Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Discrete Mathematics 29 (2007) 509–513

Electronic Notes in DISCRETE MATHEMATICS

www.elsevier.com/locate/endm

Deterministic Random Walks on Regular Trees

Joshua Cooper[a] Benjamin Doerr[b]
Tobias Friedrich[b] Joel Spencer[c]

[a] *LeConte College, Department of Mathematics, University of South Carolina, 1523 Greene Street, Columbia, SC 29208, USA*

[b] *Max-Planck-Institut für Informatik, Stuhlsatzenhausweg 85, 66123 Saarbrücken, Germany*

[c] *Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012-1185, USA*

Abstract

Jim Propp's rotor router model is a deterministic analogue of a random walk on a graph. Instead of distributing chips randomly, each vertex serves its neighbors in a fixed order. The difference between the Propp machine and a random walk has been analyzed on infinite d-dimensional grids. There, apart from a technicality, independent of the starting configuration, at each time, the number of chips on each vertex in the Propp model deviates from the expected number of chips in the random walk model by at most a constant. We show that this is not the case for the k-regular tree ($k \geq 3$), i.e., there is a starting configurations on which both models deviate by an arbitrarily large number of chips.

Keywords: random walk, rotor-router model, chip firing games, discrepancy

1 Introduction

The rotor-router model is a simple deterministic process suggested by Jim Propp. It can be viewed as an attempt to derandomize random walks on

doi:10.1016/j.endm.2007.07.082

graphs. So far, the "Propp machine" has been studied primarily on infinite grids $\mathbb{Z}^d$. There, each vertex $x \in \mathbb{Z}^d$ is equipped with a "rotor" together with a cyclic permutation (called a "rotor sequence") of the $2d$ cardinal directions of $\mathbb{Z}^d$. While a chip (particle, coin, ...) performing a random walk departs a vertex in a random direction, in the Propp model it always goes in the direction the rotor is pointing. After a chip is sent, the rotor is rotated according to the fixed rotor sequence. This rule ensures that chips are distributed quite evenly among the neighbors of a vertex.

Cooper and Spencer [1] compared the Propp machine and the random walk in terms of *single vertex discrepancy*. Apart from a technicality, they place arbitrary numbers of chips on the vertices. Then they run the Propp machine on this initial configuration for a certain number of rounds. A round consists of each chip (in arbitrary order) performing one move as directed by the Propp machine. In the consequent chip arrangement, they compare the number of chips at each vertex that the Propp machine places there with the expected number of chips that a random walk in same number of rounds and the same initial configuration places there. Cooper and Spencer showed that for all grids $\mathbb{Z}^d$, these differences can be bounded by a constant c_d independent of the initial setup (in particular, the total number of chips) and the run-time.

In this paper, we rigorously analyze the Propp machine on the infinite k-regular tree. We show that there are configurations such that the single vertex discrepancy is arbitrarily large for $k \geq 3$. This complements the results of Cooper, Doerr, Spencer, and Tardos [2], where the case of $k = 2$ (the infinite line) is analyzed.

2 Preliminaries

To bound the single vertex discrepancy between the Propp machine and a random walk on the k-regular tree we first introduce several requisite definitions and notational conventions. Let $G = (V, E)$ be the infinite k-regular tree, also known as the "Cayley tree" and the "Bethe lattice". We fix an arbitrary node to be its *origin* $\mathbf{0}$. $|\mathbf{x}|$ denotes the shortest (i.e., ordinary graphical) distance between the origin and vertex $\mathbf{x}$.

In order to avoid discussing all equations in the expected sense and thereby to simplify the presentation, one can treat the expectation of the random walk as a *linear machine* [1]. Here, in each time step a pile of ℓ chips is split evenly, with ℓ/k chips going to each neighbor. By the "harmonic property" of random walks, the (possibly non-integral) number of chips at vertex $\mathbf{x}$ at time t is exactly the expected number of chips in the random walk model.

A *configuration* describes the current "state" of the linear or Propp machine. A configuration of the linear machine is a function $V \to \mathbb{R}_+$, assigning to each vertex $\mathbf{x} \in V$ its current (possibly fractional) number of chips. A configuration of the Propp machine assigns to each vertex $\mathbf{x} \in V$ its current (integral) number of chips and the current direction of the rotor.

