ON SIGN-SYMmetric SIGNED GRAPHS

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Abstract. A signed graph is said to be sign-symmetric if it is switching isomorphic to its negation. Bipartite signed graphs are trivially sign-symmetric. We give new constructions of non-bipartite sign-symmetric signed graphs. Sign-symmetric signed graphs have a symmetric spectrum but not the other way around. We present constructions of signed graphs with symmetric spectra which are not sign-symmetric. This, in particular answers a problem posed by Belardo, Cioabă, Koolen, and Wang (2018).

1. Introduction

Let $G$ be a graph with vertex set $V$ and edge set $E$. All graphs considered in this paper are undirected, finite, and simple (without loops or multiple edges).

A signed graph is a graph in which every edge has been declared positive or negative. In fact, a signed graph $\Gamma$ is a pair $(G,\sigma)$, where $G = (V,E)$ is a graph, called the underlying graph, and $\sigma : E \to \{-1,+1\}$ is the sign function or signature. Often, we write $\Gamma = (G,\sigma)$ to mean that the underlying graph is $G$. The signed graph $(G,-\sigma) = -\Gamma$ is called the negation of $\Gamma$. Note that if we consider a signed graph with all edges positive, we obtain an unsigned graph.

Let $v$ be a vertex of a signed graph $\Gamma$. Switching at $v$ is changing the signature of each edge incident with $v$ to the opposite one. Let $X \subseteq V$. Switching a vertex set $X$ means reversing the signs of all edges between $X$ and its complement. Switching a set $X$ has the same effect as switching all the vertices in $X$, one after another.

Two signed graphs $\Gamma = (G,\sigma)$ and $\Gamma' = (G,\sigma')$ are said to be switching equivalent if there is a series of switching that transforms $\Gamma$ into $\Gamma'$. If $\Gamma'$ is isomorphic to a switching of $\Gamma$, we say that $\Gamma$ and $\Gamma'$ are switching isomorphic and we write $\Gamma \simeq \Gamma'$.

The signed graph $-\Gamma$ is obtained from $\Gamma$ by reversing the sign of all edges. A signed graph $\Gamma = (G,\sigma)$ is said to be sign-symmetric if $\Gamma$ is switching isomorphic to $(G,-\sigma)$, that is: $\Gamma \simeq -\Gamma$.

For a signed graph $\Gamma = (G,\sigma)$, the adjacency matrix $A = A(\Gamma) = (a_{ij})$ is an $n \times n$ matrix in which $a_{ij} = \sigma(v_i,v_j)$ if $v_i$ and $v_j$ are adjacent, and 0 if they are not. Thus $A$ is a symmetric matrix with entries $0,\pm 1$ and zero diagonal, and conversely, any such matrix is the adjacency matrix of a signed graph. The spectrum of $\Gamma$ is the list of eigenvalues of its adjacency matrix with their multiplicities. We say that $\Gamma$ has a

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symmetric spectrum (with respect to the origin) if for each eigenvalue $\lambda$ of $\Gamma$, $-\lambda$ is also an eigenvalue of $\Gamma$ with the same multiplicity.

Recall that (see [4]), the Seidel adjacency matrix of a graph $G$ with the adjacency matrix $A$ is the matrix $S$ defined by

$$S_{uv} = \begin{cases} 
0 & \text{if } u = v \\
-1 & \text{if } u \sim v \\
1 & \text{if } u \not\sim v 
\end{cases}$$

so that $S = J - I - 2A$. The Seidel adjacency spectrum of a graph is the spectrum of its Seidel adjacency matrix. If $G$ is a graph of order $n$, then the Seidel matrix of $G$ is the adjacency matrix of a signed complete graph $\Gamma$ of order $n$ where the edges of $G$ are precisely the negative edges of $\Gamma$.

**Proposition 1.1.** Suppose $S$ is a Seidel adjacency matrix of order $n$. If $n$ is even, then $S$ is nonsingular, and if $n$ is odd, $\text{rank}(S) \geq n - 1$. In particular, if $n$ is odd, and $S$ has a symmetric spectrum, then $S$ has an eigenvalue 0 of multiplicity 1.

**Proof.** We have $\det(S) \equiv \det(I - J)(\text{mod } 2)$, and $\det(I - J) = 1 - n$. Hence, if $n$ is even, $\det(S)$ is odd. So, $S$ is nonsingular. Now, if $n$ is odd, any principal submatrix of order $n - 1$ is nonsingular. Therefore, $\text{rank}(S) \geq n - 1$. \qed

The goal of this paper is to study sign-symmetric signed graphs as well as signed graphs with symmetric spectra. It is known that bipartite signed graphs are sign-symmetric. We give new constructions of non-bipartite sign-symmetric graphs. It is obvious that sign-symmetric graphs have a symmetric spectrum but not the other way around (see Remark 4.1 below). We present constructions of graphs with symmetric spectra which are not sign-symmetric. This, in particular answers a problem posed in [2].

