



On the Spectrum of Some Signed Complete and Complete Bipartite Graphs

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Abstract. In this paper, we obtain the spectrum of signed complete and complete bipartite graphs whose negative edges form a matching. Moreover, we construct a family of signed complete graphs having symmetric spectrum.

1. Introduction

Signed graphs were introduced by Harary in [15] as a model for social networks. Then Zaslavsky in [25] applied an equivalence on the class of signed graphs. A detailed reference of materials on signed graphs is the dynamic survey of Zaslavsky [26].

Let G be a graph with the vertex set V and the edge set E . All graphs considered in this paper are undirected, finite, and simple (without loops or multiple edges). The *complement* of a graph G is a graph on the same vertices such that two distinct vertices are adjacent if and only if they are not adjacent in G . We denote it with G^c .

A *signed graph* is a graph in which every edge has been declared positive or negative. In fact, a signed graph Γ is a pair (G, σ) , where $G = (V, E)$ is a graph, called the underlying graph, and $\sigma : E \rightarrow \{-1, +1\}$ is the sign function or signature. Often, we write $\Gamma = (G, \sigma)$ to mean that the underlying graph is G . Note that if we consider a signed graph with all edges positive, we obtain an unsigned graph. Hence, signed graphs can be seen as a generalization of unsigned version.

In signed graphs we distinguish two kinds of cycles. A cycle C is called *balanced* if the number of negative edges in C is even. Otherwise, C is called *unbalanced*. A *balanced graph* is a signed graph which all cycles are balanced. A signed graph which is not balanced is an *unbalanced graph*.

Let v be a vertex of a signed graph Γ . The *switching* at v is changing the signature of each edge incident with v to the opposite one. Let $X \subseteq V$. Switching a vertex set X means reversing the signs of all edges between X and its complement. Switching a set X has the same effect as switching all the vertices in X , one after another.

Two signed graphs $\Gamma = (G, \sigma)$ and $\Gamma' = (G, \sigma')$ are *switching equivalent*, if there is a series of switchings that transforms Γ to Γ' . Switching equivalence is an equivalence relation on signatures of a fixed graph. An

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equivalence class is called a *switching class*. If Γ' is isomorphic to a switching of Γ , we say that Γ and Γ' are *switching isomorphic* and we write $\Gamma \approx \Gamma'$. Switching isomorphism is an equivalence relation on all signed graphs. Moreover, two signed graphs are non-isomorphic when they are not switching isomorphic.

By Harary's Balance Theorem in [15], we have the following corollary. Notice that $-\Gamma$ is obtained from the signed graph Γ by reversing the sign of all edges.

Corollary 1.1. *The signed graph $-\Gamma$ is balanced if and only if Γ is bipartite.*

We call Γ *antibalanced* if $-\Gamma$ is balanced; equivalently, if all even cycles are positive and all odd cycles are negative.

For a signed graph $\Gamma = (G, \sigma)$, the adjacency matrix $A = (a_{ij})$ is an $n \times n$ matrix in which $a_{ij} = \sigma(v_i v_j)$ if v_i and v_j are adjacent, and 0 if they are not. Thus A is a symmetric matrix with entries $0, \pm 1$ and zero diagonal, and conversely, any such matrix is the adjacency matrix of a signed graph.

Next, we recall some relations between two signed graphs which are switching isomorphic. For more information we refer reader to [21] and [25].

Lemma 1.2 ([25, Corollary 3.3]). *A signed graph Γ is balanced if and only if it switches to a signed graph in which all edges are positive, and it is antibalanced if and only if it switches to a signed graph in which all edges are negative.*

Theorem 1.3 ([25, Proposition 3.2]). *Two signed graphs with the same underlying graph are switching equivalent if and only if they have the same set of positive cycles.*

The spectrum of Γ is the list of eigenvalues of its adjacency matrix with their multiplicities. We denote it by $\text{Spec}(\Gamma)$ and say that the spectrum of Γ is symmetric if for each eigenvalue $\lambda \in \text{Spec}(\Gamma)$, $-\lambda \in \text{Spec}(\Gamma)$.

