A Polynomial Kernel for Bipartite Permutation Vertex Deletion

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Abstract

In a permutation graph, vertices represent the elements of a permutation, and edges represent pairs of elements that are reversed by the permutation. In the PERMUTATION VERTEX DELETION problem, given an undirected graph G and an integer k, the objective is to test whether there exists a vertex subset $S \subseteq V(G)$ such that $|S| \leq k$ and $G - S$ is a permutation graph. The parameterized complexity of PERMUTATION VERTEX DELETION is a well-known open problem. Bożyk et al. [IPEC 2020] initiated a study on this problem by requiring that $G - S$ be a bipartite permutation graph (a permutation graph that is bipartite). They called this the BIPARTITE PERMUTATION VERTEX DELETION (BPVD) problem. They showed that the problem admits a factor 9-approximation algorithm as well as a fixed parameter tractable (FPT) algorithm running in time $\mathcal{O}(9^k |V(G)|^9)$. Moreover, they posed the question *whether* BPVD *admits a polynomial kernel.* We resolve this question in the affirmative by designing a polynomial kernel for BPVD. In particular, we obtain the following: Given an instance (G, k) of BPVD, in polynomial time we obtain an equivalent instance (G', k') of BPVD such that $k' \leq k$, and $|V(G')| + |E(G')| \leq k^{\mathcal{O}(1)}$.

Keywords Kernelization · Bipartite permutation graph · Bicliques

1 Introduction

In a *graph modification problem*, the input consists of an *n*-vertex graph G and an integer k. The objective is to determine whether k *modification operations*—such as

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vertex deletions, or edge deletions, insertions, or contractions—are sufficient to obtain a graph with prescribed structural properties such as being planar, bipartite, chordal, interval, acyclic or edgeless. Graph modification problems include some of the most basic problems in graph theory and graph algorithms. Unfortunately, most of these problems are NP-complete [26, 34]. Therefore, they have been studied intensively within various algorithmic paradigms for coping with NP-completeness [14, 17, 28], including approximation algorithms, parameterized complexity, and algorithms for restricted input classes.

Graph modification problems have played a central role in the development of parameterized complexity. Here, the number of allowed modifications, k, is considered a *parameter*. With respect to k, we seek a *fixed-parameter tractable* (FPT) algorithm, namely, an algorithm whose running time has the form $f(k)n^{\mathcal{O}(1)}$ for some computable function f. One way to obtain such an algorithm is to exhibit a *kernelization algorithm*, (or *kernel*, for short). A kernel for a graph problem Π is an algorithm that, given an instance (G, k) of Π, runs in polynomial time and outputs an equivalent instance (G', k') of Π such that $|V(G')|$ and k' are upper bounded by $f(k)$ for some computable function f. The function f is called the *size* of the kernel, and if f is a polynomial function, then we say that the kernel is a *polynomial kernel*. A kernel for a problem immediately implies that it admits an FPT algorithm, but kernels are also interesting in their own right. In particular, kernels allow us to model the performance of polynomial time pre-processing algorithms. The field of kernelization has received considerable attention, especially after the introduction of the methods for proving kernelization lower bounds [3, 7, 8, 11, 16, 20, 21]. We refer to the surveys [15, 19, 25, 27], as well as the books [6, 10, 12, 31], for a detailed treatment of the area of kernelization. In this paper, we study the kernelization complexity of the following problem.

BIPARTITE PERMUTATION VERTEX DELETION (BPVD)   **Parameter:** k
Input: A graph G and an integer k.
Question: Does there exist a subset $S \subseteq V(G)$ of size at most k such that $G - S$ is a bipartite permutation graph?

