Deterministic Random Walks on Regular Trees

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Abstract

Jim Propp's rotor router model is a deterministic analogue of a random walk on a graph. Instead of distributing chips randomly, each vertex serves its neighbors in a fixed order.

Cooper and Spencer (Comb. Probab. Comput. (2006)) show a remarkable similarity of both models. If an (almost) arbitrary population of chips is placed on the vertices of a grid $\mathbb{Z}^d$ and does a simultaneous walk in the Propp model, then at all times and on each vertex, the number of chips deviates from the expected number the random walk would have gotten there, by at most a constant. This constant is independent of the starting configuration and the order in which each vertex serves its neighbors.

This result raises the question if all graphs do have this property. With quite some effort, we are now able to answer this question negatively. For the graph being an infinite k-ary tree ($k \geq 3$), we show that for any deviation D there is an initial configuration of chips such that after running the Propp model for a certain time there is a vertex with at least D more chips than expected in the random walk model. However, to achieve a deviation of D it is necessary that at least $k^{\Theta(D)}$ vertices contribute by being occupied by a number of chips not divisible by k in a certain time interval.

1 Introduction

The rotor-router model is a simple deterministic process first introduced by Priezzhev et al. [9] and later popularized by Jim Propp. It can be viewed as an attempt to derandomize random walks on graphs. So far, the "Propp machine" has been studied primarily on infinite grids $\mathbb{Z}^d$. There, each vertex $x \in \mathbb{Z}^d$ is equipped with a "rotor" together with a cyclic permutation (called a "rotor sequence") of the $2d$ cardinal directions of $\mathbb{Z}^d$. While a chip (particle, coin, ...) performing a random walk leaves a vertex in a random direction, in the Propp model it always goes in the direction the rotor is pointing. After a chip is sent, the rotor is rotated according to the fixed rotor sequence. This rule ensures that chips are distributed quite evenly among the neighbors of a vertex.

The Propp machine has attracted considerable attention recently. It has been shown that it closely resembles a random walk in several respects. The first results were due to Levine and Peres [8] (and later Landau and Levine [7]) who compared random walk and Propp machine in an *aggregating model* called Internal Diffusion-Limited Aggregation (IDLA).

Cooper and Spencer [2] compared both models in terms of the *single vertex discrepancy*. Apart from a technicality, they place arbitrary numbers of chips on the vertices. Then they run the Propp machine on this initial configuration for a certain number of rounds. A round consists of each chip (in arbitrary order) performing one move as directed by the Propp machine. For the resulting chip arrangement, they compare the number of chips at each vertex with the expected number of chips that the random walk would have put there, starting from the same initial configuration and running for the same time. Cooper and Spencer showed that for all grids $\mathbb{Z}^d$, these differences can be bounded by a constant c_d independent of the initial setup (in particular, the total number of chips) and the run-time. For the case $d = 1$, that is, the graph being the infinite path, the optimal constant c_1 is approximately 2.29 [1], for the two-dimensional grid it is $c_2 \approx 7.87$ [3].

This raises the question whether the Propp machine on all graphs simulates the random walk that well.In this work, we show that the infinite k-regular tree behaves different. We prove that here arbitrarily large discrepancies can result from suitable initial configurations. However, to obtain a discrepancy of D, at least $k^{\Theta(D)}$ vertices have to participate by sending chips at some time.

While the work cited above and ours in this paper primarily aims at understanding random walks and their deterministic couterparts from a foundations perspective, we should note that in the meantime the rotor router mechanism also lead to improvements in applications. An example is the quasirandom analogue of the "randomized rumor spreading" protocol to broadcast information in networks [4].

2 Preliminaries

To bound the single vertex discrepancy between the Propp machine and a random walk on the k-regular tree we first introduce several requisite definitions and notational conventions. Let $G = (V, E)$ be the infinite

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k-regular tree, also known as the "Cayley tree" and the "Bethe lattice". We fix an arbitrary node to be its *origin* $\mathbf{0}$. $|\mathbf{x}|$ denotes the shortest (i. e., ordinary graphical) distance between the origin and vertex $\mathbf{x}$.