Theorem 1.4. *Switching a signed graph does not change its spectrum. Also, the switching isomorphic graphs have the same spectrum.*

Theorem 1.5 (B.D. Acharya [1]). *The signed graph $\Gamma = (G, \sigma)$ is balanced if and only if Γ and G have the same eigenvalues.*

In this paper, we give some results about the spectrum of signed graphs. In addition, we are going to determine the eigenvalues of signed complete and complete bipartite graphs when their negative edges form a matching. Also, we say that a signed graph Γ is *uniquely determined by the spectrum* if for any signed graph Γ' with the same spectrum, Γ and Γ' are switching isomorphic.

2. Signed Complete Graphs with Symmetric Spectrum

In this section, we recall some lemmas and theorems from [19] and [23], and then we present several results about the spectrum of signed graphs. Also, we consider signed complete graphs having the symmetric spectrums.

If b and u denote the number of balanced and unbalanced triangles in a signed graph $\Gamma = (G, \sigma)$, then $\hat{b} = \frac{b}{b+u}$ and $\hat{u} = \frac{u}{b+u}$ are the fractions of balanced and unbalanced triangles in Γ , respectively. So, $\hat{b} + \hat{u} = 1$.

Remark 2.1. If Γ and Γ' are signed complete graphs, then the set of unbalanced triangles in Γ and Γ' are the same if and only if Γ and Γ' are switching equivalent, see [25, Proposition 7E.1].

Theorem 2.2 ([23, Theorem 2]). *Let Γ be a signed graph (not necessarily complete) with the adjacency matrix A , and suppose that the adjacency matrix of its underlying graph is U . If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A , and μ_1, \dots, μ_n are the eigenvalues of U , then the fraction of balanced triangles in Γ can be expressed as follows:*

$$\hat{b} = \frac{1}{2} \left(1 + \frac{\sum_{i=1}^n \lambda_i^3}{\sum_{i=1}^n \mu_i^3} \right).$$

By [19], there are 7 non-isomorphic signed complete graphs of order 5, three with an odd number of negative edges and four with an even number of negative edges. Also, there are 16 non-isomorphic signed complete graphs of order 6. By these facts we obtain the following remark.

Remark 2.3. By computer computation, one can see that the spectrum of these non-isomorphic graphs with underlying graph K_5 (similarly K_6) are distinct. That is, if the spectrum of two signed complete graphs of order 5 (or 6) are equal, then those graphs are switching isomorphic. In other words, we can partition signed complete graphs of order 5 (or 6) into several classes such that the elements of each class have the same spectrum.

Corollary 2.4. Let Γ be a signed graph. If the spectrum of Γ is symmetric, then the number of balanced triangles is equal to the number of unbalanced triangles in Γ .

Proof. Apply Theorem 2.2. \square

Note that signed complete graph $\Gamma = (K_n, \sigma)$ with $n = 4t + 3$ for some positive integer t , and for any signature σ cannot have a symmetric spectrum because the number of triangles in K_n for $n = 4t + 3$ is odd. See Corollary 2.5.

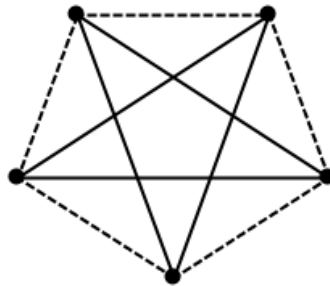
Corollary 2.5. The spectrum of signed complete graphs having odd number of triangles cannot be symmetric.

Proof. It is easy to see that the assertion holds by Corollary 2.4. \square

We have a well-known theorem in unsigned case that the spectrum of a graph is symmetric if and only if it is bipartite [8, Theorem 3.2.3]. In signed case, the spectrum of any signed bipartite graph is symmetric, but there are some examples with symmetric spectrum which are not bipartite. Consider the signed complete graph Γ given in Fig. 2, which dashed lines indicate negative edges, indeed Γ is not bipartite, but $\text{Spec}(\Gamma)$ is symmetric. Now, we propose the following question.

Question 2.6. Which signed complete graphs have the symmetric spectrum?