A graph G is a *permutation graph* if the vertices represent the elements of a permutation, and edges represent pairs of elements that are reversed by the permutation. Alternatively, a *permutation graph* can be defined as an intersection graph of line segments whose endpoints lie on two parallel lines $\mathcal{L}_1$ and $\mathcal{L}_2$, with one endpoint of each line segment lying on $\mathcal{L}_1$ and the other endpoint on $\mathcal{L}_2$. Due to their intriguing combinatorial properties and modeling power, the class of permutation graphs is one of the well-studied graph classes [5, 18]. As a subclass of perfect graphs, many problems that are NP-complete on general graphs can be solved efficiently on permutation graphs, such as CLIQUE, INDEPENDENT SET, CHROMATIC NUMBER, TREEWIDTH and PATHWIDTH. Further, there is a linear time algorithm to test whether a given graph is a permutation graph and, if so, construct a permutation representing it [30]. Whether PERMUTATION VERTEX DELETION admits an FPT algorithm has been a longstanding open problem in the area. In order to make progress on this open problem, recently, Bożyk et al. [4] studied the problem of deleting vertices to a subclass of permutation

graphs. The subclasses of permutation graphs include the classes of bipartite permutation graphs (characterized by Spinrad, Brandstädt & Stewart 1987 [32]) and cographs. While the fixed-parameter tractability of vertex deletion to cographs follows easily because of the finite forbidden characterization (as induced subgraphs) of cographs, no such result was known for vertex deletion to bipartite permutation graphs. Bożyk et al. [4] studied BPVD and showed that the problem admits a factor 9-approximation algorithm as well as an FPT algorithm running in time $\mathcal{O}(9^k n^9)$. A natural follow-up question to this work, explicitly asked in [4], is whether BPVD admits a polynomial kernel. In this paper, we resolve this question in the affirmative.

Theorem 1 BIPARTITE PERMUTATION VERTEX DELETION *admits a polynomial kernel.*

1.1 Methods

Our kernelization heavily uses the characterization of bipartite permutation graphs in terms of their *forbidden induced subgraphs*, also called *obstructions*. Specifically, a graph H is an obstruction to the class of bipartite permutation graphs if H is not a bipartite permutation graph and $H - \{v\}$ is a bipartite permutation graph for every vertex $v \in V(H)$. A graph G is a bipartite permutation graph if and only if it does not contain any obstruction as an induced subgraph. The set of obstructions to bipartite permutation graphs has been completely characterized by Spinrad et al. [32]. It consists of T_2, X_2, X_3, K_3, as well two infinite families of graphs: even cycles of length at least 6, and odd cycles of length at least 5 (see Fig. 1). We call any obstruction of size less than 45 a *small obstruction*, and call all other obstructions *large obstructions*. Note that every large obstruction is a hole (induced cycle) of length at least 45.

The first ingredient of our kernelization algorithm is the factor 9 polynomial time approximation algorithm for BPVD by Bożyk et al. [4]. We use this algorithm to obtain an approximate solution of size at most $9k$ or conclude that no solution of size at most k exists. We grow this approximate solution to a solution T of size $\mathcal{O}(k^{45})$, such that every set $Y \subseteq V(G)$ of size at most k is a minimal hitting set for small obstructions in G if and only if Y is a minimal hitting set for small obstructions in the graph induced on T. Once we have T (also called a modulator), we know that $G - T$ is a bipartite permutation graph. Let S be a minimal (or minimum) solution of size at most k. Then,
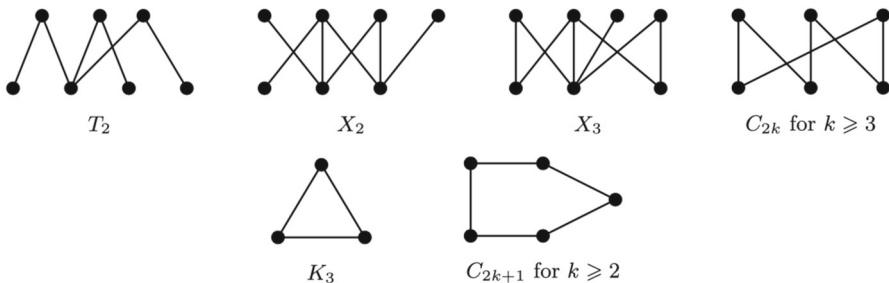

