to the result, c. f. Equation (6). Cases (a)–(c.ii) only change by a factor of $(1+o(1))$, and in the case of (c.iii) we obtain that $\Pr[U \text{ is a clique}] \leqslant (1 + \frac{\alpha k c_1^{k-1}}{(1-\alpha)k+1}) e^{-\alpha k R/2}$. When $\alpha > 1$ (i.e. when the graph has a power law exponent $\beta > 3$), the number of cliques is therefore bounded by

$$\begin{aligned}\mathbb{E}[K_k] &= \binom{n}{k} \Pr[U \text{ is clique}] \\ &\leqslant \left(\frac{ne}{k}\right)^k \frac{\alpha k c_1^{k-1}}{(\alpha-1)k+1} e^{-\frac{R}{2}(k-1)} (1+o(1)) \\ &= nk^{-k} \cdot \frac{\alpha k e (c_1 e^{-C/2+1})^{k-1}}{(\alpha-1)k+1} (1+o(1)) \\ &= n \cdot \Theta(k)^{-k},\end{aligned}$$

since $n = e^{\frac{R-C}{2}}$. Recall that $C$ is a parameter of the network model adjusting the average degree, see Section 3. For $\alpha = 1$ we obtain a similar bound $\mathbb{E}[K_k] \leqslant n \cdot \Theta(k)^{-k+1} = n \cdot \Theta(k)^{-k}$.

For networks with a dense core ($\frac{1}{2} \leqslant \alpha < 1$), we obtain

$$\begin{aligned}\mathbb{E}\left[K_k\right] &\leqslant \left(\frac{ne}{k}\right)^k \left(1 + \frac{\alpha k c_1^{k-1}}{(1-\alpha)k+1}\right) e^{-\alpha k \frac{R}{2}} \\ &= n^{(1-\alpha)k} k^{-k} \left(1 + \frac{\alpha k c_1^{k-1}}{(1-\alpha)k+1}\right) e^{(1-\alpha\frac{C}{2})k} \\ &= n^{(1-\alpha)k} \Theta(k)^{-k},\end{aligned}$$

if $k > \frac{1}{1-\alpha}$. Table 1 contains the detailed results for these cases. In the case where $k \leqslant 1/(1-\alpha)$, which is not shown in the table, our result states that there is at most a linear number of $k$-cliques. This agrees, for instance, with the known fact that for $k = 2 \leqslant \frac{1}{1-\alpha}$ there are $\Theta(n)$ many edges in $G$.

## 5 Proof of the Lower Bound

In this section, we show the lower bounds for $\mathbb{E}\left[K_k\right]$ stated in Table 1, which asymptotically match the upper bounds we proved in the previous section.

**Theorem 2** *In a hyperbolic random graph, the expected number of $k$-cliques is at least $n^{(1-\alpha)k}\Theta(k)^{-k}$ and $n \cdot \Theta(k)^{-k}$ if $\frac{1}{2} < \alpha < 1$ and $\alpha \geqslant 1$, respectively.*

To obtain these matching lower bounds, we consider two cases. In the case when $\frac{1}{2} < \alpha < 1$, we show that hyperbolic random graphs exhibit a tightly connected core and the high-degree nodes thus form a clique of polynomial size. The number of $k$-cliques in $G$ is then simply dominated by the number of distinct $k$-subsets of nodes in the core.

To be more precise, consider the ball $B_0(R/2)$. All nodes in this area have distance at most $R$ from each other by the triangle inequality. It is therefore left to bound the number of nodes in $B_0(R/2)$. By Lemma 3 we know that

$$\mu(B_0(x)) = e^{-\alpha(R-x)}(1 - \mathcal{O}(e^{-\alpha x})),$$

that is, the probability that a sampled point has at most distance $x$ from the center of $D_R$ is $e^{-\alpha(R-x)}(1+o(1))$. Consequently, we expect $ne^{-\alpha R/2}(1-o(1))$ nodes in $B_0(R/2)$. Observe that for $\frac{1}{2} < \alpha < 1$ and $R = 2\ln n + C$ this amounts to $e^{-\alpha C/2} n^{1-\alpha}(1 - o(1))$, which is polynomial. In Section 6, we will also see that this number is close to the size of the maximum clique.