Notice that if Γ is a signed graph which is switching isomorphic to $-\Gamma$. Then the spectrum of Γ is symmetric.



$$\text{Spec}(\Gamma) = \begin{pmatrix} -\sqrt{5} & 0 & \sqrt{5} \\ 2 & 1 & 2 \end{pmatrix}.$$

Figure 1:

Now, we are going to construct two types of signed complete graphs with symmetric spectrum. Partition the vertex set of K_n into two sets A and B of equal cardinality $|A| = |B|$. If $|A|$ is even we can construct

two types of signed graphs with symmetric spectrum. Choose a positive (resp. negative) sign for all edges connecting vertices in A (resp. B). For each fixed vertex v in A , there are $\frac{n}{2}$ edges connecting v and vertices in B , we negatively sign half of them. Now, if we switch all vertices in A we obtain a graph isomorphic to $-\Gamma$. A second type of graph with symmetric spectrum is obtaining by assigning a positive sign to all edges in the cut-set (A, B) . Such second type of graph with symmetric spectrum can be obtained when $|A|$ is odd as well.

By [8, Corollary 2.3.3], one can obtain the coefficients of the adjacency characteristic polynomial for unsigned graphs. Note that *elementary figures* are the graphs K_2 and C_n (i.e. the complete graph of order 2 and the cycle of order n); a *basic figure* is the disjoint union of elementary figures.

Theorem 2.7 ([4, Theorem 2.3]). *Let $\Gamma = (G, \sigma)$ and $P_\Gamma(x) = x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$ be a signed graph and its adjacency characteristic polynomial, respectively. Then*

$$a_i = \sum_{B \in \mathcal{B}_i} (-1)^{p(B)} 2^{|c(B)|} \sigma(B),$$

where \mathcal{B}_i is the set of basic figures on i vertices in G , $p(B)$ is the number of components of B , $c(B)$ the set of cycles in B , and $\sigma(B) = \prod_{C \in c(B)} \sigma(C)$.

By Theorem 2.7, if we denote the number of balanced and unbalanced ℓ -cycles with C_ℓ^+ and C_ℓ^- , respectively, then it is easy to see that $a_3 = 2(C_3^- - C_3^+)$.

In the following remark, we present an equivalent condition for signed graphs with symmetric spectrum.

Remark 2.8. Let Γ be a signed graph of order n and A be the adjacency matrix of Γ . Then the following conditions are equivalent.

- (i) The spectrum of Γ is symmetric.
- (ii) If $P_\Gamma(x) = x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$ is the characteristic polynomial of A , then $a_{2k+1} = 0$, for $k = 0, 1, \dots$

We recall that the previous theorem also holds for unsigned graphs [8, Theorem 3.2.5].

3. The spectrum of signed complete graphs

By [6], the *Seidel adjacency matrix* of a graph G with the adjacency matrix A is the matrix S defined by

$$S_{uv} = \begin{cases} 0 & \text{if } u = v \\ -1 & \text{if } u \sim v \\ 1 & \text{if } u \not\sim v \end{cases}$$

so that $S = J - I - 2A$. The Seidel adjacency spectrum of a graph is the spectrum of its Seidel adjacency matrix.

Remark 3.1. For a k -regular graph of order n with eigenvalue k and other eigenvalues θ , the Seidel spectrum consists of $n - 1 - 2k$ and the values $-1 - 2\theta$, see [6]. For more information about the Seidel matrices refer to [14, 22, 24]. Now, if G is a graph of order n , then the Seidel matrix of G is the adjacency matrix of a signed complete graph Γ of order n which the edges of G form all negative edges in Γ .

Let negative edges in signed complete graph $\Gamma = (K_n, \sigma)$ form a k -regular graph of order n . In the following theorem we state some necessary and sufficient conditions so that Γ has symmetric spectrum. Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of Γ . Suppose that a, b and p are real numbers such that $a + b = p$. The two numbers a and b are symmetric with respect $p/2$. When for each i there exists some j such that $\lambda_i + \lambda_j = -1$, we say that the spectrum is symmetric with respect to $x = -1/2$.

