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Erratum to: Reoptimization Time Analysis of Evolutionary Algorithms on Linear Functions Under Dynamic Uniform Constraints

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Abstract In the article *Reoptimization Time Analysis of Evolutionary Algorithms on Linear Functions Under Dynamic Uniform Constraints*, we claimed a worst-case runtime of $O(nD\log D)$ and $O(nD)$ for the Multi-Objective Evolutionary Algorithm and the Multi-Objective $(\mu+(\lambda,\lambda))$ Genetic Algorithm, respectively, on linear profit functions under dynamic uniform constraint. The technique used to prove these results contained an error. Instead, we correct this mistake and show a weaker bound of $O(nD^2)$ for both algorithms.

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The Multi-Objective Evolutionary Algorithm

In Theorem 9 of [3], we claimed a bound of $O(nD\log D)$ for the expected reoptimization time of the MOEA. Its proof was based on the notion of *candidate* solutions x for which there is an optimum x^* (assuming $B \le B^*$) such that $x_i = 1$ implies $x_i^* = 1$. In other words, an optimum can be created from a candidate by only flipping 0-bits. We used drift analysis on the potential function $G = B^* - h$, where h is the largest Hamming weight among all candidates in S. The analysis relied on this potential to be non-increasing during the reoptimization. This is not the case as illustrated by the following counterexample.

Suppose $n = 5$, $B = 2$, and $B^* = 4$; the weights of the profit function P shall observe the inequalities $w_1 \ge w_2 \ge w_3 \ge w_4 > w_5$ and $w_2 + w_5 > w_3 + w_4$, the population S may consist of the solutions $y = 11000$ and $z = 10110$. The unique optimum of P under constraint B^* is $x^* = 11110$, making both members of S candidates. Their largest Hamming weight h is 3. A mutation of z flipping 4 bits may result in the string $z' = 11001$, which replaces z due to the higher profit. However, z' is not a candidate anymore. The candidate of highest Hamming weight is now y and h decreases to 2. This increases the potential G in turn.

We now prove a weaker runtime bound for the MOEA with an alternative technique.

Theorem 9 *The reoptimization time of the* MOEA *on linear functions under dynamic uniform constraint is*

$$E[T] = O(nD^2).$$

Proof. We first present our analysis for the case of $B \le B^*$. For every integer $B \le u < B^*$, let $x^{(u)} = \arg\max_{|x|_1 = u} P(x)$ be a solution of maximum profit among all solutions with Hamming weight u. Assume $x^{(u)} \in S$ for some

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Profit Function	(1+1) **EA**	**MOEA**	**MOEA-S**	**MO** $(\mu+(\lambda,\lambda))$ **GA**	
OneMax	$\mathrm{O}\big(n\log\big(\frac{n-B}{n-B^*}\big)\big)$	$\mathrm{O}\big(nD\log\big(\frac{n-B}{n-B^*}\big)\big)$	$\mathrm{O}\big(n\log\big(\frac{n-B}{n-B^*}\big)\big)$	$\mathrm{O}\Big(\min\{\sqrt{nD^3},D^2\sqrt{\frac{n}{n-B^*}}\}\Big)$	if $B\le B^*$
	$\mathrm{O}\big(n\log\big(\frac{B}{B^*}\big)\big)$	$\mathrm{O}\big(nD\log\big(\frac{B}{B^*}\big)\big)$	$\mathrm{O}\big(n\log\big(\frac{B}{B^*}\big)\big)$	$\mathrm{O}\Big(\min\{\sqrt{nD^3},D^2\sqrt{\frac{n}{B^*}}\}\Big)$	if $B>B^*$
linear function	$\mathrm{O}\big(n^2\log(B^*w_{\max})\big)$	$\mathrm{O}\big(nD^2\big)$	$\mathrm{O}(n\log D)$	$\mathrm{O}\big(nD^2\big)$	