In order to avoid discussing all equations in the expected sense and thereby to simplify the presentation, one can treat the expectation of the random walk as a *linear machine* [2]. Here, in each time step a pile of ℓ chips is split evenly, with ℓ/k chips going to each neighbor. By the "harmonic property" of random walks, the (possibly non-integral) number of chips at vertex $\mathbf{x}$ at time t is exactly the expected number of chips in the random walk model.

A *configuration* describes the current "state" of the linear or Propp machine. A configuration of the linear machine is a function $V \to \mathbb{R}_+$, assigning to each vertex $\mathbf{x} \in V$ its current (possibly fractional) number of chips. A configuration of the Propp machine assigns to each vertex $\mathbf{x} \in V$ its current (integral) number of chips and the current direction of the rotor.

As pointed out in the introduction, there is one limitation without which neither the results of [1–3] nor our results hold. Note that since G is a bipartite graph, chips that start on even vertices never mix with those starting on odd vertices. It looks as if we are playing two noninteracting games at once. However, this is not true, because chips at different parity vertices may affect each other through the rotors. We therefore require the initial configuration to have chips only on *one* parity. Without loss of generality, we consider only even initial configurations, i. e., chip configurations supported on vertices an even distance from the origin.

We now describe the *Propp machine* in detail. For all $\mathbf{x} \in V$ and $t \in \mathbb{N}_0$ let $f(\mathbf{x}, t)$ denote the number of chips on vertex $\mathbf{x}$ and $\textsc{arr}(\mathbf{x}, t)$ the direction of the rotor associated with $\mathbf{x}$ after t steps of the Propp machine. In other words, $f(\cdot, t)$ is the configuration function at time t. We will use $\mathbf{x} + \textsc{arr}(\mathbf{x}, t)$ to denote the node at which the current rotor of $\mathbf{x}$ is pointing at time t. (Though written additively, this operation is in fact that of the nonabelian group $\langle \{d_i\}_{i=1}^k \mid d_i^2 = 1\rangle$ generated by the set $\textsc{dir} = \{d_i\}_{i=1}^k$ of values that $\textsc{arr}(\cdot, \cdot)$ takes on; G is then a Cayley graph with generator set $\textsc{dir}$. Previous work on $\mathbb{Z}^d$ can be viewed as the same story for the free abelian group.) $\textsc{next}(\mathbf{A})$ denotes the next position of the rotor $\mathbf{A}$.

To describe the linear machine we use the same fixed initial configuration as for the Propp machine. In one step, each vertex $\mathbf{x}$ sends a $1/k$ fraction of its (possibly fractional) number of chips to each neighbor. Let $E(\mathbf{x}, t)$ denote the number of chips at vertex $\mathbf{x}$ after t steps of the linear machine. This is equal to the expected number of chips at vertex $\mathbf{x}$ after a random walk of all chips for t steps. Note that $E(\mathbf{x}, t) = \frac{1}{k}\sum_{\mathbf{A}\in\textsc{dir}} E(\mathbf{x} + \mathbf{A}, t-1)$ by definition.

A random walk on G can be described by its probability density. By $H(x, t)$ we denote the probability that a chip from a vertex with distance x to the origin arrives at the origin after t random steps ("at time t") in a simple random walk. Then,

$$H(x,t) = k^{-t} n(x,t) \tag{2.1}$$

with $n(x, t)$ counting the number of paths between two vertices at distance x on the infinite k-regular tree. It is easy to verify the following properties of $n(x, t)$:

$$\begin{aligned} n(0,0) &= 1, \\ n(x,0) &= 0 \quad \text{for all } x \geq 1, \\ n(0,t) &= kn(1,t-1) \quad \text{for all } t \geq 1, \\ n(x,t) &= n(x-1,t-1) + \\ &\qquad (k-1)n(x+1,t-1) \quad \text{for all } x,t \geq 1. \end{aligned} \tag{2.2}$$

Finally, we write $\mathbf{x} \sim t$ to mean that $|\mathbf{x}| \equiv t \pmod 2$.

3 Mod-k-forcing Theorem

For a deterministic process like the Propp machine, it is obvious that the initial configuration (that is, the location of each chip and the direction of each rotor), determines all subsequent configurations. The following theorem shows a partial converse, namely that (roughly speaking) we may prescribe the number of chips modulo k on all vertices at all times by finding an appropriate initial configuration. It can easily be proved by induction. An analogous result for the one-dimensional Propp machine has been shown in [1].