Let $K_k(G)$ be the number of $k$-cliques in $G$. Clearly, if $G' \subseteq G$, then we have that $K_k(G') \leqslant K_k(G)$. Consider for $G$ the hyperbolic random graph and for $G'$ the graph induced on $G$ by only taking vertices $v$ with $r_v \leqslant R/2$. Then, we get

$$\mathbb{E}\left[K_k\right] = \mathbb{E}\left[K_k(G)\right] \geqslant \mathbb{E}\left[K_k(G')\right] = \mathbb{E}\left[\binom{X}{k}\right],$$

where $X$ is the random variable describing the number of nodes that drop in $B_0(R/2)$. To show the lower bound, we use the following well-known lemma, which can, for example, be found in [30, Ex. 1].

**Lemma 5** *The function $f(x) = \binom{x}{k}$ is convex on $x \geqslant k$.*

Therefore, using Jensen's inequality [16], which says $f(\mathbb{E}[X]) \leqslant \mathbb{E}[f(X)]$ for convex functions $f$, we obtain

$$\mathbb{E}\left[\binom{X}{k}\right] \geqslant \binom{\mathbb{E}[X]}{k} = \binom{e^{-\alpha C/2} n^{1-\alpha}(1 - \mathcal{O}(e^{-\alpha R/2}))}{k}$$
$$\geqslant \left(\frac{e^{-\alpha C/2} n^{1-\alpha}(1 - o(1))}{k}\right)^k .$$

Thus, we have that $\mathbb{E}[K_k] \geqslant n^{(1-\alpha)k} \cdot \Theta(k)^{-k}$, which proves the lower bound for the dense case.

## 5.1 Small Cliques Outside of the Core

So far, we have seen that for $\frac{1}{2} < \alpha < 1$, hyperbolic random graphs contain many cliques in the core. When $\alpha \geqslant 1$, however, the number of nodes in $B_0(R/2)$ is of order $\mathcal{O}(1)$. We now show that due to the underlying geometry, cliques still emerge outside of the core.

To this end, we investigate a circular sector of the disk $D_R$ with angle $\theta = a/n$, for some constant $a$, which we choose later. Clearly, there are $\frac{2\pi n}{a}$ non-overlapping sectors. As we show in the following, such a circular sector has a (geometric) diameter of $\leqslant R$, if $a$ is chosen as an appropriate constant. This means that all points in the sector have pairwise distance at most $R$ and therefore form a complete subgraph.

Since the angular coordinates of nodes are sampled uniformly, the probability that we sample a node inside one specific circular sector of angle $a/n$ is exactly $\frac{a}{2\pi n}$. Therefore, the probability that a set of $k$ independently sampled points $U$ is contained in one sector is

$$\Pr[U \text{ is clique}] \geqslant \frac{2\pi n}{a} \cdot \left(\frac{a}{2\pi n}\right)^k = \left(\frac{a}{2\pi n}\right)^{k-1} .$$

This probability is maximized by choosing $a$ as large as possible, i.e., such that for any larger value the diameter exceeds $R$. It remains to derive a suitable value for $a$. We note that a statement similar to the following lemma has been proven concurrently by Fountoulakis [10].