Table 1: Overview of Results. Upper bounds on the expected reoptimization times of the (1+1) EA, the Multi-Objective Evolutionary Algorithm (MOEA), its variant with single bit flip (MOEA-S) and the Multi-Objective $(\mu+(\lambda,\lambda))$ Genetic Algorithm (MO $(\mu+(\lambda,\lambda))$ GA) on linear functions of length-n bit strings under dynamic uniform constraint. B denotes the old and B^* the new cardinality bound, $D=|B^*-B|$ their difference. Runtimes of the form $\mathrm{O}(n\log(B/B^*))$ are to be read as $\mathrm{O}(n\log B)$ if $B^*=0$. For comparison, the (1+1) EA needs $\Omega(n)$ iterations to optimize OneMax under uniform constraint from scratch in the static setting (if B is not too close to 0, n or $n/2$) and $\Omega(n^2)$ for general linear profit functions [2].

$B\le u<B^*$, then choosing $x^{(u)}$ for mutation and flipping exactly the 0-bit of maximum weight and nothing else creates solution $x^{(u+1)}$ within an expected number of $en(D+1)$ generations. Observe that neither $x^{(u)}$ nor $x^{(u+1)}$ ever get deleted from S as they are maximal w.r.t. to the partial order $\succcurlyeq_{\text{MOEA}}$. Summing over all $D=|B^*-B|$ waiting times for the $x^{(u+1)}$, starting from the initial solution $x_{\text{orig}}=x^{(B)}\in S$, gives the claimed bound.

For the case of $B>B^*$, the solution $x^{(u)}=\arg\max_{|x|_1=u}P(x)$ of maximum profit among all solutions with Hamming weight u for every integer $B^*<u\le B$, and the probability choosing some $x^{(u)}$ in the population for mutation and flipping exactly the 1-bit of minimum weight and nothing else, are considered. Using the analysis similar to that given above yields the claimed bound. □

The Multi-Objective $(\mu+(\lambda,\lambda))$ Genetic Algorithm

The same error occurred in the analysis Multi-Objective $(\mu+(\lambda,\lambda))$ Genetic Algorithm (MO $(\mu+(\lambda,\lambda))$ GA) on general linear profit functions. In the following, we restate the corrected Subsection 5.2 of as a whole for completeness. Besides the modifications given below, the summary of the corresponding results in Table 1 of the article [3] must also be corrected, as the table given above. Both for the MOEA and the MO $(\mu+(\lambda,\lambda))$ GA on linear functions the bound is $\mathrm{O}(nD^2)$. Finally, the second and the last sentence of the last paragraph of the introduction should be omitted.

5.2 Linear Function with Dynamic Uniform Constraint

We now turn to linear functions under dynamic uniform constraints and start by lower bounding the probability of an improvement if the parameters are set in the right way.

As for OneMax, every bit in the considered bit string has the same weight (i.e., every bit string with Hamming weight $A+1$ is an optimal solution with Hamming weight $A+1$), Lemmata 13 and 14 thus study the probability of an iteration of the while-loop in the MO $(\mu+(\lambda,\lambda))$ GA to find an arbitrary solution with Hamming weight $A+1$ if $B\le B^*$ ($A-1$ if $B>B^*$), starting with a solution with Hamming weight A. Similarly, for the linear profit function, we study the probability of an iteration of the while-loop to find an optimal solution with Hamming weight $A+1$ if $B\le B^*$ ($A-1$ if $B>B^*$), starting with an optimal solution with Hamming weight A. Note again that the following two lemmata are considered based on the assumption that the optimal solution with Hamming weight A in the population is chosen for reproduction.

Lemma 16 *Choosing an optimal solution x of a linear profit function under dynamic uniform constraint with $|x|_1=A$ ($B\le A<B^*$) for reproduction, the probability of an iteration of the while-loop in the* MO $(\mu+(\lambda,\lambda))$ GA *to get an optimal solution y^* with $|y^*|_1=A+1$ from x is greater than C/λ, where $C>0$ is a constant, if $p=\lambda/n$, $c=1/\lambda$, and $\lambda=\lceil\sqrt{n}\,\rceil$.*

Proof. The reasoning runs in a similar way to that of Lemma 13. Let j be the index such that w_j is the largest element in $\{w_i\mid x_i=0,1\le i\le n\}$. As the analysis given in Lemma 13, to get the optimal solution y^* with

Hamming weight $|x|_1+1$, it is a necessary condition that the solution x' obtained by the mutation phase has $x'_j = 1$. Thus in the following, we first analyze the probability of the mutation phase to obtain such a solution.

