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Some problems concerning the Frobenius number for extensions of an arithmetic progression

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Abstract

For positive and relative prime set of integers $A = \{a_1, \ldots, a_k\}$, let $\Gamma(A)$ denote the set of integers of the form $a_1x_1 + \cdots + a_kx_k$ with each $x_i \geq 0$. It is well known that $\Gamma^c(A) = \mathbb{N} \setminus \Gamma(A)$ is a finite set, so that $\mathrm{g}(A)$, which denotes the largest integer in $\Gamma^c(A)$, is well defined. Let $A = AP(a, d, k)$ denote the set $\{a, a+d, \ldots, a+(k-1)d\}$ of integers in arithmetic progression, and let $\gcd(a, d) = 1$. We (i) determine the set $A^+ = \{b \in \Gamma^c(A) : \mathrm{g}(A \cup \{b\}) = \mathrm{g}(A)\}$; (ii) determine a subset $\overline{A^+}$ of $\Gamma^c(A)$ of largest cardinality such that $A \cup \overline{A^+}$ is an independent set and $\mathrm{g}(A \cup \overline{A^+}) = \mathrm{g}(A)$; and (iii) determine $\mathrm{g}(A \cup \{b\})$ for some class of values of b that includes results of some recent work.

Keywords Basis · Independent basis · Representable · Frobenius number

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1 Introduction

Given a finite set $A = \{a_1, \ldots, a_k\}$ of positive integers with gcd $A := \gcd(a_1, \ldots, a_k) = 1$, let $\Gamma(A) := \{a_1x_1 + \cdots + a_kx_k : x_i \geq 0\}$. We call an integer N `representable` by A if $N \in \Gamma(A)$. Sylvester showed that the set $\Gamma^c(A) := \mathbb{N} \setminus \Gamma(A)$ of non-representable positive integers is finite, asked to determine

$$\mathrm{g}(A) := \max \Gamma^c(A),$$

and showed that $\mathrm{g}(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1$. For brevity, we use $\mathrm{g}(A)$ to denote $\mathrm{g}(a_1, \ldots, a_k)$. Although the problem was proposed by Sylvester [12], the function $\mathrm{g}(A)$ is called the `Frobenius number` after Frobenius, who was largely instrumental in giving this problem the early recognition in his lectures.

There is no closed-form formula for the Frobenius number $\mathrm{g}(A)$ for $|A| \geq 3$. A lot of research has centered around improving bounds or providing improved algorithms for $\mathrm{g}(A)$, both in the general case and in special cases. One of the few cases where the Frobenius number $\mathrm{g}(A)$ has been determined is the case where the elements of A (called the `basis` elements) are in arithmetic progression.

The Frobenius number $\mathrm{g}(A)$ for a basis $A = \{a_1, \ldots, a_k\}$ is unchanged by the removal of any element, say a_k, that is representable by the other elements of the basis, that is if $a_k \in \Gamma(A \setminus \{a_k\})$. Following Selmer [11] and others, we term a basis in which there is an element which is representable by the other elements a `dependent` basis. Any basis for which $k > \min A$ is necessarily dependent. For if $\min A = a_1$, say, at least two of the $k - 1$ $(\geq a_1)$ numbers $a_2, \ldots, a_k$ must be in the same congruence class modulo a_1 (in which case the larger number is representable by the smaller number and a_1), or at least one of the numbers is a multiple of a_1. Bases which are not dependent are termed `independent`, and $k \leq a_1$ for such bases.

Towards the end of his extensive and significant paper, Selmer [11] considered the problem of the change in the Frobenius number in extending a basis $A = \{a_1, \ldots, a_k\}$ by a single- element a_{k+1}. It is clear that $\mathrm{g}(A \cup \{a_{k+1}\}) \leq \mathrm{g}(A)$, and that there is equality if $a_{k+1} \in \Gamma(A)$. Under the assumption that $a_{k+1} \notin \Gamma(A)$, he considered the problem of determining all positive integers a_{k+1} for which the Frobenius number remains unchanged. Mendelsohn [7] has shown the non-existence of such an extension when $k = 2$. Kirfel [4] gave a condition under which such an extension is possible when $k = 3$. One of the instances when such an extension is possible for $3 < k < a_1$ is the case where the basis is independent and consists of terms in an arithmetic progression satisfying the condition $\lfloor \frac{a_1-2}{k-1} \rfloor = \lfloor \frac{a_1-2}{k} \rfloor$; see [11].

There are three sections in our paper aside from this introductory section (Sect. 1). Throughout this paper, A denotes the set $AP(a, d, k) = \{a, a + d, a + 2d, \ldots, a + (k - 1)d\}$ with $\gcd(a, d) = 1$ and $2 \leq k \leq a$, and $b \in \Gamma^c(A)$. If $k > a$, each of the elements $a + (a + 1)d, \ldots, a + (k - 1)d$ is representable by a and one of the elements from $a + d, \ldots, a + (a - 1)d$. Thus, as remarked earlier, $AP(a, d, k)$ is a dependent basis when $k > a$, and $\mathrm{g}(AP(a, d, k)) = \mathrm{g}(AP(a, d, a))$ for all $k > a$. By A^+, we mean the set of all b for which $\mathrm{g}(A \cup \{b\}) = \mathrm{g}(A)$. We determine the set A^+ in Sect. 2 making use of a result of Tripathi [13] that determines the least integer in $\Gamma(A)$ in each congruence class modulo a; see Lemma 2. These results are given

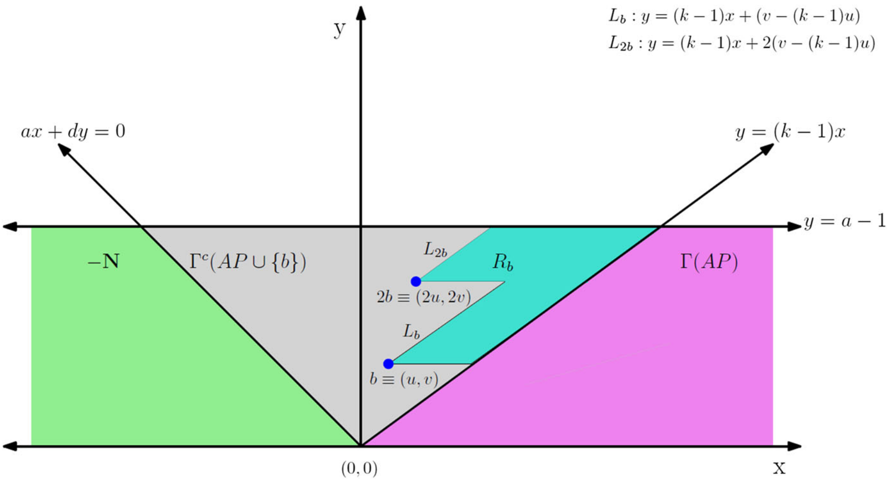

Fig. 1 A geometric depiction of $\Gamma(AP)$ and $\Gamma(AP \cup \{b\})$. Every integer n is of the form $ax + dy$ with $0 \leq y \leq a-1$, and is represented by the lattice point (x, y) lying within or on the infinite strip in this figure. Proposition 1 implies that the integers in $\Gamma(AP)$ are the lattice points in the region labeled $\Gamma(AP)$, or on the boundary. Thus any $b \in \Gamma^c(A)$ is a lattice point in the region bounded by the lines $ax + dy = 0$, $y = (k-1)x$ and $y = a-1$. Lattice points in the region R_b are the integers in $\Gamma(AP \cup \{b\}) \setminus \Gamma(AP)$. All points on the boundary except those on the line $y = (k-1)x$ are included in the set

in Theorem 1 when $k < a$ and in Lemma 3 when $k = a$. In Sect. 3, we deal with the problem of determining a subset $\overline{A^+}$ of $\Gamma^c(A)$ of largest cardinality such that $A \cup \overline{A^+}$ is an independent set and $\mathrm{g}(A \cup \overline{A^+}) = \mathrm{g}(A)$. In Sect. 4, we determine $\mathrm{g}(A \cup \{b\})$ for certain classes of b. This section is independent of Sects. 2 and 3.

We arrived at the results in Sects. 2, 3, and 4 geometrically, by capturing the set of integers as an infinite strip in the plane and representing $\Gamma(A)$, some extensions of A, and their complements. These results are then proved by well-established methods algebraically, paving the way for using geometric methods to assist in results concerning the Frobenius Problem. It is important to note that the geometric interpretations we have employed here lead to the solution of these problems in a much more transparent manner than do number theoretic means. Figures 1 and 2 capture the regions that describe $\Gamma(A)$, $\Gamma(A \cup \{b\})$, and $\Gamma^c(A \cup \{b\})$, and can be used to arrive at the results in Theorems 1 and 2.