2. Constructions of sign-symmetric graphs

We note that the property that two signed graphs $\Gamma$ and $\Gamma'$ are switching isomorphic is equivalent to the existence of a ‘signed’ permutation matrix $P$ such that $PA(\Gamma)P^{-1} = A(\Gamma')$. If $\Gamma$ is a bipartite signed graph, then we may write its adjacency matrix as

$$A = \begin{bmatrix} O & B \\ B^\top & O \end{bmatrix}.$$ 

It follows that $PAP^{-1} = -A$ for

$$P = \begin{bmatrix} -I & O \\ O & I \end{bmatrix},$$

which means that bipartite graphs are ‘trivially’ sign-symmetric. So it is natural to look for non-bipartite sign-symmetric graphs. The first construction was given in [1] as follows.
Theorem 2.1. Let \( n \) be an even positive integer and \( V_1 \) and \( V_2 \) be two disjoint sets of size \( n/2 \). Let \( G \) be an arbitrary graph with the vertex set \( V_1 \). Construct the complement of \( G \), that is \( G^c \), with the vertex set \( V_2 \). Assume that \( \Gamma = (K_n, \sigma) \) is a signed complete graph in which \( E(G) \cup E(G^c) \) is the set of negative edges. Then the spectrum of \( \Gamma \) is sign-symmetric.

Theorem 2.1 says that for an even positive integer \( n \), let \( B \) be the adjacency matrix of an arbitrary graph on \( n/2 \) vertices. Then, the complete signed graph in which the negatives edges induce the disjoint union of \( G \) and its complement, is sign-symmetric.

2.1. Constructions for general signed graphs. Let \( M_{r,s} \) denote the set of \( r \times s \) matrices with entries from \( \{-1, 0, 1\} \). We give another construction generalizing the one given in Theorem 2.1:

**Theorem 2.2.** Let \( B, C \in M_{k,k} \) be symmetric matrices where \( B \) has a zero diagonal. Then the signed graph with the adjacency matrices

\[
A = \begin{bmatrix} B & C \\ C & -B \end{bmatrix}
\]

is sign-symmetric on \( 2k \) vertices.

**Proof.**

\[
\begin{bmatrix} O & -I \\ I & O \end{bmatrix} \begin{bmatrix} B & C \\ C & -B \end{bmatrix} \begin{bmatrix} O & I \\ -I & O \end{bmatrix} = \begin{bmatrix} -B & -C \\ -C & B \end{bmatrix} = -A
\]

\( \square \)

Note that Theorem 2.2 shows that there exists a sign-symmetric graph for every even order.

We define the family \( \mathcal{F} \) of signed graphs as those which have an adjacency matrix satisfying the conditions given in Theorem 2.2. To get an impression on what the role of \( \mathcal{F} \) is in the family of sign-symmetric graphs, we investigate small complete signed graphs. All but one complete signed graphs with symmetric spectra of orders 4, 6, 8 are illustrated in Fig. 6 (we show one signed graph in the switching class of the signed complete graphs induced by the negative edges). There is only one sign-symmetric complete signed graph of order 4. There are four complete signed graphs with symmetric spectrum of order 6, all of which are sign-symmetric, and twenty-one complete signed graphs with symmetric spectrum of order 8, all except the last one are sign-symmetric, and together with the negation of the last signed graph, Fig. 6 gives all complete signed graphs with symmetric spectrum of order 4, 6 and 8. Interestingly, all of the above sign-symmetric signed graphs belong to \( \mathcal{F} \).

The following proposition shows that \( \mathcal{F} \) is closed under switching.

**Proposition 2.3.** If \( \Gamma \in \mathcal{F} \) and \( \Gamma' \) is obtained from \( \Gamma \) by switching, then \( \Gamma' \in \mathcal{F} \).