Theorem 1. (Mod-k-forcing Theorem) *For any initial direction of the rotors and any $\pi\colon V \times \mathbb{N}_0 \to \{0, 1, \ldots, (k-1)\}$ with $\pi(\mathbf{x}, t) = 0$ for all $\mathbf{x} \not\sim t$, there is an initial even configuration $f(\mathbf{x}, 0)$ that results in subsequent configurations satisfying $f(\mathbf{x}, t) \equiv \pi(\mathbf{x}, t) \pmod k$ for all $\mathbf{x}$ and $t \geq 0$.*

Proof. Analogously to the proof given in [1], we start with $f(\mathbf{x}, 0) := \pi(\mathbf{x}, 0)$ chips at location $\mathbf{x}$. Now assume that our initial (even) configuration is such that for some $T \in \mathbb{N}$ we have $f(\mathbf{x}, t) \equiv \pi(\mathbf{x}, t) \pmod k$ for all $t < T$. We modify this initial configuration by defining $f'(\mathbf{x}, 0) := f(\mathbf{x}, 0) + \varepsilon_{\mathbf{x}} k^T$ for even $\mathbf{x}$, while we have $f'(\mathbf{x}, 0) = 0$ for odd $\mathbf{x}$. Here, $\varepsilon_{\mathbf{x}} \in \{0, 1, \ldots, (k-1)\}$ are to be determined such that $f'(\mathbf{x}, t) \equiv \pi(\mathbf{x}, t) \pmod k$ for all $t \leq T$.

Observe that a pile of k^T chips splits evenly T times. Hence for all choices of the $\varepsilon_{\mathbf{x}}$, we have $f'(\mathbf{x}, t) \equiv$

$\pi(\mathbf{x},t) \pmod k$ for all $t < T$. Let $\varepsilon_{\mathbf{y}} := 0$ for all $\mathbf{y}$ with $|\mathbf{y}| < T$. By induction on $|\mathbf{x}|$, we fix the $\varepsilon_{\mathbf{x}}$ such that $f'(\mathbf{x},T) \equiv \pi(\mathbf{x},T) \pmod k$ for all $\mathbf{x}$.

Assume that for some $\theta \in \mathbb{N}_0$, the current $\varepsilon_{\mathbf{y}}$ fulfill $f'(\mathbf{x},T) \equiv \pi(\mathbf{x},T) \pmod k$ for all $\mathbf{x}$ with $|\mathbf{x}| < \theta$. For all $\mathbf{x}$ with $|\mathbf{x}| = \theta$, we choose some $\mathbf{y}$ with $|\mathbf{y}| = T + \theta$ such that the (shortest) distance between $\mathbf{x}$ and $\mathbf{y}$ is T and set $\varepsilon_{\mathbf{y}} := \big(\pi(\mathbf{x},T) - f'(\mathbf{x},T)\big) \bmod k$. For all other $\mathbf{y}$ with $|\mathbf{y}| = T+\theta$, we set $\varepsilon_{\mathbf{y}} := 0$. By the compactness principle, this yields the existence of $\varepsilon_{\mathbf{y}}$, $\mathbf{y} \in V$, such that $f'(\mathbf{x},t) \equiv \pi(\mathbf{x},t) \pmod k$ for all $t \leq T$ and $\mathbf{x} \in V$. By invoking the compactness principle again this in turn gives the sought-after initial configuration $f'(\mathbf{x},0)$. □

4 The Basic Method

In this section, we lay the foundations for our analysis of the maximal possible single-vertex discrepancy. In particular, we will see that it is possible to determine the contribution of a vertex to the discrepancy at another one independent from all other vertices.

In the following, we employ several arguments closely resembling those of [1–3]. For the moment, in addition to the notations given in Section 2, we also use the following mixed notation. By $E(\mathbf{x}, t_1, t_2)$ we denote the (possibly fractional) number of chips at location x after first performing t_1 steps with the Propp machine and then $t_2 - t_1$ steps with the linear machine.