**Lemma 6** *Let $S$ be a circular sector of $D_R$ of angle $\frac{a}{n} = \frac{2}{n} e^{-C/2}(1 - \mathcal{O}(n^{-2}))$. Then, $S$ has a (geometric) diameter of at most $R$.*

*Proof* Let $u, v$ be two points inside $S$ with maximum distance. Observe that these points lie on the boundary of $S$. Otherwise, consider the geodesic that goes through $u, v$ and intersects $S$ at $u', v'$. But then $\mathrm{d}(u', v') > \mathrm{d}(u, v)$ contradicts the assumption that $u, v$ had maximum distance. Observe further that

$$\cosh(\mathrm{d}(u, v)) := \cosh(r_u) \cosh(r_v) - \sinh(r_u) \sinh(r_v) \cos(\Delta\varphi)$$

is monotonously increasing for $0 \leqslant \Delta\varphi \leqslant \pi$. Since $S$ has an angle of $\frac{a}{n} \ll \pi$, we thus may assume that $u, v$ have a relative angle of $\Delta\varphi = \frac{a}{n}$.

We now show that if $r_u = r_v = R$, $\mathrm{d}(u, v) \leqslant R$. By Lemma 4 it follows that all other pairs of points with smaller radial coordinates also have a distance of at most $R$.

By Lemma 1, the maximum angle between $u, v$ such that their distance is at most $R$, is

$$\begin{aligned}\theta(R, R) &= 2e^{-\frac{R}{2}}(1 \pm \mathcal{O}(e^{-R})) = 2e^{-\frac{R-C}{2}-\frac{C}{2}}(1 \pm \mathcal{O}(e^{-R})) \\ &= 2e^{-C/2}\tfrac{1}{n}(1 \pm \mathcal{O}(n^{-2})).\end{aligned}$$

Thus, we set $a = 2e^{-C/2}(1 \pm \mathcal{O}(n^{-2}))$. □

Finally, the probability that a set $U$ of $k$ nodes is a clique is

$$\begin{aligned}\Pr[U \text{ is clique}] &\geqslant \left(\frac{a}{2\pi n}\right)^{k-1} \\ &\geqslant \left(\frac{e^{-C/2}}{\pi n}(1 - \mathcal{O}(n^{-2}))\right)^{k-1} \\ &= \left(\frac{e^{-C/2}}{\pi n}\right)^{k-1}(1 \pm \mathcal{O}(n^{-1})),\end{aligned}$$

since $(1 - \mathcal{O}(n^{-2}))^{k-1} \geqslant (1 - \mathcal{O}(n^{-2}))^{n} = (1 - \mathcal{O}(n^{-1}))$. For the expected number of $k$-cliques, this implies that

$$\begin{aligned}\mathbb{E}[K_k] &= \binom{n}{k}\Pr[U \text{ is } k\text{-clique}] \\ &\geqslant \left(\frac{n}{k}\right)^k \left(\frac{e^{-C/2}}{\pi n}\right)^{k-1}(1 \pm \mathcal{O}(n^{-1})) \\ &= n \cdot \Theta(k)^{-k}.\end{aligned}$$

Taken together with the result from above, we conclude

$$\mathbb{E}[K_k] \geqslant \max\{n, n^{(1-\alpha)k}\} \cdot \Theta(k)^{-k}.$$

## 6 Largest Clique

In this section, we present the bounds on the clique number $\omega(G)$, i.e., the size of the largest clique in $G$, as stated in Table 1. The asymptotic bounds are summarized in the following theorem.

**Theorem 3** *With high probability, the clique number of a hyperbolic random graph is $\Theta(n^{1-\alpha})$ and $\frac{\log n}{\log\log n}(1 \pm o(1))$ if $\frac{1}{2} < \alpha < 1$ and $\alpha \geqslant 1$, respectively.*

We use the upper bounds on the number of $k$-cliques from Theorem 1 to obtain upper bounds for $\omega(G)$ by applying the Markov inequality

$$\Pr[K_k > 1] \leqslant \mathbb{E}\left[K_k\right].$$

Let therefore $\varepsilon > 0$ and solve $\mathbb{E}\left[K_k\right] \leqslant n^{-\varepsilon}$ for $k$. If $\varepsilon$ is chosen as a constant independent of $n$, we obtain an upper bound on the clique number that holds with high probability.