For any mutant $x^{(i)} = \texttt{mutate}_\ell(x)$ that is obtained by the operator $\texttt{mutate}_\ell(x)$ on x, where $1 \leq i \leq \lambda$, the probability that $x_j^{(i)} = 0$ is at most

$$\prod_{t=0}^{\ell-1} \frac{n-1-t}{n}.$$

Thus the probability that $x_j^{(i)} = 0$ for all $1 \leq i \leq \lambda$ is

$$\left(\prod_{t=0}^{\ell-1} \frac{n-1-t}{n}\right)^{\lambda} \leq \left(\frac{n-1}{n}\right)^{\ell\lambda},$$

and the probability that there exists a mutant among $\{x^{(1)}, \ldots, x^{(\lambda)}\}$ whose j-th bit is 1, is at least

$$1-\left(\frac{n-1}{n}\right)^{\ell\lambda}.$$

Given that $|x_1| \leq B^*$, an offspring of x is called *valid* if some 0-bit was flipped by the mutation operator during its creation (if $|x_1| > B^*$, a 1-bit needs to be flipped). Now, assume that a mutant whose j-th bit is a 1 has been obtained by the operator $\texttt{mutate}_\ell(x)$ during the mutation phase. The solution is thus a valid offspring of x, but it may not be the unique valid offspring of x in $\{x^{(1)}, \ldots, x^{(\lambda)}\}$. Consequently, the solution is chosen as x' with probability $\Omega(1/\lambda)$, and the event $x'_j = 1$ happens at the end of the mutation phase with probability

$$\frac{1}{\lambda}\left(1-\left(\frac{n-1}{n}\right)^{\ell\lambda}\right).$$

Now we consider the event that the solution y' obtained by the crossover phase is an optimal solution with Hamming weight $|x|_1+1$, based on x and x', where bit x'_j is assumed to be 1. Note that the mutation operator $\texttt{mutate}_\ell(x)$ flips exactly ℓ bits in x, thus only the ℓ positions where x and x' are different need to be considered. For $y^{(i)} = \texttt{cross}_c(x, x')$, the probability that $y^{(i)}$ chooses x'_j as its j-th bit and the other $\ell-1$ bits in x as its bits in the corresponding positions is at least $c(1-c)^{\ell-1}$. Therefore, the crossover phase gets an optimal solution y^* having Hamming weight $|x|_1+1$ with probability at least

$$1-(1-c(1-c)^{\ell-1})^{\lambda}.$$

Summarizing above analysis, the probability to get an optimal solution y^* having Hamming weight $|x|_1+1$ within an iteration of the while-loop, is at least

$$\frac{1}{\lambda}\left(1-\left(\frac{n-1}{n}\right)^{\ell\lambda}\right)\left(1-\left(1-c\,(1-c)^{\ell-1}\right)^{\lambda}\right).$$

Now we give a lower bound of the above probability. W.l.o.g., assume that n is sufficiently large so that $\lambda = \lceil \sqrt{n}\, \rceil \geq 2$, and $n \geq 7\lambda/4$. Denote by L the random variable sampled by the binomial distribution $\mathrm{Bin}(n,p)$, and K the success to sample an optimal solution y^* with $|y^*|_1 = |x|_1+1$ within an iteration of the while-loop. We have

$$\Pr[K] \;\geq \sum_{\ell=\lceil\lambda/4\rceil}^{7\lambda/4} \Pr[K|L=\ell]\cdot\Pr[L=\ell],$$

where $\Pr[K|L=\ell] \geq \frac{1}{\lambda}(1-(\frac{n-1}{n})^{\ell\lambda})(1-(1-c(1-c)^{\ell-1})^{\lambda})$.