We are interested in bounding the discrepancies $|f(\mathbf{x},t) - E(\mathbf{x},t)|$ for all vertices $\mathbf{x}$ and all times t. It suffices to consider the vertex $\mathbf{x} = \mathbf{0}$. From

$$\begin{aligned} E(\mathbf{0},0,t) &= E(\mathbf{0},t), \\ E(\mathbf{0},t,t) &= f(\mathbf{0},t), \end{aligned}$$

we obtain

$$f(\mathbf{0},t) - E(\mathbf{0},t) \;=\; \sum_{s=0}^{t-1} \left(E(\mathbf{0},s+1,t) - E(\mathbf{0},s,t)\right).$$

Let $|\mathbf{x}|$ denote the distance of a vertex $\mathbf{x}$ to $\mathbf{0}$. Now $E(\mathbf{0},s+1,t) - E(\mathbf{0},s,t) = \sum_{\mathbf{x}\in V} \sum_{\ell=1}^{f(\mathbf{x},s)} \big(H(|\mathbf{x} + \textsc{next}^{\ell-1}(\textsc{arr}(\mathbf{x},s))|, t-s-1) - H(|\mathbf{x}|, t-s)\big)$ motivates the definition of the *influence* of a Propp move (compared to a random walk move) from vertex $\mathbf{x}$ in direction $A \in \{-1,+1\}$ on the discrepancy of $\mathbf{0}$ (t time steps later) by

$$\textsc{inf}(x,A,t) := H(x+A,t-1) - H(x,t).$$

In order to ultimately reduce all $\textsc{arr}$s involved to the initial arrow settings $\textsc{arr}(\cdot,0)$, we define $s_i(\mathbf{x}) := \min\left\{u \geq 0 \mid i < \sum_{t=0}^{u} f(\mathbf{x},t)\right\}$ for all $i \in \mathbb{N}_0$. Hence at time $s_i(\mathbf{x})$ the location $\mathbf{x}$ is occupied by its i-th chip (where, to be consistent with [1], we start counting with the 0-th chip).

Consider the discrepancy at $\mathbf{0}$ at time T. Then the above yields

(4.3)

$$f(\mathbf{0},T) - E(\mathbf{0},T) = \sum_{\mathbf{x}\in V} \sum_{\substack{i\geq 0,\\ s_i(\mathbf{x})<T}} \textsc{inf}(|\mathbf{x}|, \textsc{next}^i(\textsc{arr}(\mathbf{x},0)), T - s_i(\mathbf{x})).$$

Since the inner sum of equation (4.3) will occur frequently in the remainder, let us define the *contribution* of a vertex $\mathbf{x}$ to be

$$\textsc{con}(\mathbf{x}) \;:=\; \sum_{\substack{i\geq 0,\\ s_i(\mathbf{x})<T}} \textsc{inf}(|\mathbf{x}|, \textsc{next}^i(\textsc{arr}(\mathbf{x},0)), T - s_i(\mathbf{x})),$$

where we both suppress the initial configuration leading to the $s_i(\cdot)$ as well as the run-time T.

The main result of this section, summarized in the following theorem, is that it suffices to examine each vertex $\mathbf{x}$ separately.

Theorem 2. *The discrepancy between Propp machine and linear machine after T time steps is the sum of the contributions $\textsc{con}(\mathbf{x})$ of all vertices $\mathbf{x}$, i. e.,*

$$f(\mathbf{0},T) - E(\mathbf{0},T) \;=\; \sum_{\mathbf{x}\in V} \textsc{con}(\mathbf{x}).$$

5 Divergence of the models

In this section, we analyze a specific initial configuration and show that the Propp machine may deviate from the linear machine by an arbitrarily large number of chips.

For a fixed time T at which we aim to maximize the discrepancy $f(\mathbf{0},T) - E(\mathbf{0},T)$ we examine a configuration in which all vertices $\mathbf{x}$ with $0 < |\mathbf{x}| \leq T/\lambda$ and $\lambda := \frac{k}{k-2}$ are occupied by a number of chips not divisible by k only once. We assume that at that time $T - t_{|\mathbf{x}|}$ with $t_x := \lceil \lambda x \rceil$ a chip is sent in the direction of $\mathbf{0}$. Such a configuration exists by Theorem 1. We will prove the following theorem.