Fig. 1 The set of obstructions for a bipartite permutation graph (Figure from [4])
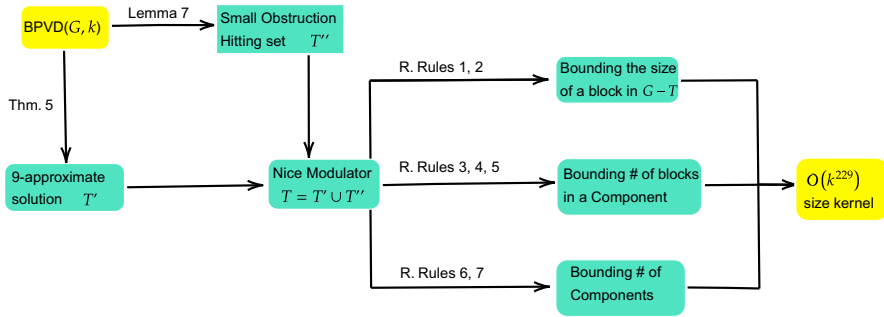
Fig. 2 An overview of the kernalization algorithm

the only purpose of vertices in $S \cap (V(G) \setminus T)$ is to hit large obstructions. Next, we analyze the graph $G - T$ and reduce its size by applying various reduction rules.

For the kernelization algorithm, we look at $G - T$ and focus on *one* connected component of $G - T$. Since $G - T$ is a bipartite permutation graph, it has a "complete bipartite decomposition" [33]. For our kernelization purpose, we heavily use this known decomposition. A *biclique* or a *complete bipartite graph* is a bipartite graph where every vertex of the first part is adjacent to every vertex of the second part. We give a semi-formal definition of a complete bipartite decomposition [33]. Let $H = G - T$. A complete bipartite decomposition of H is an ordered partition $(Q_1, R_1, Q_2, R_2, \ldots, Q_s, R_s)$, of $V(H)$, where each part is either a biclique or an independent set, and vertices of each part have neighbours only (within the part and) in the two immediately preceding and succeeding parts. The complete bipartite decomposition is similar to the clique partition used by Ke et al. [24] for designing a polynomial kernel for vertex deletion to proper interval graphs.

In the first phase, we bound the maximum biclique size in $G - T$, i.e., the size of Q_i for $i \in [s]$. Our biclique-reduction procedure builds upon the clique-reduction procedure of Marx [29], which was used in the kernelizations for CHORDAL VERTEX DELETION [1, 22] and INTERVAL VERTEX DELETION [2]. The procedure of Marx [29], as well as our procedure, are based on an "irrelevant vertex rule". In particular, we find a vertex that is unnecessary for a solution of size at most k and delete it. Moreover, after this procedure, we reduce the size of each biclique in $G - T$ by $k^{\mathcal{O}(1)}$. Next, using a simple marking procedure, we bound the size of R_i for $i \in [s]$ as well.

In the second phase, we bound the size of the connected component of $G - T$ we started with. Towards this, we first bound the number of bicliques in $Q_1, Q_2, \ldots, Q_t$ that contain a neighbor of a vertex in T (say *good bicliques*). We use small obstructions and, in particular, T_2 (the subdivided claw) and K_3 (the triangle) to bound the number of good bicliques by $k^{\mathcal{O}(1)}$. This automatically divides the biclique partition into *chunks*. Mark all the good bicliques. A maximal set of unmarked bicliques between two marked bicliques form a chunk. It is clear that the number of chunks is upper bounded by $k^{\mathcal{O}(1)}$. Finally, we use structural analysis to bound the size of each chunk, which includes the design of a reduction rule that computes a minimum cut between the two good bicliques that border the chunk. In particular, we show that each chunk can be replaced by a graph of size $k^{\mathcal{O}(1)}$. We remark that the procedure also needs to

handle the presence of independent sets $R_1, R_2, \ldots, R_s$, which we have completely ignored in the discussion.