**Proof.** Let \( \Gamma \in \mathcal{F} \). It is enough to show that if \( \Gamma' \) is obtained from \( \Gamma \) by switching with respect to its first vertex, then \( \Gamma' \in \mathcal{F} \). We may write the adjacency matrix of
\[ A = \begin{bmatrix}
0 & b^\top & c & c^\top \\
\mathbf{b} & B' & \mathbf{c} & C' \\
\mathbf{c} & \mathbf{c}^\top & 0 & -\mathbf{b}^\top \\
\mathbf{c} & C' & -\mathbf{b} & -B'
\end{bmatrix}. \]

After switching with respect to the first vertex of \( \Gamma \), the adjacency matrix of the resulting signed graph is

\[ A = \begin{bmatrix}
0 & -\mathbf{b}^\top & -\mathbf{c} & -\mathbf{c}^\top \\
-\mathbf{b} & B' & \mathbf{c} & C' \\
-\mathbf{c} & \mathbf{c}^\top & 0 & -\mathbf{b}^\top \\
-\mathbf{c} & C' & -\mathbf{b} & -B'
\end{bmatrix}. \]

Now by interchange the 1st and \((k + 1)\)-th rows and columns we obtain
which is a matrix of the form given in Theorem 2.2 and thus $\Gamma'$ is isomorphic with a signed graph in $\mathcal{F}$.

In the following we present two constructions for complete sign-symmetric signed graphs using self-complementary graphs.

2.2. Constructions for complete signed graphs. In the following, the meaning of a self-complementary graph is the same as defined for unsigned graphs. Let $G$ be a self-complementary graph so that there is a permutation matrix $P$ such that $PA(G)P^{-1} = A(G)$ and $PA(G)P^{-1} = A(G)$. It follows that if $\Gamma$ is a complete signed graph with $E(G)$ being its negative edges, then $A(\Gamma) = A(G) - A(G)$, (in other words, $A(\Gamma)$ is the Seidel matrix of $G$). It follows that $PA(\Gamma)P^{-1} = -A(\Gamma)$. So we obtain the following:

**Observation 2.4.** If $\Gamma$ is a complete signed graph whose negative edges induce a self-complementary graph, then $\Gamma$ is sign-symmetric.

We give one more construction of sign-symmetric signed graphs based on self-complementary graphs as a corollary to Observation 2.4. We remark that a self-complementary graph of order $n$ exists whenever $n \equiv 0$ or 1 (mod 4).

**Proposition 2.5.** Let $G, H$ be two self-complementary graphs, and let $\Gamma$ be a complete signed graph whose negative edges induce the join of $G$ and $H$ (or the disjoint union of $G$ and $H$). Then $\Gamma$ is sign symmetric. In particular, if $G$ has $n$ vertices, and if $H$ is a singleton, then the complete signed graph $\Gamma$ of order $n + 1$ with negative edges equal to $E(G)$ is sign-symmetric.

In the following remark we present a sign-symmetric construction for non-complete signed graphs.
Remark 2.6. Let $\Gamma'$, $\Gamma''$ be two signed graphs which are isomorphic to $-\Gamma'$, $-\Gamma''$, respectively. Consider the signed graph $\Gamma$ obtained from joining $\Gamma'$ and $\Gamma''$ whose negative edges are the union of negative edges in $\Gamma'$ and $\Gamma''$. Then, $\Gamma$ is sign-symmetric.

Remark 2.7. By Proposition 2.5, we have a construction of sign-symmetric complete signed graphs of order $n \equiv 0, 1$ or $2 \pmod{4}$. All complete sign-symmetric signed graphs of order 5 and 9 (depicted in Fig. 7) can be obtained in this way. There is just one sign-symmetric signed graph of order 5 which is obtained by joining a vertex to a complete signed graph of order 4 whose negative edges form a path of length 3 (which is self-complementary). Moreover, there exist sixteen complete signed graphs of order 9 with symmetric spectrum of which ten are sign-symmetric; the first three are not sign-symmetric, and when we include their negations we get them all. All of these ten complete sign-symmetric signed graphs can be obtained by joining a vertex to a complete signed graph of order 8 whose negative edges induce a self-complementary graph. Note that there are exactly ten self-complementary graphs of order 8.

Theorem 2.8. There exists a complete sign-symmetric signed graph of order $n$ if and only if $n \equiv 0, 1$ or $2 \pmod{4}$.

Proof. Using the previous results obviously one can construct a sign-symmetric signed graph of order $n$ whenever $n \equiv 0, 1$ or $2 \pmod{4}$. Now, suppose that there is a complete sign-symmetric signed graph $\Gamma$ of order $n$ with $n \equiv 3 \pmod{4}$. By [7, Corollary 3.6], the determinant of the Seidel matrix of $\Gamma$ is congruent to $1 - n \pmod{4}$. Since $n \equiv 3 \pmod{4}$, the determinant of the Seidel matrix (obtained from the negative edges of $\Gamma$) is not zero. Hence, we can conclude that all eigenvalues of $\Gamma$ are non-zero. Therefore, $\Gamma$ cannot have a symmetric spectrum, and also it cannot be sign-symmetric. \qed

In [9] all switching classes of Seidel matrices of order at most seven are given. There is a error in the spectrum of one of the graphs on six vertices in [9, Table 4.1] (2.37 should be 2.24), except for that, the results in [9] coincide with ours.