Theorem 3.2. Let $\Gamma = (K_n, \sigma)$. Assume that negative edges in Γ form a k -regular graph H of order n , where $k, \mu_1, \dots, \mu_{n-1}$ are all eigenvalues of H , when all edges of H are considered positive. The spectrum of Γ is symmetric if and only if one of the following conditions hold:

- (i) $n = 4t + 1, k = 2t$ for some positive integer t , and μ_1, \dots, μ_{n-1} are symmetric with respect to the axis $x = -1/2$.
- (ii) n is even, and there exists some $j, 1 \leq j \leq n - 1$ such that $\mu_j = \frac{n-2}{2} - k$. The remaining μ_i are symmetric with respect to the axis $x = -1/2$.

Proof. We know that for a k -regular graph of order n with eigenvalue k and other eigenvalues θ , the Seidel spectrum consists of $n - 1 - 2k$ and the values $-1 - 2\theta$. So, if the spectrum of Γ is symmetric, then there are two cases:

Case 1. If $n - 1 - 2k = 0$, then $k = \frac{n-1}{2}$. Since k is a positive integer, n should be odd. By Corollary 2.5, $n = 4t + 1$ for some positive integer t . Also, other eigenvalues of $\Gamma, -1 - 2\mu_1, \dots, -1 - 2\mu_{n-1}$ should be symmetric with respect to $x = -1/2$.

Case 2. If $n - 1 - 2k \neq 0$, then there exists some $j, 1 \leq j \leq n - 1$ such that $\mu_j = \frac{n-2}{2} - k$, and $-1 - \mu_j$ is an eigenvalue. Moreover, the remaining μ_i are symmetric with respect to the axis $x = -1/2$. \square

Remark 3.3. One can check that if $\Gamma = (K_8, \sigma)$ which negative edges are two distinct P_4 , then the spectrum of Γ is symmetric. It is easy to see that the spectrum of $P_4 \cup P_4$ is symmetric with respect to the axis $x = -1/2$, but $P_4 \cup P_4$ is not regular.

In the following theorem we present a big family of signed complete graphs having symmetric spectrum.

Theorem 3.4. Let n be an even positive integer and V_1 and V_2 be two disjoint sets of size $\frac{n}{2}$. Let G be an arbitrary graph with the vertex set V_1 . Construct the complement of G , that is G^c , with the vertex set V_2 . Assume that $\Gamma = (K_n, \sigma)$ is a signed complete graph in which $E(G) \cup E(G^c)$ is the set of negative edges. Then the spectrum of Γ is symmetric.

Proof. It is sufficient to switch at the elements of V_1 . One can see that $\Gamma \simeq -\Gamma$. Hence, the spectrum of Γ is symmetric. \square

Let I_n and J_n for some positive integer n , be the $n \times n$ matrices identity and all 1's, respectively.

We consider a matching of negative edges M_t of size t for some $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$, in K_n with the vertex set $\{v_1, \dots, v_n\}$, as follows,

$$M_t = \{v_1v_2, v_3v_4, \dots, v_{2t-1}v_{2t}\}.$$

Theorem 3.5. Let Γ_t be a signed complete graph of order n whose negative edges form a matching of size t .

- (i) $t < \lfloor n/2 \rfloor$, then

$$\text{Spec}(\Gamma_t) = \left(\begin{array}{ccccc} -3 & a_t & -1 & 1 & b_t \\ t-1 & 1 & n-(2t+1) & t & 1 \end{array} \right),$$

where a_t and b_t are as follows,

$$a_t = \frac{n-4 - \sqrt{n^2 + 4s}}{2}, \quad b_t = \frac{n-4 + \sqrt{n^2 + 4s}}{2}, \quad \text{s.t. } s = n - 4t + 1.$$

- (ii) If n is even and $t = \frac{n}{2}$, then $\Gamma_{\frac{n}{2}}$ has the following spectrum:

$$\text{Spec}(\Gamma_{\frac{n}{2}}) = \left(\begin{array}{ccc} -3 & 1 & n-3 \\ \frac{n}{2}-1 & \frac{n}{2} & 1 \end{array} \right).$$