Theorem 3. *For any initial direction of the rotors and any $T > 0$, there is an even initial configuration such that the single vertex discrepancy between the Propp machine and linear machine after T time steps is $\Omega(\sqrt{k\,T})$.*

By Theorem 2, the discrepancy at the origin at time T of the above described initial configuration is

(5.4)

$$\begin{aligned} f(\mathbf{0},T) - E(\mathbf{0},T) &= \sum_{\substack{\mathbf{x}\in V,\\ |\mathbf{x}|\le T/\lambda}} \big(H(|\mathbf{x}|-1, t_{|\mathbf{x}|}-1) - H(|\mathbf{x}|, t_{|\mathbf{x}|})\big) \\ &= \sum_{x=1}^{\lfloor T/\lambda\rfloor} k(k-1)^{x-1} \left(H(x-1,t_x-1) - H(x,t_x)\right) \\ &= \sum_{x=1}^{\lfloor T/\lambda\rfloor} k(k-1)^{x-1} \left(\frac{n(x-1,t_x-1)}{k^{(t_x-1)}} - \frac{n(x,t_x)}{k^{t_x}}\right) \\ &= \sum_{x=1}^{\lfloor T/\lambda\rfloor} \frac{(k-1)^{x-1}}{k^{t_x-1}} \big(k\,n(x-1,t_x-1) - n(x,t_x)\big) \\ &= \sum_{x=1}^{\lfloor T/\lambda\rfloor} \frac{(k-1)^{x-1}}{k^{t_x-1}}\, i(x,t_x) \end{aligned}$$

with

$$i(x,t) := k\,n(x-1,t-1) - n(x,t)$$

for all $x,t \ge 1$. Let us define $i(x,t) = 0$ otherwise. For $x,t \ge 2$ we get

$$\begin{aligned} i(x,t) &= k\,n(x-1,t-1) - n(x,t) \\ &= kn(x-2,t-2) + k(k-1)n(x,t-2) \\ &\quad - n(x-1,t-1) - (k-1)n(x+1,t-1) \\ &= i(x-1,t-1) + (k-1)i(x+1,t-1). \end{aligned}$$

It remains to examine the cases $(x,t) \in (\mathbb{N}\times\mathbb{N}) \setminus (\mathbb{N}_{\ge 2}\times\mathbb{N}_{\ge 2})$. Right from the definition, we get $i(1,1) = k-1$. For $x \ge 2$ and $t = 1$ we have

$$\begin{aligned} i(x,1) &= k\,n(x-1,0) - n(x,1) \\ &= 0 \\ &= i(x-1,0) + (k-1)i(x+1,0). \end{aligned}$$

Also for $x = 1$ and $t \ge 2$ we have

$$\begin{aligned} i(1,t) &= k\,n(0,t-1) - n(1,t) \\ &= (k-1)\big(n(0,t-1) - n(2,t-1)\big) \\ &= (k-1)\big(k\,n(1,t-2) - n(2,t-1)\big) \\ &= i(0,t-1) + (k-1)\,i(2,t-1). \end{aligned}$$

Summarizing the above, we see that $i(x,t)$ can be defined recursively as follows.

$$\begin{aligned} &i(x,0) = 0 \quad \text{for all } x \ge 0, \\ &i(0,t) = 0 \quad \text{for all } t \ge 0, \\ &i(1,1) = k-1, \\ &i(x,t) = i(x-1,t-1) + (k-1)i(x+1,t-1) \\ &\qquad\qquad\qquad \text{for } (x,t) \in \mathbb{N}^2_{\ge 1} \setminus \{(1,1)\}. \end{aligned}$$

This recursive view of $i(x,t)$ reveals another interpretation of these quantities. Apart from a factor $(k-1)^{(\frac{t-x}{2}+1)}$, $i(x,t)$ counts the number of lattice paths from the origin $(0,0)$ to (x,t) of steps $(+1,+1)$ and $(-1,+1)$ which do not cross the line $x = 0$. This can be described by the well-known *Ballot numbers*. The classical description is as follows. Suppose A and B are candidates for president. Let A receive a votes and B b votes with $a \ge b$. The probability that A stays ahead of B as the votes are counted is $(a-b+1)/(a+1)$ [6]. This implies that for given positive integers a, b with $a > b$, the number of lattice paths starting at the origin and consisting of a upsteps $(+1,+1)$ and b downsteps $(+1,-1)$ such that no step ends on the x-axis is $\frac{a-b}{a+b}\binom{a+b}{a}$. Therefore,

$$i(x,t) = (k-1)^{(\frac{t-x}{2}+1)}\, \frac{x}{t} \binom{t}{\frac{t+x}{2}} \tag{5.5}$$

for $x,t > 0$. Note that this can be continued to all $(x,t) \in (\mathbb{R}_+)^2$.