The extension problem to determine A^+ we discuss in Sect. 2 was solved by Ritter [8, Theorem 1] by using an algorithm of Rødseth [10] to compute $\mathrm{g}(A \cup \{b\})$. We give a much shorter and more direct proof of the result, leading to a much cleaner formula to describe the set A^+ in Theorem 1 for the case $k < a$, and in Lemma 3 for the case $k = a$. We also show the equivalence of our result with that obtained by Ritter in [8, Theorem 1] for $k < a$. For the case $k = a$, we give a simplified version of the result obtained by Ritter in [8, Theorem 1] and show this to be incorrect in Remark 3 and Remark 4.

The maximum extension problem to determine $\overline{A^+}$ that we discuss in Sect. 3 was also solved by Ritter [8, Theorem 2] by extending the argument given in his proof of [8, Theorem 1]. Our proof is short and direct, and the equivalence of our result with

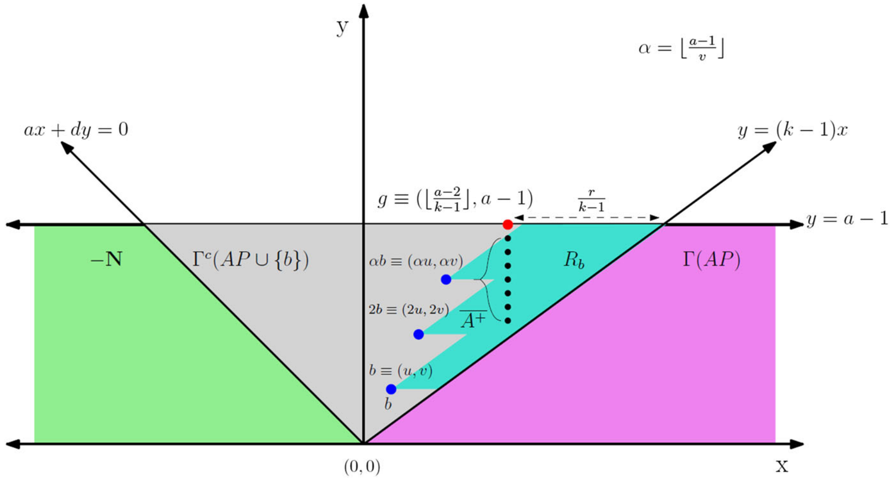

Fig. 2 A geometric depiction of A^+ and $\overline{A^+}$. The Frobenius number $g(AP)$ is represented by (red-filled circle). Integers in A^+ are those b in the region $\Gamma^c(AP)$ for which the corresponding region R_b does not contain the (red-filled circle) point. A candidate for the set $\overline{A^+}$ is the set of (black-filled circle) points (Color figure online)

that obtained by Ritter in [8, Theorem 2] is much more transparent in this case. We also use Fig. 2 to geometrically explain the error in the case $k = a$ of [8, Theorem 1].

In Sect. 4 we determine the Frobenius number of extensions of the set A by some elements b that do not belong to $\Gamma(A)$. These results are guided by geometric interpretation, and include some results of Dulmage and Mendelsohn [3], Kan et al. [5], Mathews [6], and Rødseth [10].

Let A be any set of positive integers with $\gcd(A) = 1$, and let $a \in A$. Let $m_{\mathbf{C}}$ denote the least positive integer in $\Gamma(A)$ which is also in the congruence class $\mathbf{C}$ modulo a. The function g is easily determined from the values of $m_{\mathbf{C}}$ via following the well-known theorem due to Brauer and Shockley.

Lemma 1 (Brauer and Shockley [2]) *Let $a \in A$. Then*

$$g(A) = \max_{\mathbf{C}} m_{\mathbf{C}} - a,$$

where the maximum is taken over all non-zero classes $\mathbf{C}$ *modulo* a.

2 Characterization of single-element extensions of APs that do not change the Frobenius number

For arithmetic progressions, Roberts [9] determined $g(A)$, later simplified by Bateman [1]. A simple proof for both these results using Lemma 1 can be found in [13].

Henceforth let $A = AP(a, d, k) = \{a, a+d, a+2d, \ldots, a+(k-1)d\}$ with $\gcd(a, d) = 1$ and $k \geq 2$. In view of the fact the set of representable integers remains the same for all $k \geq a$, we restrict our attention to $k \leq a$ in what follows. Thus $\mathrm{g}(A)$ denotes the largest N such that

$$ax_0 + (a+d)x_1 + (a+2d)x_2 + \cdots + (a+kd)x_k = a\sum_{i=0}^{k} x_i + d\sum_{i=1}^{k} ix_i = N \quad (1)$$

has no solution in non-negative integers $x_0, x_1, \ldots, x_k$, and $\mathrm{n}(A)$ the number of such N.

Lemma 2 (Tripathi [13]) *Let $A = AP(a, d, k)$. For $y \in \{1, \ldots, a-1\}$, the least integer in $\Gamma(A) \cap (dy)$ is given by*

$$m_{dy} = a\left(1 + \left\lfloor \frac{y-1}{k-1} \right\rfloor\right) + dy.$$

From Lemmas 1 and 2, it easily follows that

$$\mathrm{g}(A) = a\left\lfloor \frac{a-2}{k-1} \right\rfloor + d(a-1) \quad (2)$$

and that

$$\Gamma^c(A) = \left\{ax + dy : 1 \leq y \leq a-1, -\frac{dy-1}{a} \leq x \leq \left\lfloor \frac{y-1}{k-1} \right\rfloor\right\}. \quad (3)$$

In this section, following the question posed by Selmer [11], we look at the problem of determining all positive integers b for which $\mathrm{g}(A \cup \{b\}) = \mathrm{g}(A)$. We may assume that $b \in \Gamma^c(A)$ since $\Gamma(A \cup \{b\}) = \Gamma(A)$ if $b \in \Gamma(A)$.

Proposition 1 *Let k, m, n be non-negative integers, $k \geq 1$. Then there exist non-negative integers $x_0, x_1, \ldots, x_k$ such that*

$$\sum_{i=0}^{k} x_i = m, \quad \sum_{i=1}^{k} ix_i = n \quad (4)$$

if and only if $n \leq km$. Moreover, $n = km$ if and only if $x_k = m$ and $x_i = 0$ for $i \neq k$.

Proof Suppose there exist non-negative integers $x_0, x_1, \ldots, x_k$ satisfying (4). Then $n = \sum_{i=1}^{k} ix_i \leq k\sum_{i=1}^{k} x_i \leq km$.

Conversely, suppose $n \leq km$. If $n = km$, $x_k = m$, and $x_i = 0$ for $i \neq k$ satisfies (4). If $n < km$, write $n = qk + r$, with $q \in \{0, 1, \ldots, m-1\}$ and $r \in \{0, 1, \ldots, k-1\}$. Then $x_k = q$, $x_r = 1$, $x_0 = m - q - 1$, and all other $x_i = 0$ satisfies (4).

Observe that $n = km$ is equivalent to $\sum_{i=1}^{k}(k-i)x_i = 0$. Since $(k-i)x_i \geq 0$ for each i, the last equation holds if and only if $(k-i)x_i = 0$ for each i, so that $x_i = 0$ for $i < k$. Thus $x_k = m$. □

Definition 1 For any set of positive integers A with $\gcd(A) = 1$, define

$$A^+ = \left\{b \in \Gamma^c(A) : \mathrm{g}(A \cup \{b\}) = \mathrm{g}(A)\right\}.$$

Observe that $\mathrm{g}(A) \notin A^+$. The case when $|A| = 2$ was solved by Mendelsohn [7]. However, we include a short and simple proof to show that $A^+ = \emptyset$ in this case for the sake of completeness.

Proposition 2 (Mendelsohn [7]) *If* $|A| = 2$, *then* $A^+ = \emptyset$.

Proof Let $A = \{a, b\}$, with $\gcd(a, b) = 1$, and let $c \in \Gamma^c(A)$. Then $c = by - ax$ with $1 \le y \le a-1$ and $x \ge 1$. But then $\mathrm{g}(A) = ab - a - b = a(x-1) + b(a-1-y) + c \in \Gamma(A \cup \{c\})$. Therefore $\mathrm{g}(a, b, c) < \mathrm{g}(a, b)$. □

Proposition 3 *If* $b = ax + dy \in A^+$, *then* $x \ge 0$.