Proof. (i) We would like to give the independent eigenvectors corresponding to each eigenvalue. The eigenspaces associated with the eigenvalues a_t and b_t are respectively generated by

$$\begin{bmatrix} \frac{-(s+1) - \sqrt{n^2 + 4s}}{4t} \\ \vdots \\ \frac{-(s+1) - \sqrt{n^2 + 4s}}{4t} \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{-(s+1) + \sqrt{n^2 + 4s}}{4t} \\ \vdots \\ \frac{-(s+1) + \sqrt{n^2 + 4s}}{4t} \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

where the number of

$$\frac{-(s+1) - \sqrt{n^2 + 4s}}{4t} \text{ and } \frac{-(s+1) + \sqrt{n^2 + 4s}}{4t},$$

in the above vectors is $2t$. Also, the eigenvalue 1 has t independent eigenvectors as follows:

$$\text{For } 1 \leq i \leq t, \text{ define } X_i = [x_1, \dots, x_n]^T, x_{2i-1} = -1, x_{2i} = 1, \text{ and otherwise } x_j = 0.$$

Furthermore, the $n - (2t + 1)$ independent eigenvectors of the eigenvalue -1 are as follows:

$$\text{For } 1 \leq i \leq n - (2t + 1), \text{ define } Y_i = [y_1, \dots, y_n]^T, y_{2t+1} = -1, y_{2t+i+1} = 1,$$

and otherwise $y_j = 0$. Note that the first $2t$ entries of each vector are 0. Finally, we have $t - 1$ independent eigenvectors corresponding to the eigenvalue -3 as follows:

$$\text{For } 1 \leq i \leq t - 1, \text{ define } Z_i = [z_1, \dots, z_n]^T, z_1 = z_2 = -1, z_{2i+1} = z_{2(i+1)} = 1,$$

and otherwise $z_j = 0$.

(ii) Without loss of generality, one may write the adjacency matrix of $\Gamma_{\frac{n}{2}}$ as follows,

$$A = \begin{pmatrix} 0 & -1 & 1 & 1 & 1 & 1 & 1 & \dots \\ -1 & 0 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 1 & 0 & -1 & 1 & 1 & 1 & \dots \\ 1 & 1 & -1 & 0 & 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & 1 & 0 & -1 & 1 & \dots \\ 1 & 1 & 1 & 1 & -1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

So, the following holds:

$$A = (J_{n/2} - I_{n/2}) \otimes J_2 + I_{n/2} \otimes B,$$

where

$$B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

For the definition of the Kronecker product \otimes , see [8, Definition 2.5.2]. Since $J_2 B = B J_2$, it is sufficient to consider the sum of eigenvalues in $(J_{n/2} - I_{n/2}) \otimes J_2$ and $I_{n/2} \otimes B$ with the same eigenvectors. Hence, it is easy to see that the assertion holds. \square

4. The Spectrum of Signed Complete Bipartite Graphs

In this section, we determine the spectrum of signed complete bipartite graphs $K_{n,n}$, whose negative edges form a matching. Consider $K_{n,n}$ with two parts $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$ as a partition of its vertex set. Without loss of generality, we choose a matching of negative edges M_t of size t , for some $1 \leq t \leq n$ as follows,

$$M_t = \{u_1v_1, \dots, u_tv_t\}.$$

Theorem 4.1. Let Γ_t be a signed complete bipartite graph $K_{n,n}$ such that its negative edges form a matching of size t .

(i) If $t < n$, then the spectrum is

$$\text{Spec}(\Gamma_t) = \begin{pmatrix} -a_t & -2 & -b_t & 0 & b_t & 2 & a_t \\ 1 & t-1 & 1 & 2n-(2t+2) & 1 & t-1 & 1 \end{pmatrix},$$

where a_t and b_t are as follows:

$$a_t = \frac{n-2 + \sqrt{n^2 + 4s}}{2}, \quad b_t = \frac{-(n-2) + \sqrt{n^2 + 4s}}{2}, \quad \text{s.t. } s = n - 2t + 1.$$