Since $c = 1/\lambda$, and that only the values $\ell \in [\lambda/4, 7\lambda/4]$ are considered, we can get

$$\left(1-c\,(1-c)^{\ell-1}\right)^{\lambda} \;\leq\; \left(1-\frac{1}{\lambda}\left(1-\frac{1}{\lambda}\right)^{\frac{7\lambda}{4}}\right)^{\lambda} \;\leq\; \left(1-\frac{1}{8\sqrt{2}\lambda}\right)^{\lambda} \;\leq\; e^{-\frac{1}{8\sqrt{2}}}.$$

The second inequality above always holds because of the fact that $(1-1/a)^a \geq 1/4$ for any $a \geq 2$. For the term $1-(\frac{n-1}{n})^{\ell\lambda}$, we have

$$1-\left(\frac{n-1}{n}\right)^{\ell\lambda} \geq 1-\left(\frac{n-1}{n}\right)^{\frac{\lambda^2}{4}} \geq 1-(1-\frac{1}{n})^{\frac{\sqrt{n}^2}{4}} \geq 1-(1-\frac{1}{n})^{\frac{n}{4}} \geq 1-e^{-\frac{1}{4}}.$$

Thus $\Pr[K|L=\ell]$ is greater than α/λ, where $\alpha=(1-e^{-\frac{1}{8\sqrt{2}}})(1-e^{-\frac{1}{4}})$ is a positive constant , and

$$\Pr[K] \geq \sum_{\ell=\lceil\lambda/4\rceil}^{7\lambda/4} \Pr[K|L=\ell]\cdot\Pr[L=\ell] \geq \frac{\alpha}{\lambda}\cdot\sum_{\ell=\lceil\lambda/4\rceil}^{7\lambda/4}\Pr[L=\ell].$$

If n is sufficiently large, Chebyshev's inequality [1] can be used to estimate the fraction of values that are more than a certain distance from the mean $np=\lambda$ of the binomial distribution $\mathrm{Bin}(n,p)$. Specifically, no more than $1/k^2$ of the binomial distribution's values can be more than k standard deviations away from the mean (the standard deviation of the binomial distribution $\mathrm{Bin}(n,p)$ is $\sqrt{np(1-p)}$). Consider the case $k=1.05$, thus more than 9% of values drawn from $\mathrm{Bin}(n,p)$ are within 1.05 standard deviations ($1.05\sqrt{np(1-p)}$) from the mean $np=\lambda$, and

$$\sum_{\ell=\lceil\lambda/4\rceil}^{7\lambda/4}\Pr[L=\ell] \geq \sum_{\ell=\lceil\lambda-1.05\sqrt{np(1-p)}\rceil}^{\lambda+1.05\sqrt{np(1-p)}}\Pr[L=\ell] \geq 0.09\ .$$

The first inequality holds because $1.05\sqrt{np(1-p)} = 1.05\sqrt{\lambda(1-\frac{\lambda}{n})} \leq 1.05\sqrt{\lambda} \leq \frac{3\lambda}{4}$ for any $\lambda \geq 2$.

Combining all above discussion, $\Pr[K]$ is greater than C/λ, where $C=0.09\cdot\alpha>0$. □

Now we consider the case of $B>B^*$, for which we have the lemma below.

Lemma 17 *Choosing an optimal solution x of a linear profit function under dynamic uniform constraint with $|x|_1=A$ ($B\geq A>B^*$) for reproduction, the probability of an iteration of the while-loop in the* MO $(\mu+(\lambda,\lambda))$ GA *to get an optimal solution y^* with $|y^*|_1=A-1$ from x is greater than C/λ, where $C>0$ is a constant, if $p=\lambda/n$, $c=1/\lambda$, and $\lambda=\lceil\sqrt{n}\,\rceil$.*

Proof. For the case of $B>B^*$, the smallest element w_j with index j in $\{w_i \mid x_i=1, 1\leq i\leq n\}$ is considered. As the analysis given in Lemma 14, to get an optimal solution y^* with Hamming weight $|x|_1-1$, it is a necessary condition that the solution x' obtained by the mutation phase has $x'_j=0$.