### 6.1 Dense Core

Let us first consider the case when $\frac{1}{2} < \alpha < 1$ and there exists a dense core in the center of $D_R$. Due to Theorem 1, there exists some constant $c$ such that

$$\Pr[K_k > 1] \leqslant \mathbb{E}\left[K_k\right] \leqslant n^{(1-\alpha)k} \cdot (ck)^{-k}.$$

We set $k = \frac{2}{c} n^{1-\alpha}$ to obtain

$$\begin{aligned}\mathbb{E}\left[K_k\right] &\leqslant n^{(1-\alpha)k} \cdot (ck)^{-k}\\ &= n^{(1-\alpha)\frac{2}{c}n^{1-\alpha}} \cdot (2n^{1-\alpha})^{-\frac{2}{c}n^{1-\alpha}}\\ &= 2^{-\frac{2}{c}n^{1-\alpha}}.\end{aligned}$$

This term is asymptotically smaller than $n^{-\varepsilon}$ for any constant $\varepsilon$, since

$$2^{-\frac{2}{c}n^{1-\alpha}} \leqslant n^{-\varepsilon} \Leftrightarrow \frac{2}{c}n^{1-\alpha} \geqslant \varepsilon \log_2 n$$

for large enough $n$. Therefore, we know that $\omega(G) \leqslant \Theta(n^{1-\alpha})$ in this case. The precise leading constant for this approach depends on $c_1$, see Equation (7). Since $c_1 > 1$, we have for $k = \omega(1)$

$$\mathbb{E}\left[K_k\right] \leqslant n^{(1-\alpha)k}\left(\tfrac{1}{c_1} e^{\alpha\frac{C}{2}-1}(1+o(1))\right)^{-k}.$$

Using a similar approach as above, we can compute that

$$\omega(G) \leqslant c_1 e^{-\alpha\frac{C}{2}+1} n^{1-\alpha}(1+o(1)) = \mathcal{O}(n^{1-\alpha})$$

holds with high probability.

To compute a matching lower bound, recall that Section 5 states that $B_0(R/2)$ contains $e^{-\alpha C/2} n^{1-\alpha}(1-o(1))$ nodes in expectation. Let $X$ be the number of nodes in $B_0(R/2)$. Since each node is sampled independently from all others, we may apply a multiplicative Chernoff bound (see e.g. [9]) to obtain that

$$\Pr[X \leqslant (1 - \tfrac{1}{\log n})\mathbb{E}\left[X\right]] \leqslant \exp(-\Theta(1) \cdot \log^{-2} n e^{-\alpha C/2} n^{1-\alpha}).$$

As this tail probability decreases faster than any polynomial, we have with high probability that the largest clique is of size

$$\omega(G) \geqslant e^{-\alpha C/2} n^{1-\alpha}(1-o(1)) = \Omega(n^{1-\alpha}).$$

We note that a similar observation has been made by Candellero and Fountoulakis [6], where the authors computed bounds on the number of nodes close to the center.

## 6.2 Sparse Core

For $\alpha \geqslant 1$, when a dense core is not present, we have proven that $\mathbb{E}[K_k] = n \cdot \Theta(k)^{-k}$. Thus, there exists a constant $c$ such that $\mathbb{E}[K_k] \leqslant n \cdot (ck)^{-k}$. Again, we apply a Markov bound to upper bound the probability that a large clique occurs. Thus, we need to choose $k$ such that $\Pr[K_k \geqslant 1] \leqslant n^{-\varepsilon}$. Since it holds

$$\Pr[K_k \geqslant 1] \leqslant \mathbb{E}[K_k] \leqslant n \cdot (ck)^{-k},$$

it suffices to find a $k$ such that $n \cdot (ck)^{-k} \leqslant n^{-\varepsilon}$, which is equivalent to $(ck)^{-k} \leqslant n^{-1-\varepsilon}$. By taking $k := (1+\varepsilon)\frac{\log n}{\log\log n}$, we obtain for large enough $n$