Equations (5.4) and (5.5) now give

(5.6)

$$\begin{aligned} f(\mathbf{0},T) - E(\mathbf{0},T) &= \sum_{x=1}^{\lfloor T/\lambda\rfloor} \frac{(k-1)^{x-1}}{k^{(t_x-1)}}\ (k-1)^{(\frac{t_x-x}{2}+1)}\, \frac{x}{t_x} \binom{t_x}{\frac{t_x+x}{2}} \\ &= \sum_{x=1}^{\lfloor T/\lambda\rfloor} \frac{(k-1)^{(\frac{t_x+x}{2})}}{k^{(t_x-1)}}\ \frac{x}{t_x} \binom{t_x}{\frac{t_x+x}{2}} \\ &> \sum_{x=1}^{\lfloor T/\lambda\rfloor} \frac{(k-1)^{\frac{\lambda+1}{2}x}}{2\lambda\, k^{\lambda x}} \binom{t_x}{\frac{t_x+x}{2}}. \end{aligned}$$

It remains to bound the binomial coefficient. From the Stirling formula we know that

$$\frac{5}{2}\sqrt{n}\left(\frac{n}{e}\right)^n < n! < \sqrt{\frac{15}{2}}\sqrt{n}\left(\frac{n}{e}\right)^n \tag{5.7}$$

for $n > 0$ and get

$$\binom{n}{k} > \frac{n^{n+1/2}}{3\,(n-k)^{n-k+1/2}\,k^{k+1/2}}. \tag{5.8}$$

for $n, k > 0$. Note that equation (5.8) also holds for the generalized binomial coefficient which is defined for nonintegral n, k by using the Gamma function Γ. The following lemma shows that inequality (5.6) also holds without the ceiling function.

Lemma 4. *$\binom{2x}{x+y}$ is monotonic nondecreasing in x for $0 \le y \le x$.*

Proof. With Ψ denoting the logarithmic derivative of the gamma function, i.e., the Digamma function, we get

$$\frac{\partial \binom{2x}{x+y}}{\partial x} = \big(2\,\Psi(2x+1) - \Psi(x+y+1) - \Psi(x-y+1)\big)\binom{2x}{x+y}.$$

The lemma follows by the fact that above binomial coefficient is positive for $x, y \geq 0$ and that $\Psi(x)$ is monotonic nondecreasing for $x > 0$ ([5, Thm. 7]). □

By the definitions of λ and t_x we now get with Lemma 4

$$\begin{aligned}\binom{t_x}{(t_x+x)/2} &\geq \binom{\lambda x}{\frac{\lambda+1}{2}x} \\ &= \binom{\frac{k}{k-2}x}{\frac{1}{k-2}x} \\ &> \frac{k^{\frac{k}{k-2}x+\frac{1}{2}}\,(k-2)^{\frac{1}{2}}}{3\,(k-1)^{\frac{k-1}{k-2}x+\frac{1}{2}}\,\sqrt{x}}\end{aligned}$$

for $x > 0$, $k > 2$.

Using this we obtain for all $k > 2$

$$\begin{aligned}f(\mathbf{0},T) - E(\mathbf{0},T) &> \sum_{x=1}^{\lfloor T/\lambda \rfloor} \frac{(k-2)^{\frac{3}{2}}}{6\,k^{\frac{1}{2}}\,(k-1)^{\frac{1}{2}}\,\sqrt{x}} \\ &= \Omega(\sqrt{k\,T}).\end{aligned}$$

Using the same arguments as the following section one can also prove that for all even initial configurations the single vertex discrepancy after T time steps is at most $O(\sqrt{kT})$.

6 Convergence of the models

The previous section showed that for very special configurations the single vertex discrepancy can be unbounded. In this section we show that, on the other hand, many configurations have a bounded discrepancy. We will sketch the proof of the following theorem.