As pointed out in the introduction, there is one limitation without which neither the results of [1–3] nor our results hold. Note that since G is a bipartite graph, chips that start on even vertices never mix with those starting on odd vertices. It looks as if we are playing two noninteracting games at once. However, this is not true, because chips at different parity vertices may affect each other through the rotors. We therefore require the initial configuration to have chips only on *one* parity. Without loss of generality, we consider only even initial configurations, i.e., chip configurations supported on vertices an even distance from the origin.

We now describe the *Propp machine* in detail. For all $\mathbf{x} \in V$ and $t \in \mathbb{N}_0$ let $f(\mathbf{x}, t)$ denote the number of chips on vertex $\mathbf{x}$ and $\textsc{arr}(\mathbf{x}, t)$ the direction of the rotor associated with $\mathbf{x}$ after t steps of the Propp machine. In other words, $f(\cdot, t)$ is the configuration function at time t. We will use $\mathbf{x} + \textsc{arr}(\mathbf{x}, t)$ to denote the node at which the current rotor of $\mathbf{x}$ is pointing at time t. (Though written additively, this operation is in fact that of the nonabelian group $\langle \{d_i\}_{i=1}^k \,|\, d_i^2 = 1 \rangle$ generated by the set $\textsc{dir} = \{d_i\}_{i=1}^k$ of values that $\textsc{arr}(\cdot, \cdot)$ takes on; G is then a Cayley graph with generator set $\textsc{dir}$. Previous work on $\mathbb{Z}^d$ can be viewed as the same story for the free abelian group.) $\textsc{next}(\mathbf{A})$ denotes the next position of the rotor $\mathbf{A}$.

To describe the linear machine we use the same fixed initial configuration as for the Propp machine. In one step, each vertex $\mathbf{x}$ sends a $1/k$ fraction of its (possibly fractional) number of chips to each neighbor. Let $E(\mathbf{x}, t)$ denote the number of chips at vertex $\mathbf{x}$ after t steps of the linear machine. This is equal to the expected number of chips at vertex $\mathbf{x}$ after a random walk of all chips for t steps. Note that $E(\mathbf{x}, t) = \frac{1}{k} \sum_{\mathbf{A} \in \textsc{dir}} E(\mathbf{x} + \mathbf{A}, t - 1)$ by definition.

A random walk on G can be described by its probability density. By $H(x, t)$ we denote the probability that a chip from a vertex with distance x to the origin arrives at the origin after t random steps ("at time t") in a simple random walk. Then,

$$H(x, t) = k^{-t} n(x, t) \tag{1}$$

with $n(x, t)$ counting the number of paths between two vertices at distance x on the infinite k-regular tree. It is easy to describe $n(x, t)$ with some recursive equations.

Finally, we write $\mathbf{x} \sim t$ to mean that $|\mathbf{x}| \equiv t \pmod 2$.

3 Some Proof Ideas

Due to space limitations, it is impossible even to sketch the proofs of the results stated above. We therefore focus on particular aspects.

For a deterministic process like the Propp machine, it is obvious that the initial configuration (that is, the location of each chip and the direction of each rotor), determines all subsequent configurations. The following theorem shows a partial converse, namely that (roughly speaking) we may prescribe the number of chips modulo k on all vertices at all times by finding an appropriate initial configuration. An analogous result for the one-dimensional Propp machine has been shown in [2].