$$\begin{aligned}
(ck)^{-k} &= \left((1+\varepsilon)\frac{c\log n}{\log\log n}\right)^{-(1+\varepsilon)\frac{\log n}{\log\log n}} \overset{!}{\leqslant} n^{-1-\varepsilon} \\
&\Leftarrow \log\left((1+\varepsilon)\frac{c\log n}{\log\log n}\right) \cdot \left((-1-\varepsilon)\frac{\log n}{\log\log n}\right) \leqslant (-1-\varepsilon)\log n \\
&\Leftarrow \log\log n \cdot (1-o(1)) \cdot \left((-1-\varepsilon)\frac{\log n}{\log\log n}\right) \leqslant (-1-\varepsilon)\log n.
\end{aligned}$$

Therefore, there is no larger clique than $(1+\varepsilon)\frac{\log n}{\log\log n}$ with probability $1 - n^{-\varepsilon}$. Setting $\varepsilon > 0$ to any constant yields a result with high probability. We may, however, obtain an even tighter result by choosing, for example, $\varepsilon = \frac{1}{\log\log n}$. Then, $n^{-\varepsilon} = o(1)$ and therefore the largest clique is of size at most $\frac{\log n}{\log\log n}(1+o(1))$ asymptotically almost surely, that is, with probability $1 - o(1)$.

To obtain a matching lower bound, observe that the analysis in Section 5 corresponds to a balls-into-bins experiment. There are $\frac{2\pi n}{a}$ circular sectors (bins), and each node (ball) is uniformly sampled in one of those. Since there are $n$ balls and $\Theta(n)$ bins, an application of [27, Theorem 1] yields the desired result. For reasons of completeness, we restate the relevant part of the theorem:

**Theorem 4 ([27])** *Let $M$ be the random variable that counts the maximum number of balls in any bin. If we throw $n$ balls independently and uniformly at random into $m = \Theta(n)$ bins, then*

$$\Pr\left[M \geqslant \frac{\log m}{\log\frac{m\log m}{n}}\left(1 + 0.99 \cdot \frac{\log\log\frac{m\log m}{n}}{\log\frac{m\log m}{n}}\right)\right] = 1 - o(1).$$

Observe that since $m = \Theta(n)$, we have $\log m = (1 \pm o(1))\log n$. Furthermore,

$$\log\frac{m\log m}{n} = \log(\Theta(\log m)) = (1 \pm o(1))\log\log n.$$

Plugging this into the theorem, we obtain that with probability $1-o(1)$, there is a clique of size at least

$$\frac{\log n}{\log\log n}(1 \pm o(1))\left(1 + 0.99\frac{\log\log\log n}{\log\log n}(1 \pm o(1))\right) \geqslant \frac{\log n}{\log\log n}(1 - o(1)).$$

This proves the lower bound for the maximum clique in Table 1.

## 7 Algorithms for Finding Cliques

So far, we showed bounds on the size of cliques in hyperbolic random graphs, but did not yet investigate on how to find them algorithmically. For the case $\alpha \geqslant 1$ we showed that there are only a few cliques in the graph. Therefore, a simple enumeration algorithm finds the largest clique in polynomial time. In fact, it is even possible to find the largest clique in linear time, as shown by the following theorem.