Proof Suppose $b = ax + dy \in \Gamma^c(A)$, with $x < 0$ and $1 \le y \le a - 1$. Then

$$k\left(\left\lfloor \frac{a-2}{k-1} \right\rfloor - x\right) > (k-1)\left(\left\lfloor \frac{a-2}{k-1} \right\rfloor + 1\right) \ge a - 1 > a - 1 - y.$$

Hence there exist non-negative integers $x_0, x_1, \ldots, x_k$ such that

$$\sum_{i=0}^{k} x_i = \left\lfloor \frac{a-2}{k-1} \right\rfloor - x, \quad \sum_{i=1}^{k} i x_i = a - 1 - y$$

has a simultaneous solution by Proposition 1. Therefore $g(A) - b = a(\lfloor \frac{a-2}{k-1} \rfloor - x) + d(a - 1 - y)$ is representable by the form given by the LHS of (1), and hence belongs to $\Gamma(A)$. Thus $b \notin A^+$. □

Proposition 4 *Let* $A = AP(a, d, k)$. *Then* $b = au + dv \in A^+$ *if and only if the equation*

$$a\left(\sum_{i=0}^{k-1} x_i + uy\right) + d\left(\sum_{i=1}^{k-1} i x_i + vy\right) = a\left\lfloor \frac{a-2}{k-1} \right\rfloor + d(a-1) \tag{5}$$

has no solution in non-negative integers $x_0, x_1, \ldots, x_{k-1}$, y, *and* u, v *satisfy* $0 \le u \le \lfloor \frac{v-1}{k-1} \rfloor$, $1 \le v \le a - 1$.

Proof The term on the LHS of (5) represents a typical element of $\Gamma(A \cup \{b\})$. The proposition now follows directly from (1), (2), and the definition of A^+. Note that the restriction on u being non-negative is a consequence of Proposition 3. □

Proposition 5 *Let* $A = AP(a, d, k)$. *Then* $A^+ = \emptyset$ *if* $a = 1, 2$.

Proof If $a = 1$, then $\Gamma^c(A) = \emptyset$. If $a = 2$, then d is odd and $\Gamma^c(A) = \{1, 3, 5, \ldots, d\}$. For $b \in \Gamma^c(A)$, $\mathrm{g}(A \cup \{b\}) = b - 2 < d = \mathrm{g}(A)$. □

Proposition 6 *Let* $A = AP(a, d, k)$. *Then* $A^+ = \emptyset$ *if* $a \equiv 2 \mod k - 1$.

Proof If $a \equiv 2 \mod k-1$, then $g(A) = a\frac{a-2}{k-1} + d(a-1)$ by (2). By Proposition 4, $A^+ = \emptyset$ precisely when Eq. (5) admits a solution in non-negative integers. Thus it suffices to show that

$$\sum_{i=0}^{k-1} x_i + ry = \frac{a-2}{k-1}, \quad \sum_{i=1}^{k-1} ix_i + sy = a-1$$

has a solution in non-negative integers $x_0, x_1, \ldots, x_{k-1}, y$, given that r, s satisfy $0 \le r \le \lfloor \frac{s-1}{k-1} \rfloor$, $1 \le s \le a-1$. Set $y = 1$. Then the equations reduce to

$$\sum_{i=0}^{k-1} x_i = \frac{a-2}{k-1} - r, \quad \sum_{i=1}^{k-1} ix_i = a-1-s.$$

Now $a-1-s \le (k-1)(\frac{a-2}{k-1} - r) = a-2-(k-1)r$ is the same as $(k-1)r \le s-1$, and this is true by the assumption on r, s. So the equations above have a simultaneous solution by Proposition 1. This proves $A^+ = \emptyset$. □

Remark 1 Proposition 6 implies Proposition 2 and the case $a = 2$ in Proposition 5.

Proposition 7 *Let $A = AP(a, d, k)$. Then* $\max A^+ = g(A) - d$ *if $A^+ \neq \emptyset$.*

Proof Suppose $A^+ \neq \emptyset$. If $b \in A^+$, then $b \le m_{dy} - a$ for some $y \in \{1, \ldots, a-1\}$. Since $b \neq g(A)$, it follows from Lemma 2 that $b \le g(A) - d$.

We show that $b = g(A) - d \in A^+$. If this were not the case, then $g(A) \in \Gamma(A \cup \{b\})$. Since $g(A) \notin \Gamma(A)$ and $2b > g(A)$ (as $a > 2$ by Proposition 2), we must have $g(A) - b \in \Gamma(A)$. But $g(A) - b = d \notin \Gamma(A)$. Therefore $g(A) - d \in A^+$. □

Theorem 1 *Let $A = AP(a, d, k)$. Let $3 \le k \le a-1$, and let*

$$r = \begin{cases} (a-1) \bmod (k-1) & \text{if } (k-1) \nmid (a-1); \\ k-1 & \text{if } (k-1) \mid (a-1). \end{cases}$$

Then

$$A^+ = \left\{ au + dv : u \ge 1, 1 \le v \le a-1, 0 < \left\lfloor \frac{a-1}{v} \right\rfloor (v - (k-1)u) < r \right\}.$$

In particular, if $r = 1$ then $A^+ = \emptyset$.

Proof For non-negative integers $x_0, x_1, \ldots, x_{k-1}$, we write $m = \sum_{i=0}^{k-1} x_i$ and $n = \sum_{i=0}^{k-1} ix_i$. Thus by Proposition 4, $au + dv \notin A^+$ if and only if

$$a(m + uy) + d(n + vy) = a\frac{a-r-1}{k-1} + d(a-1) \tag{6}$$

has a solution in non-negative integers $x_0, x_1, \ldots, x_{k-1}, y$, where u, v satisfy $0 \le u \le \lfloor \frac{v-1}{k-1} \rfloor$, $1 \le v \le a-1$.

Observe that $au + dv \in \Gamma(A)$ if $v = 0$ or if $1 \leq v \leq a-1$ and $\left\lfloor \frac{a-1}{v} \right\rfloor (v-(k-1)u) \leq 0$ by (3). Henceforth we may assume $1 \leq v \leq a-1$. We show that in each of the cases (i) $u = 0$; (ii) $\left\lfloor \frac{a-1}{v} \right\rfloor (v-(k-1)u) \geq r$, (6) has a solution, and that if cases (i) and (ii) do not simultaneously hold, then (6) has no solution.

Fix $v \in \{1, \ldots, a-1\}$. We first show that there is a solution to (6) with $u = 0$. We claim that $r \leq \frac{a-1}{2}$. If $(k-1) \mid (a-1)$, then $r = k-1 \leq \frac{a-1}{2}$ since $k < a$. Otherwise $(k-1) \nmid (a-1)$; write $a-1 = q(k-1)+r$. If $q = 1$, then $2(k-1) > a-1$ and $r = a-k$, so that $r < \frac{a-1}{2}$. If $q > 1$, then $2r < 2(k-1) \leq q(k-1) < a-1$, so that $r < \frac{a-1}{2}$. Hence the claim.

Therefore, we may choose a non-negative integer y such that vy lies in the interval $[r, a-1]$. Then $m = \frac{a-r-1}{k-1}$, $n = (a-1) - vy \leq a-1-r$ provide a solution to (6) by Proposition 1.

If $\left\lfloor \frac{a-1}{v} \right\rfloor (v-(k-1)u) \geq r$, then $y = \lfloor \frac{a-1}{v} \rfloor$, $n = a-1-v\lfloor \frac{a-1}{v} \rfloor$, $m = \frac{a-r-1}{k-1} - u\lfloor \frac{a-1}{v} \rfloor$ gives a solution to (6). Since

$$
\begin{aligned}
(k-1)m - n &= (a-r-1) - (k-1)u\left\lfloor \frac{a-1}{v} \right\rfloor - (a-1) + v\left\lfloor \frac{a-1}{v} \right\rfloor \\
&= \left\lfloor \frac{a-1}{v} \right\rfloor (v-(k-1)u) - r \geq 0,
\end{aligned}
$$

m, n simultaneously exist by Proposition 1. Hence the solution satisfies the necessary constraints.

Suppose neither of the cases (i), (ii) hold. We show that (6) has no solution under the given constraints. Any solution to (6) must have the form

$$
m + uy = \frac{a-r-1}{k-1} - dt, \quad n + vy = a-1+at, \tag{7}
$$

where $t \in \mathbb{Z}$. Multiplying the first equation in (7) by $k-1$ and subtracting from the second gives

$$
(n-(k-1)m) + (v-(k-1)u)\, y = (a+(k-1)d)\, t + r. \tag{8}
$$

Since $n \geq 0$ and $v \nmid (a-1)$ by Remark 2 (at the end of this proof), from (7) we have

$$
\begin{aligned}
y &\leq \frac{a-1+at}{v} = \frac{(a-1)(t+1)+t}{v} \leq \left(\left\lfloor \frac{a-1}{v} \right\rfloor + \frac{v-1}{v}\right)(t+1) + \frac{t}{v} \\
&< \left\lfloor \frac{a-1}{v} \right\rfloor (t+1) + t + 1.
\end{aligned}
$$

Since $u \geq 0$, $v \leq a-1$, and $r \leq k-1$, we have

$$
\begin{aligned}
(v-(k-1)u)\, y &\leq (v-(k-1)u)\left\lfloor \frac{a-1}{v} \right\rfloor (t+1) + (v-(k-1)u)\, t \\
&< r(t+1) + at \\
&< (a+(k-1)d)\, t + r.
\end{aligned}
$$

But this contradicts (8) since $n-(k-1)m \leq 0$ by Proposition 1.