3. Positive and negative cycles

A graph whose connected components are $K_2$ or cycles is called an elementary graph. Like unsigned graphs, the coefficients of the characteristic polynomial of the adjacency matrix of a signed graph $\Gamma$ can be described in terms of elementary subgraphs of $\Gamma$.

Theorem 3.1 ([3, Theorem 2.3]). Let $\Gamma = (G, \sigma)$ be a signed graph and

\[ P_{\Gamma}(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \]

be the characteristic polynomial of the adjacency matrix of $\Gamma$. Then

\[ a_i = \sum_{B \in B_i} (-1)^{p(B)}2^{c(B)}|\sigma(B)|, \]

where $B_i$ is the set of elementary subgraphs of $G$ on $i$ vertices, $p(B)$ is the number of components of $B$, $c(B)$ the set of cycles in $B$, and $\sigma(B) = \prod_{C \in c(B)} \sigma(C)$. 

Remark 3.2. It is clear that $\Gamma$ has a symmetric spectrum if and only if in its characteristic polynomial (1), we have $a_{2k+1} = 0$, for $k = 1, 2, \ldots$.

In a signed graph, a cycle is called positive or negative if the product of the signs of its edges is positive or negative, respectively. We denote the number of positive and negative $\ell$-cycles by $c^+\ell$ and $c^-\ell$, respectively.

Observation 3.3. For sign-symmetric signed graph, we have

$$c^+2k+1 = c^-2k+1$$

for $k = 1, 2, \ldots$.

Remark 3.4. If in a signed graph $\Gamma$, $c^+2k+1 = c^-2k+1$ for all $k = 1, 2, \ldots$, then it is not necessary that $\Gamma$ is sign-symmetric. See the complete signed graph given in Fig. 3. For this complete signed graph we have $c^+2k+1 = c^-2k+1$ for all $k = 1, 2, \ldots$, but it is not sign-symmetric. Moreover, one can find other examples among complete and non-complete signed graphs. For example, the signed graph given in Fig. 2 is a non-complete signed graph with the property that $c^+2k+1 = c^-2k+1$ for all $k = 1, 2, \ldots$, but it is not sign-symmetric.

By Theorem 3.1, we have that $a_3 = 2(c^-3 - c^+3)$. By Theorem 3.1 and Remark 3.2 for signed graphs having symmetric spectrum, we have $c^+3 = c^-3$. Further, for each complete signed graph with a symmetric spectrum, it can be seen that $c^+_5 = c^-_5$. However, the equality $c^+_2k+1 = c^-_2k+1$ does not necessarily hold for $k \geq 3$. The complete signed graph in Fig. 1 has a symmetric spectrum for which $c^+_7 \neq c^-_7$.

![Figure 1](image_url)

**Figure 1.** The graph induced by negative edges of a complete signed graph on 9 vertices with a symmetric spectrum but $c^+_7 \neq c^-_7$.

Remark 3.5. There are some examples showing that for a non-complete signed graph we have $c^+_2k+1 = c^-_2k+1$ for all $k = 1, 2, \ldots$, but their spectra are not symmetric. As an example see Fig. 2 (dashed edges are negative; solid edges are positive).

Now, we may ask a weaker version of the result mentioned in Remark 3.4 as follows.

**Question 3.6.** Is it true that if in a complete signed graph $\Gamma$, $c^+_2k+1 = c^-_2k+1$ for all $k = 1, 2, \ldots$, then $\Gamma$ has a symmetric spectrum?
Figure 2. A signed graph with $c_{2k+1}^+ = c_{2k+1}^-$ for $k = 1, 2, \ldots$, but its spectrum is not symmetric.

4. Sign-symmetric vs. symmetric spectrum

Remark 4.1. Consider the complete signed graph whose negative edges induces the graph of Fig. 3. This graph has a symmetric spectrum, but it is not sign-symmetric. Note that this complete signed graph has the minimum order with this property. Moreover, for this complete signed graph we have the equalities $c_{2k+1}^+ = c_{2k+1}^-$ for $k = 1, 2, 3$.

Figure 3. The graph induced by negative edges of a complete signed graph on 8 vertices with a symmetric spectrum but not sign-symmetric.