**Theorem 5** *The largest clique of a hyperbolic random graph with power-law exponent $\beta \geqslant 3$ can be found in expected time $\mathcal{O}(n)$.*

*Proof* Let $X$ be the number of neighbors of a node $v$ with radial coordinate smaller than $r_v$. By Lemma 3, this amounts in expectation to

$$\begin{aligned}
\mathbb{E}[X] &= n \cdot \mu(B_0(r_v) \cap B_{r_v}(R)) \\
&= \Theta(1) \cdot \exp\left(\frac{R}{2} + \frac{R - r_v - r_v}{2} - \alpha(R - r_v)\right) \\
&= \Theta(1) \cdot \exp\left((\alpha - 1)r_v - (\alpha - 1)R\right) \\
&= O(1),
\end{aligned}$$

if $r_v \geqslant \frac{R}{2}$, and

$$\mathbb{E}[X] = \Theta(1) \cdot e^{\frac{R}{2} - \alpha(R - r_v)} \leqslant \Theta(1) \cdot e^{\frac{R}{2} - \alpha(R - \frac{R}{2})} = \mathcal{O}(1)$$

otherwise. Thus, every node only has (in expectation) a constant number of neighbors with larger degree, and the largest clique can be found by exhaustively searching all node neighborhoods. For this, in each step, pick the node $v$ of smallest degree in the graph, and find the largest clique that $v$ is a part of. Then, delete $v$ and recurse. The technical analysis of this process is the same as in [11, Theorem 1], which reveals that the largest clique can be determined in $\mathcal{O}(n)$ expected time. □

This algorithm is the same as in the Chung-Lu model with $\beta \geqslant 3$ [11]. In this model, however, no algorithm is known for finding the largest clique in polynomial time when $2 < \beta < 3$. In contrast, we now show that given the geometric representation of a hyperbolic random graph sampled from the step model, a polynomial runtime is also achievable for the case $2 \leqslant \beta < 3$. The proof is similar to [7, Section 3] and works roughly as follows.

Consider two connected nodes $u, v$ with distance $\mathrm{d}(u, v) = d \leqslant R$. We denote by $S^{u,v}$ the set of all nodes that have distance at most $d$ to both nodes $u, v$. By definition, it holds $S^{u,v} \subset B_u(d) \cap B_v(d)$. Consider now the largest clique $C$ in the graph, and let $x, y \in C$ be the two nodes with maximum distance in $C$. It is then easy to see that $C \subseteq S^{x,y}$. Thus, it suffices to find the largest clique in $S^{u,v}$ for all connected node pairs $u, v \in V$.

In the following, we prove that the graph induced by the nodes $S^{u,v}$ is complement to a bipartite graph. Finding the largest clique then boils down

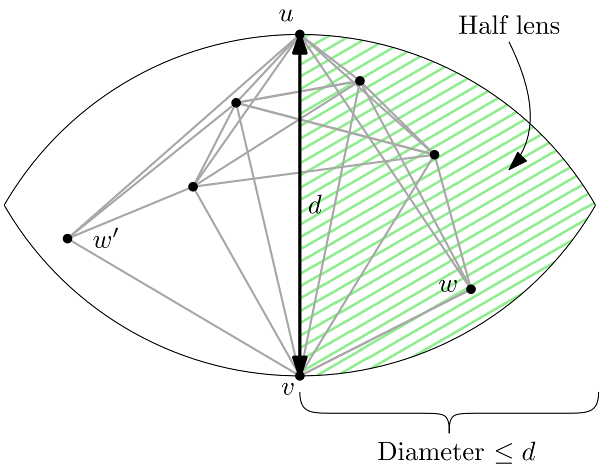



**Figure 2** Illustration of the clique algorithm. Every two nodes $u, v$ with distance $\mathrm{d}(u, v) = d \leqslant R$ define a lens $B_u(d) \cap B_v(d)$. A half lens has the geometric diameter $d \leqslant R$ and the nodes within thus form a clique. The union of the two half lenses is not necessarily a clique, since some nodes $w, w'$ might have a distance $> d$.