(ii) If $t = n$, i.e. its negative edges form a perfect matching, then Γ_n has the following spectrum:

$$\text{Spec}(\Gamma_n) = \begin{pmatrix} -(n-2) & -2 & 2 & n-2 \\ 1 & n-1 & n-1 & 1 \end{pmatrix}.$$

Proof. (i) In the following, we determine all independent eigenvectors corresponding to all eigenvalues. The independent eigenvectors associated with eigenvalue 0 are of two types as follows:

$$\text{For } 2 \leq i \leq n-t, \text{ define } X_i = [x_1, \dots, x_{2n}]^T, \quad x_{t+1} = -1, \quad x_{t+i} = 1,$$

and otherwise $x_j = 0$, and $Y_i = [y_1, \dots, y_{2n}]^T, \quad y_{n+t+1} = -1, \quad y_{n+t+i} = 1,$

and otherwise $y_j = 0$. Also, we can determine the independent eigenvectors of 2 as follows:

$$\text{For } 2 \leq i \leq t, \text{ define } Z_i = [z_1, \dots, z_{2n}]^T, \quad z_1 = 1, z_{n+1} = -1, z_i = -1, z_{n+i} = 1,$$

and otherwise $z_j = 0$. The independent eigenvectors corresponding to the eigenvalue -2 are as follows:

$$\text{For } 2 \leq i \leq t, \text{ define } U_i = [u_1, \dots, u_{2n}]^T, \quad u_1 = 1, u_{n+1} = 1, u_i = -1, u_{n+i} = -1,$$

and otherwise $u_j = 0$. Finally, the following vectors are the independent eigenvectors of $a_t, b_t, -a_t$ and $-b_t$, respectively.

$$\begin{aligned} & \left[\frac{-(s+1) + \sqrt{n^2 + 4s}}{2t}, 1, \dots, 1, \frac{-(s+1) + \sqrt{n^2 + 4s}}{2t}, 1, \dots, 1 \right]^T, \\ & \left[\frac{(s+1) + \sqrt{n^2 + 4s}}{2t}, -1, \dots, -1, \frac{-(s+1) - \sqrt{n^2 + 4s}}{2t}, 1, \dots, 1 \right]^T, \\ & \left[\frac{(s+1) - \sqrt{n^2 + 4s}}{2t}, -1, \dots, -1, \frac{-(s+1) + \sqrt{n^2 + 4s}}{2t}, 1, \dots, 1 \right]^T, \\ & \left[\frac{-(s+1) - \sqrt{n^2 + 4s}}{2t}, 1, \dots, 1, \frac{-(s+1) - \sqrt{n^2 + 4s}}{2t}, 1, \dots, 1 \right]^T, \end{aligned}$$

where the $(n + 1)$ -entry is

$$\frac{-(s + 1) + \sqrt{n^2 + 4s}}{2t} \text{ or } \frac{-(s + 1) - \sqrt{n^2 + 4s}}{2t}.$$

(ii) Without loss of generality, the adjacency matrix of Γ can be written as follows,

$$A = \begin{pmatrix} 0 & J_n - 2I_n \\ J_n - 2I_n & 0 \end{pmatrix}.$$

Thus, we have

$$A^2 = \begin{pmatrix} (J_n - 2I_n)^2 & 0 \\ 0 & (J_n - 2I_n)^2 \end{pmatrix},$$

and $(J_n - 2I_n)^2 = (n - 4)J_n + 4I_n$ and the spectrum of A^2 can be easily computed,

$$\text{Spec}(A^2) = \begin{pmatrix} 4 & (n - 2)^2 \\ 2n - 2 & 2 \end{pmatrix}.$$

Now, using Remark 2.8, the proof is complete. \square

5. Weighted Directed Graphs versus Signed Graphs

One can see signed graphs as a special type of *weighted directed graphs*. The notion of weighted directed graph was introduced by Bapat et al., see [2]. A weighted directed graph is a directed graph with a simple underlying undirected graph and edges having complex weights of unit modulus.

Let G be a graph on the vertex set $\{v_1, \dots, v_n\}$. We denote a directed edge from v_i to v_j by (v_i, v_j) .