Remark 4.2. A conference matrix $C$ of order $n$ is an $n \times n$ matrix with zero diagonal and all off-diagonal entries $\pm 1$, which satisfies $CC^\top = (n-1)I$. If $C$ is symmetric, then $C$ has eigenvalues $\pm \sqrt{n-1}$. Hence, its spectrum is symmetric. Conference matrices are well-studied; see for example [4, Section 10.4]. An important example of a symmetric conference matrix is the Seidel matrix of the Paley graph extended with an isolated vertex, where the Paley graph is defined on the elements of a finite field $F_q$, with $q \equiv 1 \pmod{4}$, where two elements are adjacent whenever the difference is a nonzero square in $F_q$. The Paley graph is self-complementary. Therefore, by Proposition 2.5, $C$ is the adjacency matrix of a sign-symmetric complete signed graph. However, there exist many more symmetric conference matrices, including several that are not sign-symmetric (see [5]).

In [2], the authors posed the following problem on the existence of the non-complete signed graphs which are not sign-symmetric but have symmetric spectrum.
Problem 4.3 (2). Are there non-complete connected signed graphs whose spectrum is symmetric with respect to the origin but they are not sign-symmetric?

We answer this problem by showing that there exists such a graph for any order $n \geq 6$. For $s \geq 0$, define the signed graph $\Gamma_s$ to be the graph illustrated in Fig. 4.

The graph $\Gamma_s$

Theorem 4.4. For $s \geq 0$, the graph $\Gamma_s$ has a symmetric spectrum, but it is not sign-symmetric.

Proof. Let $S$ be the set of $s$ vertices adjacent to both 1 and 5. The positive 5-cycles of $\Gamma_s$ are 123461 together with $u1645u$ for any $u \in S$, and the negative 5-cycles are $u1465u$ for any $u \in S$. Hence, $c_5^+ = s + 1$ and $c_5^- = s$. In view of Observation 3.3, this shows that $\Gamma_s$ is not sign-symmetric.

Next, we show that $\Gamma_s$ has a symmetric spectrum. It suffices to verify that $a_{2k+1} = 0$ for $k = 1, 2, \ldots$

The graph $\Gamma_s$ contains a unique positive cycle of length 3: 4564 and a unique negative cycle of length 3: 1461. It follows that $a_3 = 0$.

As discussed above, we have $c_5^+ = s + 1$ and $c_5^- = s$. We count the number of positive and negative copies of $K_2 \cup C_3$. For the negative triangle 1461, there are $s + 1$ non-incident edges, namely 23 and $5u$ for any $u \in S$ and for the positive triangle 4564, there are $s + 2$ non-incident edges, namely 12, 23 and $1u$ for any $u \in S$. It follows that

$$a_5 = -2((s + 1) - s) + 2((s + 2) - (s + 1)) = 0.$$

Now, we count the number of positive and negative elementary subgraphs on 7 vertices:

- $C_7$: $s$ positive: $u123465u$ for any $u \in S$, and no negative;
- $K_2 \cup C_5$: $2s$ positive: $u5 \cup 123461$, and $23 \cup u1645u$ for any $u \in S$, and $s$ negative: $23 \cup u1465u$ for any $u \in S$;
$2K_2 \cup C_3$: $s+1$ positive: $u1 \cup 23 \cup 4564$ for any $u \in S$, and $s+1$ negative: $u5 \cup 23 \cup 1461$ for any $u \in S$;

$C_4 \cup C_3$: none.

Therefore,

$$a_7 = -2(s - 0) + 2(2s - s) - 2((s + 1) - (s + 1)) = 0.$$ 

The graph $\Gamma_s$ contains no elementary subgraph on 8 vertices or more. The result now follows.

More families of non-complete signed graphs with a symmetric spectrum but not sign-symmetric can be found. Consider the signed graphs $\Gamma_{s,t}$ depicted in Fig. 5, in which the number of upper repeated pair of vertices is $s \geq 0$ and the number of upper repeated pair of vertices is $t \geq 1$. In a similar fashion as in the proof of Theorem 4.4 it can be verified that $\Gamma_{s,t}$ has a symmetric spectrum, but it is not sign-symmetric.

![Figure 5. The family of signed graphs $\Gamma_{s,t}$](image)

**References**


Figure 6. Complete signed graphs (up to switching isomorphism and negation) of order 4, 6, 8 having symmetric spectrum. The numbers next to the graphs are the non-negative eigenvalues. Only the last graph on the right is not sign-symmetric.
Figure 7. Complete signed graphs (up to switching isomorphism and negation) of order 5, 9 having symmetric spectrum. The numbers next to the graphs are the non-negative eigenvalues. The first three signed graphs of order 9 are not sign-symmetric.