Theorem 3.1 (Mod-k-forcing Theorem) *For any initial direction of the rotors and any* $\pi \colon V \times \mathbb{N}_0 \to \{0, 1, \ldots, (k-1)\}$ *with* $\pi(\mathbf{x}, t) = 0$ *for all* $\mathbf{x} \not\sim t$*, there is an initial even configuration* $f(\mathbf{x}, 0)$ *that results in subsequent configurations satisfying* $f(\mathbf{x}, t) \equiv \pi(\mathbf{x}, t) \pmod k$ *for all* $\mathbf{x}$ *and* $t \geq 0$*.*

The discrepancy on $\mathbf{0}$ at time T is determined by what has happened t steps before time T at all vertices $\mathbf{x}$ with $|\mathbf{x}| < t$. This motivates the definition of the *influence* of a Propp move (compared to a random walk move) from a vertex with distance x in the direction of $\mathbf{y}$ on the discrepancy of $\mathbf{0}$ (t time steps later) by

$$\textsc{inf}(x, t) := H(x-1, t-1) - H(x, t).$$

Note that if the number of chips on a vertex $\mathbf{x}$ at some time t is a multiple of k, the Propp machine distributes the chips to the respective neighbors exactly as the linear machine and therefore the sum of the influences of the Propp moves of all chips at this vertex $\mathbf{x}$ at time t is zero.

It is easy to verify the following properties of $H(x, t)$ as defined in equation (1):

$$\begin{aligned} H(0,0) &= 1, \\ H(x,0) &= 0 \quad \text{for all } x \geq 1, \\ H(0,t) &= H(1,t-1) \quad \text{for all } t \geq 1, \\ H(x,t) &= \tfrac{1}{k}\, H(x-1,t-1) + \tfrac{k-1}{k}\, H(x+1,t-1) \quad \text{for all } x, t \geq 1. \end{aligned}$$

Unfortunately, there is no simple closed-form expression for $H(x, t)$. However, a closer look on $\textsc{inf}(x, t)$ reveals a similar recursive definition, which can be solved with the help of the well-known Ballot numbers. This yields for $x, t \geq 1$

$$\textsc{inf}(x, t) = \frac{(k-1)^{(\frac{t-x}{2}+1)}}{k^t} \, \frac{x}{t} \binom{t}{\frac{t+x}{2}}. \tag{2}$$

For a fixed time T at which we aim to maximize the discrepancy $f(\mathbf{0}, T) - E(\mathbf{0}, T)$ we examine a configuration which sends exactly one odd chip from all vertices $\mathbf{x}$ with $0 < |\mathbf{x}| \leq T/\lambda$ and $\lambda := \frac{k}{k-2}$. This chip is sent in the direction of $\mathbf{0}$ at time $T - t_{|\mathbf{x}|}$ with $t_x := \lceil \lambda x \rceil$. We further assume that no other odd chips are sent. Such a configuration exists by Theorem 3.1. Using equation (2) we can prove the following theorem.

Theorem 3.2 *There is an even initial configuration such that the single vertex discrepancy between the Propp machine and linear machine after T time steps is $\Omega(\sqrt{kT})$.*

The configurations assured by Theorem 3.2 appear to be enormous. To achieve a discrepency of D they have a number of chips which is possibly hyperexponential in D^2. Furthermore, “most” configuration have a bounded discepancy. To prove this, the crucial step is to show that $\textsc{inf}(x, \cdot)$ is unimodal. This implies that the discrepancy can already be maximized if there is only a single time t for each vertex $\mathbf{x}$ at which the number of chips is not divisible by k. Some further observations give the following theorem.

Theorem 3.3 *If $f(\mathbf{x}, t) \equiv 0 \pmod{k}$ for all $\mathbf{x}$ and t such that $(1 - \varepsilon)\lambda|\mathbf{x}| < T - t < (1 + \varepsilon)\lambda|\mathbf{x}|$ with $\lambda := \frac{k}{k-2}$, then the discrepancy between Propp machine and linear machine at time T and vertex 0 is bounded by a constant depending only on $\varepsilon > 0$.*

References

[1] Joshua Cooper and Joel Spencer. Simulating a random walk with constant error. *Combinatorics, Probability and Computing.* To appear, preliminary version available from arXiv:math/0402323.

[2] Joshua Cooper, Benjamin Doerr, Joel Spencer, and Gábor Tardos. Deterministic random walks on the integers. *European Journal of Combinatorics.* To appear, preliminary version available from arXiv:math/0602300.

[3] Benjamin Doerr and Tobias Friedrich. Deterministic random walks on the two-dimensional grid. Submitted, preliminary version available from arXiv:math/0703453.