For any mutant $x^{(i)}=\mathtt{mutate}_\ell(x)$ that is obtained by the operator $\mathtt{mutate}_\ell(x)$ on x, where $1\leq i\leq\lambda$, the probability that $x^{(i)}_j=1$ is at most $\prod_{t=0}^{\ell-1}\frac{n-1-t}{n}$. Thus the probability that $x^{(i)}_j=1$ for all $1\leq i\leq\lambda$ is $\left(\prod_{t=0}^{\ell-1}\frac{n-1-t}{n}\right)^\lambda \leq \left(\frac{n-1}{n}\right)^{\ell\lambda}$, and the probability that there exists a mutant among $\{x^{(1)},\ldots,x^{(\lambda)}\}$ whose j-th bit is 0, is at least $1-\left(\frac{n-1}{n}\right)^{\ell\lambda}$. Using the reasoning same as that given in Lemma 16, the probability of the mutation phase choosing a mutant whose j-th bit is a 0 as x' is $\frac{1}{\lambda}\left(1-\left(\frac{n-1}{n}\right)^{\ell\lambda}\right)$, and the probability of the crossover phase finally getting an optimal solution y^* with Hamming weight $|x|_1-1$ is at least

$$\frac{1}{\lambda}\left(1-\left(\frac{n-1}{n}\right)^{\ell\lambda}\right)\left(1-\left(1-c\,(1-c)^{\ell-1}\right)^{\lambda}\right).$$

The remaining analysis about the lower bound of the above probability is the same as that given in Lemma 16, except that the variable K denotes the success to sample an optimal solution y^* with $|y^*|_1=|x|_1-1$ within an iteration of the while-loop. □

Finally, we show the upper bound of the expected reoptimization time for the MO $(\mu+(\lambda,\lambda))$ GA on linear functions with dynamic uniform constraints.

Theorem 18 *The expected reoptimization time of the* MO $(\mu+(\lambda,\lambda))$ GA *on a linear profit function under dynamic uniform constraint is* $O(nD^2)$*. Hereby, the parameters adapt to the solution x chosen for mutation: the mutation probability is $p=\lambda/n$, the crossover probability is $c=1/\lambda$, and $\lambda=\lceil\sqrt{n}\,\rceil$.*

Proof. The following proof for the expected reoptimization time runs in a similar way to that of Theorem 15. For $B \leq B^*$, let $x^{(A)}$ be the solution in the population S that has the maximum profit among all solutions with Hamming weight A, where $B \leq A = |x^{(A)}|_1 < B^*$. Combining Lemma 16 and the fact that probability of choosing $x^{(A)}$ for reproduction is $\Omega(1/D)$ (note that $x^{(A)}$ may not be the solution in S with the maximum Hamming weight, hence the size of S cannot be bounded by $A-B+1$, as that given in Theorem 15), the MO $(\mu+(\lambda,\lambda))$ GA takes an expected number of $\mathrm{O}(D\lambda^2) = O(nD)$ fitness evaluations to find an optimal solution with Hamming weight $A+1$. By considering all possible values of A ($B \leq A \leq B^*-1$) and summing over the waiting times from A to $A+1$, the MO $(\mu+(\lambda,\lambda))$ GA takes expected runtime $O(nD^2)$ to find an optimal solution with Hamming weight B^* starting with the initial population $\{x_{\text{orig}}\}$, where x_{orig} is an optimal solution with Hamming weight B.

For the case of $B > B^*$, the solution $x^{(A)}$ in the population S that has the maximum profit among all solutions with Hamming weight A where $B^* < A = |x^{(A)}|_1 \leq B$ is considered. Using the similar reasoning to that given above and Lemma 17 yields the same expected reoptimization runtime $\mathrm{O}(nD^2)$. □

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