Theorem 5. *If $f(\mathbf{x},t) \equiv 0 \pmod{k}$ for all $\mathbf{x}$ and t such that $(1-\varepsilon)\lambda|\mathbf{x}| < T - t < (1+\varepsilon)\lambda|\mathbf{x}|$ with $\lambda := \frac{k}{k-2}$, then the discrepancy between Propp machine and linear machine at time T and vertex 0 is bounded by a constant depending only on $\varepsilon > 0$.*

Our aim is to bound equation (4.3). To that end, we further examine INF. From its definition and equations (5.5) and (2.2) we know for $x, t > 0$

$$\begin{aligned}\mathrm{INF}(x,-1,t) &= \frac{i(x,t)}{k^t}, \\ \mathrm{INF}(x,+1,t) &= \frac{k\,n(x+1,t-1) - n(x,t)}{k^t} \\ &= \frac{\frac{1}{k-1}n(x,t) - \frac{k}{k-1}n(x+1,t-1)}{k^t} \\ &= \frac{-i(x,t)}{(k-1)\,k^t}.\end{aligned}$$

This also shows

$$\mathrm{INF}(x,-1,t) + (k-1)\,\mathrm{INF}(x,1,t) = 0. \tag{6.9}$$

Therefore, the absolute value of the influence $|\mathrm{INF}(x,A,t)|$ of sending one chip *towards* $\mathbf{0}$ is $(k-1)$ times larger than sending one chip in the opposite direction.

Note that

$$\mathrm{CON}(\mathbf{x}) = \sum_{\substack{i \geq 0, \\ s_i(\mathbf{x}) < T}} A^{(i)}\,\frac{i(|\mathbf{x}|,t_i)}{k^{t_i}}$$

with

$$\begin{aligned}t_i &:= T - s_i(\mathbf{x}) \\ A^{(i)} &:= \begin{cases}\frac{-1}{k-1} & \text{for } \mathrm{NEXT}^i(\mathrm{ARR}(\mathbf{x},0)) = +1 \\ 1 & \text{for } \mathrm{NEXT}^i(\mathrm{ARR}(\mathbf{x},0)) = -1.\end{cases}\end{aligned}$$

To bound this alternating sum, we use the following elementary fact.

Lemma 6. *Let $f\colon X \to \mathbb{R}$ be non-negative and monotone nondecreasing with $X \subseteq \mathbb{R}$. Let $A^{(0)}, \ldots, A^{(n)} \in \mathbb{R}$ and $t_0, \ldots, t_n \in X$ such that $t_0 \leq \ldots \leq t_n$ and $|\sum_{i=a}^{b} A^{(i)}| \leq 1$ for all $0 \leq a \leq b \leq n$. Then*

$$\left|\sum_{i=0}^{n} A^{(i)} f(t_i)\right| \leq \max_{x \in X} f(x).$$

Let $X \subseteq \mathbb{R}$. We call a mapping $f\colon X \to \mathbb{R}$ *unimodal*, if there is a $t_1 \in X$ such that $f|_{x \leq t_1}$ as well as $f|_{x \geq t_1}$ are monotone. The following lemma shows that $\mathrm{INF}(x,A,t)$ is unimodal.

Lemma 7. *The function $i(x,t)/k^t$ is unimodal in t for all x and A, and it is maximized over all $t \in \mathbb{R}_+$ at $t_{\max}(x) = \lambda\,x + O(1)$.*

Proof. The lemma holds for general $t \in \mathbb{R}_+$. However, to avoid longer calculations we only prove it for $t \in \mathbb{N}$.

Using $\binom{n+2}{k+1} = \frac{n^2+3n+2}{(k+1)\,(n-k+1)} \binom{n}{k}$ we get

$$\frac{i(x,t+2)}{k^{t+2}} - \frac{i(x,t)}{k^t} = \frac{(k-1)^{\frac{t-x}{2}}\, x\, p_x(t)}{(t+x+2)\,(t-x+2)\,k^{t+2}\,t} \binom{t}{\frac{t+x}{2}}$$

with $p_x(t) := -(k^3-5k^2+8k-4)t^2 - (4k^3-8k^2+8k-4)t + (k^3-k^2)(x^2-4)$. Hence, the above difference is non-negative if

$$t \geq \left(\sqrt{8k^3-4k^2-8k+4+k^2\,(k-2)^2\,x^2} - 2k^2+2k-2\right) / \,(k-2)^2 \tag{6.10}$$

and non-positive otherwise. Thus we have unimodality with $i(x,t)/k^t$ taking its maximum when t is the smallest integer having the same parity as x and satisfying equation (6.10). □

Armed with Lemmas 6 and 7, we can now prove Theorem 5.