In particular, for $r=1$ we get $A^+=\emptyset$. Indeed, we can simultaneously solve $m+u=\frac{a-2}{k-1}$ and $n+v=a-1$ by Proposition 1 since

$$(k-1)m = a-2-(k-1)u \geq (a-2)-(v-1) = n.$$

Hence (6) has a solution with $y=1$. □

Remark 2 Suppose $0 \leq u \leq \left\lfloor \frac{v-1}{k-1} \right\rfloor$ and $v \mid (a-1)$. From $u \leq \left\lfloor \frac{v-1}{k-1} \right\rfloor$, we have $v > (k-1)u$. Hence $\frac{a-1}{k-1} > \frac{a-1}{v}u$, so that $\frac{a-r-1}{k-1} = \left\lfloor \frac{a-1}{k-1} \right\rfloor \geq \frac{a-1}{v}u$ since $\frac{a-1}{v}$ is an integer. Thus $v(a-r-1) \geq (a-1)(k-1)u$, or $(a-1)\,(v-(k-1)u) \geq vr$. Hence the constraint on u, v to describe the elements in A^+ in Theorem 1 implies $v \nmid (a-1)$.

Equivalence of the results in Theorem 1 and [8, Theorem 1] for $k < a$.

We use the notations in [8, Theorem 1]. Since $\frac{x}{\lambda} + \frac{\rho+1-s\lambda}{(k-1)\lambda} = \frac{a-s\lambda}{(k-1)\lambda}$, the first inequality in Eq. (6) is equivalent to $\lambda \geq \frac{a}{(k-1)r+s}$. Using $s \leq \lfloor \frac{\rho-1}{\lambda-1} \rfloor \leq \frac{\rho-1}{\lambda-1}$ in the second inequality in Eq. (6), we get $r \leq \lfloor \frac{a-1-\rho}{(\lambda-1)(k-1)} \rfloor \leq \frac{a-1-\rho}{(\lambda-1)(k-1)} \leq \frac{a-s(\lambda-1)-2}{(\lambda-1)(k-1)}$. This is equivalent to $\lambda \leq \frac{a-2}{(k-1)r+s}+1$. Thus $\lambda = \lceil \frac{a}{(k-1)r+s} \rceil$ and $\lambda - 1 = \lfloor \frac{a-1}{(k-1)r+s} \rfloor$.

Suppose $k < a$. If $x=1$, then $2(k-1) \geq a-1$, so that $\rho = a-k \leq \frac{a-1}{2}$. If $x \geq 2$, then $2(k-1) < a-1$, so that $\rho \leq k-1 < \frac{a-1}{2}$. Therefore $\rho \leq \frac{a-1}{2}$ in any case. Hence $s \leq \frac{\rho-1}{\lambda-1} < \frac{a-1}{2(\lambda-1)} < \frac{a}{\lambda}$, since the last inequality is equivalent to $2a < \lambda(a+1)$ and $\lambda \geq 2$. Therefore $a - s\lambda > 0$.

Set $r=u$ and $(k-1)r+s=v$. Then $u \geq 1$ (since $a-s\lambda > 0$) and $1 \leq v \leq a-1$ (the upper bound follows from $\lfloor \frac{a-1}{(k-1)r+s} \rfloor = \lambda - 1 \geq 1$). Moreover, $1 \leq s \leq (\lambda-1)s \leq \rho - 1$ translates to $0 < \lfloor \frac{a-1}{v} \rfloor\,(v-(k-1)u) < r$ since $\rho = r$ in the notation of Theorem 1.

Lemma 3 *If $A = AP(a,d,a)$, then $A^+ = \left\{dv : 1 \leq v \leq a-1, v \nmid (a-1)\right\}$.*

Proof For $A = AP(a,d,a)$, we have $\mathtt{m}_{dy} = a + dy$ for $1 \leq y \leq a-1$ by Lemma 2, $\mathrm{g}\,(A) = d(a-1)$ by (2), and $A^+ \subseteq \{dv : 1 \leq v \leq a-1\}$ by (3) and Proposition 3. □

Fix $v \in \{1,\ldots,a-1\}$. If $v \mid (a-1)$, then $dv \mid \mathrm{g}\,(A)$, and so $dv \notin A^+$. Suppose $v \nmid (a-1)$, and suppose by way of contradiction that $dv \notin A^+$. Then $\mathrm{g}\,(A) \in \Gamma\,(A \cup \{dv\})$, so that $\mathrm{g}\,(A) - dvy = d(a-1-vy) \in \Gamma\,(A)$ for some $y \geq 0$. But this is impossible since $\mathtt{m}_{dx} = a + dx$ for each $x \in \{1,\ldots,a-1\}$. Hence $dv \in A^+$ when $v \nmid (a-1)$.

Remark 3 The case $k=a$ in [8, Theorem 1] may be simplified. Following the notation in [8, Theorem 1], $x=0$ and $\rho = k-1 = a-1$. Since $1 \leq s \leq \lfloor \frac{a-2}{\lambda-1} \rfloor$ and $2 \leq \lambda \leq a-1$, s may assume any value between 1 and $a-2$. Hence $a+\lambda(a-1-s) > 0$ for each s, so that $-1 < \frac{a-s\lambda}{\lambda(a-1)} \leq r \leq 0$. Thus $r=0$, and Eq. 6 in [8, Theorem 1] reduces to $\Lambda = \{(0,s) : 1 \leq s \leq a-2\}$. Equation 7 in [8, Theorem 1] now reduces to $A^+ = \left\{dv : 1 \leq v \leq a-2\right\}$, contrary to the result of Lemma 3.

Remark 4 Let $k \leq a$. Figure 2 may be used to explain the fact that no point $b \equiv (u, v)$ lying in the region $\Gamma^c(A)$ can belong to A^+ if $v \mid (a-1)$. If $k < a$, the point $\alpha b \equiv (\alpha u, a-1)$ for $\alpha = \frac{a-1}{v}$ lies to the left of the red point corresponding to $\mathrm{g}(A)$. This is because $(k-1)u < v$ (since (u, v) lies in the region $\Gamma^c(A)$) is equivalent to $\alpha u < \frac{a-1}{k-1}$, so that $\alpha u \leq \lfloor \frac{a-2}{k-1} \rfloor$. Hence the red point lies within the region $\Gamma(A \cup \{b\})$, so that $b \notin A^+$. This gives a geometric interpretation of Remark 2. If $k = a$, the same argument applies since $u = 0$, furthering our claim that the result of [8, Theorem 1] for the case $k = a$ is incorrect.

3 Maximum extensions of APs that do not change the Frobenius number

In this section, we deal with the problem of determining a subset B of $\Gamma^c(A)$ of largest cardinality such that $A \cup B$ is an independent set and $\mathrm{g}(A \cup B) = \mathrm{g}(A)$. Recall that S is an independent set if, for each $m \in S$, $m \notin \Gamma(S \setminus \{m\})$.

Definition 2 Let $A = AP(a, d, k)$, $3 \leq k \leq a$. Then $\overline{A^+}$ is any subset of $\Gamma^c(A)$ satisfying the following three conditions:

(i) $\mathrm{g}\left(A \cup \overline{A^+}\right) = \mathrm{g}(A)$;
(ii) $A \cup \overline{A^+}$ is an independent set;
(iii) if S is an independent set of integers containing A and if $\mathrm{g}(S) = \mathrm{g}(A)$, then $|S \setminus A| \leq |\overline{A^+}|$.

Note that, in fact, $\overline{A^+}$ is a subset of A^+.

Theorem 2 *Let $A = AP(a, d, k)$. Let $3 \leq k \leq a - 1$, and let*

$$r = \begin{cases} (a-1) \bmod (k-1) & \text{if } (k-1) \nmid (a-1); \\ k-1 & \text{if } (k-1) \mid (a-1). \end{cases}$$

Then

$$\overline{A^+} = \left\{ a\frac{a-r-1}{k-1} + dv : a - r \leq v \leq a - 2 \right\}.$$

In particular, if $r = 1$ then $\overline{A^+} = \emptyset$.