Definition 5.1. Let G be a weighted directed graph. Let us denote the weight of the edge (v_i, v_j) by w_{ij} , and \bar{w}_{ji} be the complex conjugate of w_{ij} . The adjacency matrix $A = [a_{ij}]$ of G is defined as follows,

$$a_{ij} = \begin{cases} w_{ij} & \text{if } (v_i, v_j) \in E(G), \\ \bar{w}_{ji} & \text{if } (v_j, v_i) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

A *mixed graph* is a graph with some directed and some undirected edges, (see [10–13]). Let G be a mixed graph. We write $v_i v_j \in E(G)$ if there exists an undirected edge between the vertices v_i and v_j . The adjacency matrix $A = [a_{ij}]$ of G is the matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G), \\ -1 & \text{if } (v_i, v_j) \in E(G) \text{ or } (v_j, v_i) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Now, let G be a directed graph with edges having colours red, blue or green. We assign weight 1 to each red edge, weight -1 to each blue edge, and weight i to each green edge. We call this graph a *3-colored digraph*. This is a very small subclass of weighted directed graphs, and mixed graphs are properly contained in it. The notion of a 3-colored digraph, introduced by Bapat et al. in [2], generalizes the one of mixed graph. Signed graphs might be seen as weighted graphs with edge weights equal to ± 1 , however the theory of signed graphs is different from that of weighted graphs in view of the cycle sign. One can see that a signed graph $\Gamma = (G, \sigma)$ might be considered as a weighted directed graph (or a mixed digraph) if we replace the

negative edges of Γ with an arbitrary directed edge of weight -1 , and assign weight $+1$ to the positive edges of Γ .

Note that some study on the Laplacian spectrum of a mixed graph has been done by Zhang and Li in [27], Zhang and Luo in [28] and Fan in [9–12]. Also one can find a lot of nice results about the Laplacian spectrum of signed graphs, (see for example [3–5]). However, about the spectrum of the adjacency matrix of signed graphs there are a few papers. Bapat, Kalita and Pati in [2, 16–18] proved interesting results about weighted directed graphs, the spectrum and characteristic polynomial of the adjacency matrix of 3-colored digraphs. Hence, this motivates us to study more about the spectrum of adjacency matrix of signed graphs.

Let G be a 3-colored digraph with underlying graph K_n for even $n \geq 4$. Consider a perfect matching M of green edges in G with the vertex set $\{v_1, \dots, v_n\}$, as follows,

$$M = \{(v_1, v_2), (v_3, v_4), \dots, (v_{n-1}, v_n)\}.$$

Theorem 5.2. *Let G be a 3-colored digraph with underlying graph K_n for even $n \geq 4$. If G has only red and green edges whose green edges form a perfect matching, then G has the following spectrum,*

$$\text{Spec}(G) = \left(\begin{array}{cccc} -\sqrt{2}-1 & a & \sqrt{2}-1 & b \\ \frac{n}{2}-1 & 1 & \frac{n}{2}-1 & 1 \end{array} \right),$$

where $a = \frac{n-2-\sqrt{n^2-4n+8}}{2}$ and $b = \frac{n-2+\sqrt{n^2-4n+8}}{2}$.

Proof. One may consider the adjacency matrix of G as follows,

$$A = \begin{pmatrix} 0 & i & 1 & 1 & 1 & 1 & 1 & \dots \\ -i & 0 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 1 & 0 & i & 1 & 1 & 1 & \dots \\ 1 & 1 & -i & 0 & 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & 1 & 0 & i & 1 & \dots \\ 1 & 1 & 1 & 1 & -i & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

So, the following holds:

$$A = (J_{n/2} - I_{n/2}) \otimes J_2 + I_{n/2} \otimes D,$$

where

$$D = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Since two matrices $(J_{n/2} - I_{n/2}) \otimes J_2$ and $I_{n/2} \otimes D$ commute, it is not hard to see that every eigenvalue of A is the sum of one eigenvalue of $(J_{n/2} - I_{n/2}) \otimes J_2$ and one eigenvalue of $I_{n/2} \otimes D$ in a suitable order. Hence, the assertion is proved. \square

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