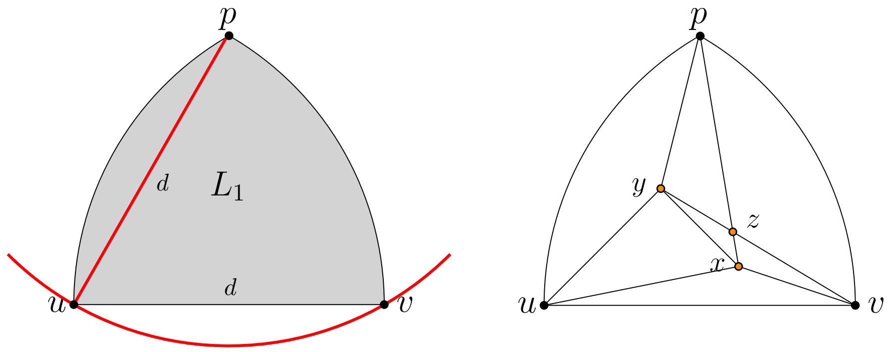
to finding the largest independent set in a bipartite graph, which is possible in polynomial time.

To show that $S^{u,v}$ is a complement of a bipartite graph, we partition the lens $B_u(d) \cap B_v(d)$ into two symmetrical areas (half lenses) and show that each half lens has geometric diameter at most $d$. Thus, both half lenses form complete subgraphs; while edges crossing the two half lenses may or may not be present. Figure 2 contains an illustration. Without loss of generality, we assume that $u = (0, 0)$ and $v = (d, 0)$. The statement generalizes to arbitrary positions by a simple coordinate transformation.

**Lemma 7** *Consider a lens of the form* $L = B_0(d) \cap B_d(d)$ *in the hyperbolic plane. Then, the half lens* $L_1 = \{(r, \varphi) \in L \mid 0 \leqslant \varphi < \pi\}$ *has geometric diameter at most* $d$.

*Proof* Let us denote with $p$ the point where the two discs of radius $d$ intersect. We first show that $p$ has distance at most $d$ to all points in the half lens. Figure 3a contains an illustration of this statement.

Consider a circle of radius $d$ around $p$, i.e. $B_p(d)$. Since $p$ has distance $d$ to both $u, v$, they lie on this circle. Since circles in hyperbolic space are convex, the geodesic between $u, v$ lies inside the circle as well. It is still necessary to show that the two circular arcs from $p$ to $u, v$ also lie within $B_p(d)$. To this end, we use the basic fact that distinct circles in the hyperbolic plane meet at most twice. Due to symmetry, it suffices to show that the arc from $p$ to $u$ is contained in $B_p(d)$. Since $u$ is on the boundary of $B_p(d)$, this leaves at most one more intersection. By symmetry of the lens, every such intersection in $L_1$ must also occur in $L_2 = \{(r, \varphi) \in L \mid \pi \leqslant \varphi < 2\pi\}$. Thus, the circular arc




(a) The point $p$ has distance at most $d$ to all points in $L_1$. In particular, the arcs from $p$ to $u, v$ are fully contained in $B_p(d)$.

(b) Two arbitrary points $x, y \in L_1$ have distance at most $d$, which can be deduced from this construction and the triangle inequality.

**Figure 3** Proof illustration for Lemma 7.

from $p$ to $u$ cannot intersect $B_p(d)$, as otherwise there would be at least three intersections.

Therefore, we know that $u, v$ and $p$ have at most distance $d$ to all nodes in the half lens $L_1$. Consider now two arbitrary points $x, y \in L_1$. We consider the three triangles obtained by using $\overline{xy}$ as base, and $u, v$ or $p$ as the third point (see Figure 3b). Since these three triangles use the same base, at least two of them intersect. W. l. o. g., we assume that $\overline{px}$ intersects $\overline{vy}$, the other cases are analog. We call the intersection point $z$. We observe now the following:

$$\overline{vy} = \overline{vz} + \overline{yz} \leqslant d \quad \text{since } v \text{ has distance at most } d \text{ to all points in } L_1\ , \tag{11}$$
$$\overline{px} = \overline{pz} + \overline{xz} \leqslant d \quad \text{since } p \text{ has distance at most } d \text{ to all points in } L_1, \tag{12}$$
$$\overline{vz} + \overline{pz} \geqslant d \quad \text{by triangle inequality, since } \overline{vp} = d. \tag{13}$$