Proof Let $B = \{a\frac{a-r-1}{k-1} + dv : a - r \leq v \leq a - 2\}$. We first show that $\mathrm{g}(A \cup B) = \mathrm{g}(A)$. Write the elements of B as $b_i = a\frac{a-r-1}{k-1} + d(a - r + i)$, $0 \leq i \leq r - 2$. Thus we need to show that

$$\begin{aligned} &a\sum_{i=0}^{k-1} x_i + d\sum_{i=1}^{k-1} i x_i + a\frac{a-r-1}{k-1}\sum_{i=0}^{r-2} y_i + d\sum_{i=0}^{r-2}(a-r+i)y_i \\ &\quad = a\frac{a-r-1}{k-1} + d(a-1) = \mathrm{g}(A) \end{aligned} \tag{9}$$

has no solution in non-negative integers x_i, y_i.

Since $\gcd(a, d) = 1$, reducing modulo a and modulo d gives

$$\sum_{i=1}^{k-1} i x_i + \sum_{i=0}^{r-2} (a-r+i) y_i \equiv a-1 \mod a;$$
$$\sum_{i=0}^{k-1} x_i + \frac{a-r-1}{k-1} \sum_{i=0}^{r-2} y_i \equiv \frac{a-r-1}{k-1} \mod d.$$

For convenience, we write $\sum_{i=0}^{k-1} x_i = m$, $\sum_{i=1}^{k-1} i x_i = n$, $\sum_{i=0}^{r-2} y_i = m'$, and $\sum_{i=1}^{r-2} i y_i = n'$. Thus

$$m = -\frac{a-r-1}{k-1}(m'-1) + dt, \quad n + (a-r)m' + n' = (a-1) - at$$

for some $t \in \mathbb{Z}$. Since (9) has no solution if each $y_i = 0$, $m' - 1 \geq 0$. Hence $t \geq 0$ from the first equation and $t \leq 0$ from the second equation. But then $t = 0$, and this is possible only if $m = 0$ and $m' = 1$ from the first equation. Hence each $x_i = 0$, so that $n = 0$, and the second equation reduces to $a - r + n' = a - 1$, or to $n' = r - 1$. However $m' = 1$ implies $n' \leq r - 2$, thereby proving that (9) has no solution in non-negative integers x_i, y_i.

We next show that $A \cup B$ is an independent set, in the sense that $n \notin \Gamma((A \cup B) \setminus \{n\})$, for each $n \in A \cup B$. Thus, we need to show that for $j \in \{0, \ldots, k-1\}$,

$$a \sum_{\substack{0 \leq i \leq k-1 \\ i \neq j}} x_i + d \sum_{\substack{0 \leq i \leq k-1 \\ i \neq j}} i x_i + a \frac{a-r-1}{k-1} \sum_{i=0}^{r-2} y_i + d \sum_{i=0}^{r-2} (a-r+i) y_i = a + jd \quad (10)$$

has no solution in non-negative integers x_i, y_i, and for $j \in \{0, \ldots, r-2\}$,

$$a \sum_{i=0}^{k-1} x_i + d \sum_{i=0}^{k-1} i x_i + a \frac{a-r-1}{k-1} \sum_{\substack{0 \leq i \leq r-2 \\ i \neq j}} y_i + d \sum_{\substack{0 \leq i \leq r-2 \\ i \neq j}} (a-r+i) y_i = a \frac{a-r-1}{k-1} + d(a-r-j) \quad (11)$$

has no solution in non-negative integers x_i, y_i.

Since $\gcd(a, d) = 1$, reducing modulo a and modulo d gives

$$\sum_{i \neq j} i x_i + (a-r)m' + n' \equiv j \mod a;$$
$$\sum_{i \neq j} x_i + \frac{a-r-1}{k-1} m' \equiv 1 \mod d;$$

and

$$n + (a-r)\sum_{i \neq j} y_i + \sum_{i \neq j} i y_i \equiv a - r - j \mod a;$$
$$m + \frac{a-r-1}{k-1}\sum_{i \neq j} y_i \equiv \frac{a-r-1}{k-1} \mod d.$$

The first pair of congruences gives

$$\sum_{i \neq j} i x_i + (a-r)m' + n' = j - at, \quad \sum_{i \neq j} x_i + \frac{a-r-1}{k-1} m' = 1 + dt,$$

for some $t \in \mathbb{Z}$. The first of these is only possible when $t \leq 0$ and the second only when $t \geq 0$, forcing $t = 0$. But then exactly one of $x_0, \ldots, x_{k-1}, y_0, \ldots, y_{r-2}$ equals 1 and all other x_i and all other y_i are 0, and this is clearly impossible.

The second pair of congruences gives

$$n + (a-r)\sum_{i \neq j} y_i + \sum_{i \neq j} i y_i = a - r - j - at,$$
$$m + \frac{a-r-1}{k-1}\sum_{i \neq j} y_i = \frac{a-r-1}{k-1} + dt,$$

for some $t \in \mathbb{Z}$. The first of these is only possible when $t \leq 0$ and the second only when $t \geq 0$ provided $\sum_{i \neq j} y_i \neq 0$, forcing $t = 0$ in this case. But then exactly one of $x_0, \ldots, x_{k-1}, y_0, \ldots, y_{r-2}$ equals 1 and all other x_i and all other y_i are 0, and this is clearly impossible. On the other hand, if $\sum_{i \neq j} y_i = 0$, then each $y_i = 0$. But then $a\frac{a-r-1}{k-1} + d(a-r-j) \in \Gamma(A)$, contradicting $a\frac{a-r-1}{k-1} + d(a-r-j) \in A^+$.

We finally show that if S is an independent set containing A satisfying $\mathrm{g}(S) = \mathrm{g}(A)$, then $|S \setminus A| \leq |B|$. Since $\mathrm{g}(S) = \mathrm{g}(A)$, we must have $S \setminus A \subseteq A^+$. Suppose, by way of contradiction, that $|S \setminus A| > |B| = r - 1$. By Theorem 1, part (c), integers in A^+ (hence in $S \setminus A$) are of the form $au + dv$, with $0 < v - (k-1)u < r$ for $u \geq 1$. By Pigeonhole Principle, there exist distinct pairs (u_1, v_1) and (u_2, v_2) such that $v_1 - (k-1)u_1 = v_2 - (k-1)u_2$. Hence $v_1 - v_2 = (k-1)(u_1 - u_2)$. Assuming $u_1 > u_2$ without loss of generality, we now have $au_1 + dv_1 = au_2 + dv_2 + a(u_1 - u_2) + d(v_1 - v_2) = au_2 + dv_2 + (a + (k-1)d)(u_1 - u_2)$, contradicting the independence of S.

This completes the proof. □

Remark 5 We note that in Theorem 2

$$\overline{A^+} = \left\{(a + (k-1)d)\,\frac{a-r-1}{k-1} + dy : 1 \le y \le r-1\right\}$$

since $a\frac{a-r-1}{k-1} + dv = (a + (k-1)d)\,\frac{a-r-1}{k-1} + d\,(v - (a-r-1))$. This is the case $k < a$ in [8, Theorem 2].

Lemma 4 *If $A = AP(a, d, a)$, then $\overline{A^+} = \emptyset$.*

Proof The elements of A form a complete residue system modulo a. Adding any element to this set will make the resultant set a dependent set, resulting in $\overline{A^+} = \emptyset$. □

Remark 6 The result of Lemma 4 also appears in [8, Theorem 2].

4 The Frobenius number for some extensions of APs

Let $A = AP(a, d, k)$ with $\gcd(a, d) = 1$ and $k \ge 2$. If b is any integer, there is a unique $v \in \{0, 1, \ldots, a-1\}$ such that $b \equiv dv \mod a$. Hence b is of the form $au + dv$, where $0 \le v \le a-1$. In this section, we determine $g\,(A \cup \{b\})$ where $b = au + dv \notin \Gamma\,(A)$, $u \ge 0$, $0 \le v \le a-1$, and u, v satisfy certain conditions. We close this section with several applications of our result.

Theorem 3 *Let $A = AP(a, d, k)$. Let $b = au + dv \notin \Gamma(A)$, with $u \ge 0$, $0 \le v \le a-1$. For $1 \le y \le a-1$, let*

$$f(y) = a\left(\left\lceil \frac{y \bmod v}{k-1} \right\rceil + u\left\lfloor \frac{y}{v} \right\rfloor\right) + dy.$$

(i) *If*

$$u\left(\left\lfloor \frac{a}{v} \right\rfloor + 1\right) + d \ge \left\lfloor \frac{v - (a \bmod v) - 1}{k-1} \right\rfloor + 1 \text{ or } v \mid a, \tag{12}$$

the least integer in $\Gamma(A)$ that is congruent to dy modulo a is given by

$$m_{dy} = f(y)$$

for $1 \le y \le a-1$.