Proof. [Proof of Thm. 5] By Lemmas 6 and 7 we know

$$\text{CON}(\mathbf{x}) \leq \frac{i(|\mathbf{x}|,(1-\varepsilon)\lambda|\mathbf{x}|)}{k^{(1-\varepsilon)\lambda|\mathbf{x}|}} + \frac{i(|\mathbf{x}|,(1+\varepsilon)\lambda|\mathbf{x}|)}{k^{(1+\varepsilon)\lambda|\mathbf{x}|}}$$

By Theorem 2 it therefore remains to show that

$$\begin{aligned} &f(\mathbf{0},T) - E(\mathbf{0},T) \\ &\quad \leq \sum_{x>0} \left(\frac{(k-1)^{x-1}}{k^{(1-\varepsilon)\lambda x-1}}\, i(x,(1-\varepsilon)\lambda x) \right. \\ &\qquad\qquad \left. + \frac{(k-1)^{x-1}}{k^{(1+\varepsilon)\lambda x-1}}\, i(x,(1+\varepsilon)\lambda x) \right) \end{aligned}$$

is bounded. By equation (5.5),

$$\begin{aligned} &\frac{(k-1)^{x-1}}{k^{(1+\varepsilon)\lambda x-1}}\, i(x,(1+\varepsilon)\lambda x) \\ &\quad = \frac{(k-1)^{((1+\varepsilon)\lambda+1)x/2}\, x}{k^{(1+\varepsilon)\lambda x-1}((1+\varepsilon)\lambda x)} \binom{(1+\varepsilon)\lambda x}{((1+\varepsilon)\lambda+1)\,x/2} \end{aligned}$$

To bound the binomial coefficient, we again use equation (5.7) and get

$$\binom{n}{k} < \frac{n^{n+1/2}}{2\,(n-k)^{n-k+1/2}\,k^{k+1/2}}.$$

Therefore,

$$\begin{aligned} &\frac{(k-1)^{x-1}}{k^{(1+\varepsilon)\lambda x-1}}\, i(x,(1+\varepsilon)\lambda x) \\ &\quad < \sqrt{\frac{k\,(k-2)^3}{(1+\epsilon)(k\epsilon+2)(2\,k-2+k\epsilon)}}\, p(\varepsilon)^{\frac{x}{k-2}} \end{aligned}$$

with

$$\begin{aligned} p(\varepsilon) :=& \frac{2k-2+k\varepsilon}{(k\varepsilon+2)(k-1)} \left(\frac{k-1}{(k\varepsilon+2)(2k-2+k\varepsilon)}\right)^{k\varepsilon/2} \\ & \left(\frac{(k-1)\,(2+2\varepsilon)^{\varepsilon+1}}{2k-2+k\varepsilon}\right)^k \end{aligned}$$

As $0 < p(\varepsilon) < 1$ for all $\varepsilon > 0$, $k \geq 3$, we have shown that $\sum_{x>0} \frac{(k-1)^{x-1}}{k^{(1+\varepsilon)\lambda x-1}}\, i(x,(1+\varepsilon)\lambda x)$ is bounded above by a constant depending on $\varepsilon > 0$. The same arguments demonstrate that $\sum_{x>0} \frac{(k-1)^{x-1}}{k^{(1-\varepsilon)\lambda x-1}}\, i(x,(1-\varepsilon)\lambda x)$ can also be bounded above by a constant depending on $\varepsilon > 0$, which finishes the proof. □

7 Conclusion

In this paper we succeeded in showing that k-ary trees ($k \geq 3$) do not admit a constant bound on the single vertex discrepancies (as was observed by Cooper and Spencer [2] for higher-dimensional grids). It remains an open problem to exhibit broader graph classes displaying the one or the other behavior, or to give a characterization of those graphs having constant bounds for the single vertex discrepancies.

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