Adding Equations (11) and (12) and subtracting Equation (13) yields

$$\overline{yz} + \overline{xz} \leqslant d.$$

Thus, by triangle inequality, we have $\overline{xy} \leqslant \overline{yz} + \overline{xz} \leqslant d$. □

Using this result, we may show that there exists a polynomial time algorithm for finding the maximum clique in a hyperbolic random graph drawn from the step model. Note that, similar to the euclidean case, this result holds with probability 1. That is, the proof is fully deterministic and does not use the distribution of nodes. Using Lemma 7, the proof is analogous to the euclidean case [7]. We reprove it here for completeness.

**Theorem 6** *Let $G$ be a graph sampled from the hyperbolic random graph in the step model. Given the geographic position of the nodes, the clique number $\omega(G)$ can be computed in worst-case $\mathcal{O}(m \cdot n^{2.5})$ time, where $m < n^2$ is a random variable describing the number of edges in $G$.*

*Proof* Let $C$ be the largest clique in $G$. Then, there must exist two nodes $u, v \in C$ such that $u, v$ have maximum geometric distance among all node pairs in $C$. Let $d := \mathrm{d}(u, v)$. Observe that $d \leqslant R$, as otherwise $u, v$ are not connected.

Consider now the induced subgraph $G[S^{u,v}]$ on all nodes $S^{u,v}$ that lie within the lens $B_u(d) \cap B_v(d)$. This subgraph can be found using the geometric representation, and, as shown in Lemma 7, $S^{u,v}$ may be partitioned in two sets $S_1, S_2$, such that both sets form a clique. Finding the largest clique in $G[S^{u,v}]$ is then equivalent to finding the largest independent set in the complement graph $\overline{G}[S^{u,v}]$. Since $S_1, S_2$ both form a clique in $G[S^{u,v}]$, they are independent in $\overline{G}[S^{u,v}]$ and therefore, $\overline{G}[S^{u,v}]$ is a bipartite graph.

Finding a maximum independent set is again equivalent to finding a minimum vertex cover. By Kőnig's Theorem (see, e. g., [8]), the size of the maximum matching in a bipartite graph is equal to the size of the minimum vertex cover. Thus, it suffices to compute the size $k$ of a maximum matching in $\overline{G}[S^{u,v}]$ and return $|S^{u,v}| - k$. Using e. g. the Hopcroft-Karp algorithm [14], this may be done in time $\mathcal{O}(|S^{u,v}|^{2.5})$.

Thus, an algorithm needs to simply check for each connected pair of nodes $u, v$ for the largest clique in $S^{u,v}$, which takes at most $\mathcal{O}(m \cdot n^{2.5})$. □

## 8 Conclusion

We present an analysis of the emergence of cliques in hyperbolic random graphs and suggest how to find them algorithmically. We found that the large clustering coefficient of these graphs strongly affects the clique number when $\beta > 3$. Previous models with independent edge probabilities predicted a clique number of 3 in this case, whereas the hyperbolic random graph contains a $\frac{\log n}{\log \log n}$ size clique.

Further, we show two algorithms for computing the largest clique in a hyperbolic random graph drawn from the step model. For graphs with power law exponent $\beta \geqslant 3$, the largest clique can be found in expected linear time. On the other hand, if the node coordinates are known, the largest clique in any hyperbolic random graph may be found in time $\mathcal{O}(m \cdot n^{2.5})$ with probability 1. It is, however, an open problem to find the largest clique when given only the graph structure—but not the geometric locations of the nodes. Moreover, it is open how our results extend to the binomial model, which allows long edges and short non-edges with a small probability.

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