(ii) *If*

$$u\left(\left\lfloor \frac{a}{v} \right\rfloor + 1\right) + d \ge \left\lfloor \frac{v - (a \bmod v) - 1}{k-1} \right\rfloor + 1,$$

then

$$g\,(A \cup \{b\}) = \max\left\{f(a-1),\ f\left(v\left\lfloor \frac{a-1}{v} \right\rfloor - 1\right)\right\} - a.$$

(iii) *If $v \mid a$, then*

$$g(A \cup \{b\}) = f(a-1) - a.$$

Proof For $y \in \{1, \ldots, a-1\}$, let $\mathtt{m}_{dy}$ denote the least integer in $\Gamma(A)$ that is congruent to dy modulo a. Under the conditions stated in (12), we claim that

$$\mathtt{m}_{dy} = a\left(\left\lceil \frac{y \bmod v}{k-1} \right\rceil + u\left\lfloor \frac{y}{v} \right\rfloor\right) + dy.$$

Recall that $\mathtt{m}_{dy}$ is the least positive integer of the form $dy + at$, with $t \geq 0$, such that

$$a\left(\sum_{i=0}^{k-1} x_i + ux\right) + d\left(\sum_{i=1}^{k-1} ix_i + vx\right) = dy + at \tag{13}$$

has a solution in non-negative integers x_i, x, and t. Hence we must minimize

$$X = \sum_{i=0}^{k-1} x_i + ux$$

subject to the constraint

$$Y = \sum_{i=1}^{k-1} ix_i + vx \equiv y \bmod a.$$

We must choose $x_0 = 0$ for minimum value. If the minimum is attained at $x = x^\star$, we must simultaneously have

$$X - ux^\star = \sum_{i=1}^{k-1} x_i, \quad Y - vx^\star = \sum_{i=1}^{k-1} ix_i.$$

By Proposition 1, this implies $Y - vx^\star \leq (k-1)(X - ux^\star)$, so that $X \geq \lceil \frac{Y-vx^\star}{k-1} \rceil + ux^\star$. Given $\sum_{i=1}^{k-1} ix_i$, in order to minimize $\sum_{i=1}^{k-1} x_i$, we must choose $x_{k-1} = \lfloor \frac{Y-vx^\star}{k-1} \rfloor$; at most one other x_i can be non-zero. If $(k-1) \mid (Y - vx^\star)$, all other $x_i = 0$; otherwise $x_r = 1$ where $r \equiv Y - vx^\star \mod k-1$. In either case, the minimum value of X is $\lceil \frac{Y-vx^\star}{k-1} \rceil + ux^\star$.

Since $b \in \Gamma^c(A)$, $u \leq \lfloor \frac{v-1}{k-1} \rfloor$ by Lemma 2. With $f_1(x^\star) = \lceil \frac{Y-vx^\star}{k-1} \rceil + ux^\star$, we have

$$\begin{aligned} f_1(x^\star) - f_1(x^\star+1) &= \left(\left\lceil \frac{Y - vx^\star}{k-1} \right\rceil - \left\lceil \frac{Y - v(x^\star+1)}{k-1} \right\rceil\right) - u \\ &\geq \left\lceil \frac{v}{k-1} \right\rceil - u - 1 \geq 0. \end{aligned}$$

Since $Y = \sum_{i=1}^{k-1} i x_i + v x^\star$ and $x_i \geq 0$ for each i, we must choose $x^\star = \lfloor \frac{Y}{v} \rfloor$ in order to minimize X. Thus we are left to minimize

$$aX + dY = a\left(\left\lceil \frac{Y \bmod v}{k-1} \right\rceil + u \left\lfloor \frac{Y}{v} \right\rfloor\right) + dY$$

subject to $Y \equiv y \mod a$. With $f_2(t) = a(\lceil \frac{(y+at) \bmod v}{k-1} \rceil + u \lfloor \frac{y+at}{v} \rfloor) + d(y+at)$, we have

$$\begin{aligned} f_2(t+1) - f_2(t) = a\Bigg(&\left\lceil \frac{(y+a(t+1)) \bmod v}{k-1} \right\rceil + u \left\lfloor \frac{y+a(t+1)}{v} \right\rfloor \\ &- \left\lceil \frac{(y+at) \bmod v}{k-1} \right\rceil - u \left\lfloor \frac{y+at}{v} \right\rfloor\Bigg) + ad. \end{aligned}$$

We show that $f_2(t+1) \geq f_2(t)$ for $t \geq 0$ when the condition (12) is satisfied.

Fix y and t, and write $y + at \equiv R \bmod v$ and $a \equiv r \bmod v$. Thus the above difference reduces to

$$f_2(t+1) - f_2(t) = a\left(\left\lceil \frac{(a+R) \bmod v}{k-1} \right\rceil - \left\lceil \frac{R}{k-1} \right\rceil + u \left\lfloor \frac{a+R}{v} \right\rfloor + d\right). \quad (14)$$

Since $\lfloor \frac{a+R}{v} \rfloor = \lfloor \frac{(a-r)+(r+R)}{v} \rfloor = \frac{a-r}{v} + \lfloor \frac{r+R}{v} \rfloor = \lfloor \frac{a}{v} \rfloor + \lfloor \frac{r+R}{v} \rfloor$, the condition $\lfloor \frac{a+R}{v} \rfloor = \lfloor \frac{a}{v} \rfloor$ is equivalent to $r + R \leq v - 1$. Hence $(a+R) \bmod v = r + R \geq R$, and so $f_2(t+1) > f_2(t)$ if $\lfloor \frac{a+R}{v} \rfloor = \lfloor \frac{a}{v} \rfloor$.

Otherwise $\lfloor \frac{a+R}{v} \rfloor = \lfloor \frac{a}{v} \rfloor + 1$, and $v \leq r + R \leq 2(v-1)$. Hence

$$\begin{aligned} \left\lceil \frac{(a+R) \bmod v}{k-1} \right\rceil - \left\lceil \frac{R}{k-1} \right\rceil &= \left\lceil \frac{r+R-v}{k-1} \right\rceil - \left\lceil \frac{R}{k-1} \right\rceil \geq \left\lceil \frac{r-v}{k-1} \right\rceil - 1 \\ &= -\left\lfloor \frac{v-r}{k-1} \right\rfloor - 1 = -\left\lfloor \frac{v-r-1}{k-1} \right\rfloor - 1, \end{aligned}$$

the last equality not holding only when $(k-1) \mid (v-r)$. On the other hand, in this exceptional case

$$\begin{aligned} \left\lceil \frac{(a+R) \bmod v}{k-1} \right\rceil - \left\lceil \frac{R}{k-1} \right\rceil &= \left\lceil \frac{R-(v-r)}{k-1} \right\rceil - \left\lceil \frac{R}{k-1} \right\rceil \\ &= -\frac{v-r}{k-1} = -\left\lfloor \frac{v-r-1}{k-1} \right\rfloor - 1. \end{aligned}$$

Thus if the first part of condition (12) is satisfied in this case, then $f_2(t+1) \geq f_2(t)$. On the other hand, if $v \mid a$, it is easy to see that $f_2(t+1) - f_2(t) \geq ad > 0$. Thus the minimum value of $aX + dY$ is given by $f_2(0) = a\left(\lceil \frac{y \bmod v}{k-1} \rceil + u \lfloor \frac{y}{v} \rfloor\right) + dy$. This completes the proof of part (i).

Since $\lfloor \frac{y}{v} \rfloor$ is increasing in y, $\mathtt{m}_{dy}$ attains its maximum either at $y = a - 1$ or at the largest y for which $y \bmod v = v - 1$. Since the largest value of y for which

$y \bmod v = v-1$ is $v\lfloor \frac{a-1}{v} \rfloor - 1$, the result in part (ii) follows from the formula $\mathrm{g}(A \cup \{b\}) = \max_{1 \le y \le a-1} \mathrm{m}_{dy} - a$.

If $v \mid a$, the term $\lceil \frac{y \bmod v}{k-1} \rceil$ returns the same value for $y = a-1$ and $y = v\lfloor \frac{a-1}{v} \rfloor - 1$. Hence $f\left(v\lfloor \frac{a-1}{v} \rfloor - 1\right) \le f(a-1)$, thereby proving part (iii). □

Remark 7 Let $A = AP(a,d,k)$, and let $b = a + Kd$ with $K \ge k$. Observe that $k \ge a$ implies $b = (a + rd) + a(qd) \in \Gamma(A)$, where $K = qa + r$, $0 \le r \le a-1$. Thus $\mathrm{g}(A \cup \{b\}) = \mathrm{g}(A)$ in this case. Therefore $b = a + Kd \notin \Gamma(A)$ implies $k \le a-1$.

Remark 8 Let $A = AP(a,d,k)$, and let $b = a + kd$ with $k \le a-1$. Using notation of Theorem 3, $u = 1$ and $v = k$. Thus (12) is satisfied since

$$u\left(\left\lfloor \frac{a}{v} \right\rfloor + 1\right) + d = \left\lfloor \frac{a}{k} \right\rfloor + 1 + d \ge d + 2 > \left\lfloor \frac{k - (a \bmod k) - 1}{k-1} \right\rfloor + 1.$$

Hence Theorem 3 gives

$$\begin{aligned}
\mathrm{g}(A \cup \{b\}) &= \max\left\{ f(a-1),\, f\left(k\left\lfloor \frac{a-1}{k} \right\rfloor - 1\right)\right\} - a \\
&= \max\left\{ a\left(\left\lceil \frac{(a-1) \bmod k}{k-1} \right\rceil + \left\lfloor \frac{a-1}{k} \right\rfloor - 1\right)\right. \\
&\qquad \left. + d(a-1),\, a\left(\left\lfloor \frac{a-1}{k} \right\rfloor - 1\right) + d\left(k\left\lfloor \frac{a-1}{k} \right\rfloor - 1\right)\right\} \\
&= a\left(\left\lceil \frac{(a-1) \bmod k}{k-1} \right\rceil + \left\lfloor \frac{a-1}{k} \right\rfloor - 1\right) + d(a-1) \\
&= a\left\lfloor \frac{a-2}{k} \right\rfloor + d(a-1) \\
&= \mathrm{g}(AP(a,d,k+1)).
\end{aligned}$$

Corollary 1 *Let a, m be relatively prime positive integers, and let $m = qa + r$, $0 \le r \le a-1$.*

(i) $a + m \in \Gamma(\{a, a+1\})$ *if and only if* $q \ge r-1$.
(ii) *If* $q \le r-2$ *and* $(q+1)\left(\lfloor \frac{a}{r} \rfloor + 1\right) \ge r - (a \bmod r) - 1$, *then*

$$\begin{aligned}
&g(a, a+1, a+m) \\
&\quad = \max\left\{ a\left((a-1) \bmod r + (q+1)\left\lfloor \frac{a-1}{r} \right\rfloor\right) - 1,\right. \\
&\qquad \left. a\left(r - 2 + (q+1)\left(\left\lfloor \frac{a-1}{r} \right\rfloor - 1\right)\right) + r\left\lfloor \frac{a-1}{r} \right\rfloor - 1\right\}.
\end{aligned}$$

(iii) *If* $q \le r-2$ *and* $r \mid a$, *then*

$$g(a, a+1, a+m) = a\left(r - 1 + (q+1)\left(\frac{a}{r} - 1\right)\right) - 1.$$

Proof Since $a+m \equiv r \mod a$ and the least non-negative integer congruent to r modulo a in $\Gamma(\{a, a+1\})$ is $r(a+1)$, $a+m \in \Gamma(\{a, a+1\})$ if and only if $a+m \geq (a+1)r$, which is the same as $q+1 \geq r$. This proves part (i).

The elements of $\{a, a+1, a+m\}$ pertain to the case $d=1, k=2, u=q+1, v=r$ in Theorem 3. In order to apply Theorem 3, we need to assume $a+m \notin \Gamma(\{a, a+1\})$. Parts (ii) and (iii) are direct consequences of Theorem 3. □

Remark 9 Kan et al. [5] gave an exact formula for $\mathrm{g}(a, a+1, a+m)$ when $2 \leq m \leq 5$ and $a > m(m-4)+1$, and also an upper bound for general m, although no proofs were given.

Corollary 2 (Dulmage and Mendelsohn [3]) *For* $a \geq 1$,

(i) $g(a, a+1, a+2, a+4) = (a+1)\left\lfloor \frac{a}{4} \right\rfloor + \left\lfloor \frac{a+1}{4} \right\rfloor + 2\left\lfloor \frac{a+2}{4} \right\rfloor - 1$;
(ii) $g(a, a+1, a+2, a+5) = \left\lfloor \frac{a}{5} \right\rfloor + (a+1)\left\lfloor \frac{a+1}{5} \right\rfloor + \left\lfloor \frac{a+2}{5} \right\rfloor + 2\left\lfloor \frac{a+3}{5} \right\rfloor - 1$;
(iii) $g(a, a+1, a+2, a+6) = (a+2)\left\lfloor \frac{a}{6} \right\rfloor + 2\left\lfloor \frac{a+1}{6} \right\rfloor + 5\left\lfloor \frac{a+2}{6} \right\rfloor + \left\lfloor \frac{a+3}{6} \right\rfloor + \left\lfloor \frac{a+4}{6} \right\rfloor + \left\lfloor \frac{a+5}{6} \right\rfloor - 1$.

Proof We show that a more general result follows from Theorem 3, like in Corollary 1. We explore the Frobenius number $\mathrm{g}(a, a+1, a+2, a+m)$ with $m \geq 4$. With $m = qa+r$, $0 \leq r \leq a-1$, $a+m \equiv r \mod a$, and the least non-negative integer congruent to r modulo a in $\Gamma(\{a, a+1, a+2\})$ is $a\lfloor \frac{r+1}{2} \rfloor + r$, by Lemma 2. Thus $a+m \in \Gamma(\{a, a+1, a+2\})$ if and only if $q \geq \lfloor \frac{r-1}{2} \rfloor$.

To apply Theorem 3, we must therefore assume $q < \lfloor \frac{r-1}{2} \rfloor$. The elements of $\{a, a+1, a+2, a+m\}$ pertain to the case $d=1, k=3, u=q+1, v=r$ in Theorem 3. Thus if

$$(q+1)\left(\left\lfloor \frac{a}{r} \right\rfloor + 1\right) \geq \left\lfloor \frac{r-(a \bmod r)-1}{2} \right\rfloor,$$

then

$$\mathrm{g}(a, a+1, a+2, a+m) = \max\left\{f(a-1), f\left(r\left\lfloor \frac{a-1}{r} \right\rfloor - 1\right)\right\} - a,$$

and if $r \mid a$, then

$$\mathrm{g}(a, a+1, a+2, a+m) = f(a-1) - a,$$

where $f(y) = a\left(\lceil \frac{y \bmod r}{2} \rceil + (q+1)\lfloor \frac{y}{r} \rfloor\right) + dy$.

When $a > m$, $q = 0$, and $r = m$, and the condition in Theorem 3 reduces to

$$\left\lfloor \frac{a}{m} \right\rfloor + 1 \geq \left\lfloor \frac{m-(a \bmod m)-1}{2} \right\rfloor. \tag{15}$$

For such a,

$$\begin{aligned} &\mathrm{g}(a, a+1, a+2, a+m) \\ &= \max\left\{a\left(\left\lceil\frac{(a-1) \bmod m}{2}\right\rceil + \left\lfloor\frac{a-1}{m}\right\rfloor\right) - 1,\right. \\ &\left. a\left(\left\lceil\frac{m-1}{2}\right\rceil + \left\lfloor\frac{a-1}{m}\right\rfloor - 2\right) + m\left\lfloor\frac{a-1}{m}\right\rfloor - 1\right\}. \end{aligned} \tag{16}$$

Now

$$\begin{aligned} &a\left(\left\lceil\frac{(a-1) \bmod m}{2}\right\rceil + \left\lfloor\frac{a-1}{m}\right\rfloor\right) - 1 \\ &\geq a\left(\left\lceil\frac{m-1}{2}\right\rceil + \left\lfloor\frac{a-1}{m}\right\rfloor - 2\right) + m\left\lfloor\frac{a-1}{m}\right\rfloor - 1 \end{aligned}$$

if and only

$$\left\lceil\frac{(a-1) \bmod m}{2}\right\rceil > \left\lceil\frac{m-1}{2}\right\rceil - 2 = \left\lceil\frac{m-5}{2}\right\rceil.$$

The above inequality holds precisely when $(a-1) \bmod m \in \{m-1, m-2, m-3\}$ when m is even and $(a-1) \bmod m \in \{m-1, m-2, m-3, m-4\}$ when m is odd. So if $a > m$ and (15) holds, then

$$\mathrm{g}(a, a+1, a+2, a+m) = \begin{cases} a\left(\left\lceil\frac{(a-1) \bmod m}{2}\right\rceil + \left\lfloor\frac{a-1}{m}\right\rfloor\right) - 1 & \text{if } a \equiv 0, -1, -2 \bmod m,\ m \text{ even,} \\ & \text{or } a \equiv 0, -1, -2, -3 \bmod m,\ m \text{ odd;} \\ a\left(\left\lceil\frac{m-1}{2}\right\rceil + \left\lfloor\frac{a-1}{m}\right\rfloor - 2\right) + m\left\lfloor\frac{a-1}{m}\right\rfloor - 1 & \text{if } a \not\equiv 0, -1, -2 \bmod m,\ m \text{ even,} \\ & \text{or } a \not\equiv 0, -1, -2, -3 \bmod m,\ m \text{ odd.} \end{cases}$$

We note that $a > m$ implies (15) for $m \in \{4, 5, 6\}$. The result of Corollary 2 may be verified to be equivalent to the above formula in these cases, thus verifying the result for $a > m$. □

Corollary 3 (Matthews [6]) *Let a, d be relative prime positive integers such that $a > \mathcal{F}_i$, where $\mathcal{F}_i$ denotes the ith Fibonacci number. Then with $S = \{a, a+d, a\mathcal{F}_{i-1}+d\mathcal{F}_i\}$ such that the elements in S are pairwise relatively prime, we have*

$$\begin{aligned} g(S) = \max\left\{\left(a + d - \mathcal{F}_{i-2}\left\lfloor\frac{a}{\mathcal{F}_i}\right\rfloor - 2\right)a - d,\ (\mathcal{F}_{i-2} - 2)a\right. \\ \left. + \left\lfloor\frac{a}{\mathcal{F}_i}\right\rfloor(a\mathcal{F}_{i-1} + d\mathcal{F}_i) - d\right\}. \end{aligned}$$

Proof The elements in S pertain to the case $k = 2$, $u = \mathcal{F}_{i-1}$, $v = \mathcal{F}_i$ in Theorem 3. Observe that $\mathcal{F}_i \nmid a$ since $\gcd(a, a\mathcal{F}_{i-1} + d\mathcal{F}_i) = 1$. Hence $\lfloor \frac{a-1}{\mathcal{F}_i} \rfloor = \lfloor \frac{a}{\mathcal{F}_i} \rfloor$ and $(a-1) \bmod \mathcal{F}_i = (a \bmod \mathcal{F}_i) - 1$. Since $a > \mathcal{F}_i$, the elements in S satisfy condition (12) as

$$\mathcal{F}_{i-1}\left(1 + \left\lfloor \frac{a}{\mathcal{F}_i} \right\rfloor\right) + d \geq 2\mathcal{F}_{i-1} + 1 > \mathcal{F}_i > \mathcal{F}_i - (a \bmod \mathcal{F}_i).$$

With notations used in Theorem 3,

$$\begin{aligned} f(a-1) &= a\left((a-1) \bmod \mathcal{F}_i + \mathcal{F}_{i-1}\left\lfloor \frac{a-1}{\mathcal{F}_i} \right\rfloor - 1\right) + d(a-1) \\ &= a\left(a \bmod \mathcal{F}_i + (\mathcal{F}_i - \mathcal{F}_{i-2})\left\lfloor \frac{a}{\mathcal{F}_i} \right\rfloor + d - 2\right) - d \\ &= a\left(a + d - \mathcal{F}_{i-2}\left\lfloor \frac{a}{\mathcal{F}_i} \right\rfloor - 2\right) - d, \end{aligned}$$

and

$$\begin{aligned} f\left(v\left\lfloor \frac{a-1}{v} \right\rfloor - 1\right) &= a\left(\mathcal{F}_i - 1 + \mathcal{F}_{i-1}\left(\left\lfloor \frac{a-1}{\mathcal{F}_i} \right\rfloor - 1\right) - 1\right) + d\left(\mathcal{F}_i \lfloor \tfrac{a-1}{\mathcal{F}_i} \rfloor - 1\right) \\ &= a\left(\mathcal{F}_i - 1 + \mathcal{F}_{i-1}\left(\left\lfloor \frac{a}{\mathcal{F}_i} \right\rfloor - 1\right) - 1\right) + d\mathcal{F}_i\left\lfloor \frac{a}{\mathcal{F}_i} \right\rfloor - d \\ &= a(\mathcal{F}_{i-2} - 2) + \left\lfloor \frac{a}{\mathcal{F}_i} \right\rfloor (a\mathcal{F}_{i-1} + d\mathcal{F}_i) - d. \end{aligned}$$

The result now follows from Theorem 3. $\square$

Corollary 4 (Rødseth [10]) *Let $A = AP(a, d, k)$. For $K \geq k$, let $a = \alpha K + \beta$, $0 \leq \beta \leq K - 1$. If $\beta = 0$ or $\alpha + d \geq \lfloor \frac{K-\beta-1}{k-1} \rfloor$, then*

$$g(A \cup \{a + Kd\}) = (a + Kd)\alpha - d + \max\left\{a\left\lfloor \frac{\beta - 2}{k - 1} \right\rfloor + d\beta, a\left\lfloor \frac{K - 2}{k - 1} \right\rfloor - a\right\}.$$

Proof If $k \geq a$, then $a + Kd \in \Gamma(A)$ by Remark 7, so that $g(A \cup \{a + Kd\}) = g(A) = a\lfloor \frac{a-2}{k-1} \rfloor + d(a-1)$. Rødseth's formula gives this result for $d \geq \lfloor \frac{K-a-1}{k-1} \rfloor$ if $k > a$ but gives $a\lfloor \frac{K-2}{a-1} \rfloor + d(a-1)$ for $d \geq \lfloor \frac{K-1}{a-1} \rfloor - 1$ if $k = a$.

Suppose $k \leq a - 1$. If $K \leq a - 1$, $b = a + Kd$ gives $u = 1$ and $v = K$ in the notation of Theorem 3, and it is easily verified that the condition (12) is identical to the condition in Corollary 4. Thus

$$
\begin{aligned}
f(a-1) &= a\left(\left\lceil \frac{(a-1) \bmod K}{k-1} \right\rceil + \left\lfloor \frac{a-1}{K} \right\rfloor - 1\right) + d(a-1) \\
&= a\left(\left\lceil \frac{\beta-1}{k-1} \right\rceil + \alpha - 1\right) + d(a-1) \text{ if } \beta \neq 0 \\
&= a\left(\left\lfloor \frac{\beta-2}{k-1} \right\rfloor + \alpha\right) + d(\alpha K + \beta - 1) \text{ if } \beta \neq 0 \\
&= (a+Kd)\alpha + d(\beta-1) + a\left\lfloor \frac{\beta-2}{k-1} \right\rfloor \text{ if } \beta \neq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
f(a-1) &= a\left(\left\lceil \frac{(a-1) \bmod K}{k-1} \right\rceil + \left\lfloor \frac{a-1}{K} \right\rfloor - 1\right) + d(a-1) \\
&= a\left(\left\lceil \frac{K-1}{k-1} \right\rceil + \alpha - 2\right) + d(a-1) \text{ if } \beta = 0 \\
&= a\left(\left\lfloor \frac{K-2}{k-1} \right\rfloor + \alpha - 1\right) + d(\alpha K + \beta - 1) \text{ if } \beta = 0 \\
&= (a+Kd)\alpha + a\left\lfloor \frac{K-2}{k-1} \right\rfloor - a - d \text{ if } \beta = 0.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
f\left(K\left\lfloor \frac{a-1}{K} \right\rfloor - 1\right) &= a\left(\left\lceil \frac{K-1}{k-1} \right\rceil + \left\lfloor \frac{a-1}{K} \right\rfloor - 2\right) + d\left(K\left\lfloor \frac{a-1}{K} \right\rfloor - 1\right) \\
&= a\left(\left\lfloor \frac{K-2}{k-1} \right\rfloor + \alpha - 1\right) + d\left(\alpha K - 1\right) \text{ if } \beta \neq 0 \\
&= (a+Kd)\alpha + d(\beta-1) + a\left\lfloor \frac{\beta-2}{k-1} \right\rfloor \text{ if } \beta \neq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
f\left(k\left\lfloor \frac{a-1}{k} \right\rfloor - 1\right) &= a\left(\left\lceil \frac{(a-1) \bmod K}{k-1} \right\rceil + \left\lfloor \frac{a-1}{K} \right\rfloor - 1\right) + d(a-1) \\
&= a\left(\left\lceil \frac{K-1}{k-1} \right\rceil + \alpha - 2\right) + d(a-1) \text{ if } \beta = 0 \\
&= a\left(\left\lfloor \frac{K-2}{k-1} \right\rfloor + \alpha - 1\right) + d(\alpha K + \beta - 1) \text{ if } \beta = 0 \\
&= (a+Kd)\alpha + a\left\lfloor \frac{K-2}{k-1} \right\rfloor - a - d \text{ if } \beta = 0.
\end{aligned}
$$

□

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