Until now, we have assumed that $G - T$ is connected. Finally, again using the obstructions T_2 and K_3, we show that the number of connected components in $G - T$ is upper bounded by $k^{\mathcal{O}(1)}$. Using this bound, together with the facts that $|T| \leq k^{\mathcal{O}(1)}$, and that each connected component is of size $k^{\mathcal{O}(1)}$, we can deduce our polynomial kernel for BPVD.

2 Preliminaries

In this section, we define some notations and list some properties of bipartite permutation graphs.

Standard Notation: For a positive integer n, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$. For a graph G, $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively. Two vertices u, v are said to be *adjacent* if there is an edge (denoted as uv) between u and v. Given vertex subsets $X, Y \subseteq V(G)$, such that $X \cap Y = \emptyset$, $E(X, Y)$ denotes the set of edges with one endpoint in X and the other in Y. The neighborhood of a vertex v, denoted by $N_G(v)$, is the set of vertices adjacent to v. The subscript in the notation for the neighborhood is omitted if the graph under consideration is clear. For a set $M \subseteq V(G)$ and a vertex $u \in V(G)$, by $M(u)$ we denote $N(u) \cap M$. For a set $S \subseteq V(G)$, $G - S$ denotes the graph obtained by deleting S from G and $G[S]$ denotes the subgraph of G induced on S. A *path* $P = v_1, \ldots, v_\ell$ is a sequence of distinct vertices where every consecutive pair of vertices is adjacent. We say that P *starts* at v_1 and *ends* at v_ℓ. The vertices (or vertex set) of P, denoted by $V(P)$, is the set $\{v_1, \ldots, v_\ell\}$. The *endpoints* of P is the set $\{v_1, v_\ell\}$ and the *internal vertices* of P is the set $V(P) \setminus \{v_1, v_\ell\}$. The *length* of P is defined as $|V(P)|$. A *cycle* is a sequence $v_1, \ldots, v_\ell$ of vertices such that $v_1, \ldots, v_\ell$ is a path and $v_\ell v_1$ is an edge. A set $Q \subseteq V(G)$ of pairwise adjacent vertices in G is called a *clique*. For graph theoretic terms and definitions not stated explicitly here, we refer to [9].

2.1 Bipartite Permutation Graph

The characterization of bipartite permutation graphs presented below was proposed by Spinrad et al. [32]. Let G be a connected bipartite graph with vertex bipartition (A, B). A linear order $(B, <_B)$ satisfies the *adjacency property* if for each vertex $u \in A$ the set $N(u)$ consists of vertices that are consecutive in $(B, <_B)$. A linear order $(B, <_B)$ satisfies the *enclosure property* if for every pair of vertices $u, u' \in A$ such that $N(u)$ is a subset of $N(u')$, vertices in $N(u') \setminus N(u)$ occur consecutively in $(B, <_B)$. A *strong ordering* of the vertices of $A \cup B$ consists of linear orders $(A, <_A)$ and $(B, <_B)$ such that for every $(u, w'), (u', w)$ in $E(G)$, where u, u' are in A and w, w' are in B, $u <_A u'$ and $w <_B w'$ imply that $(u, w) \in E(G)$ and $(u', w') \in E(G)$. Note that whenever $(A, <_A)$ and $(B, <_B)$ form a strong ordering of $A \cup B$, then $(A, <_A)$ and $(B, <_B)$ satisfy the adjacency and enclosure properties.

Theorem 2 ([32]) *The following three statements are equivalent for a connected bipartite graph $G = (A, B, E)$:*

1. *(A, B, E) is a bipartite permutation graph.*
2. *There exists a strong ordering of $A \cup B$.*
3. *There exists a linear order $(B, <_B)$ of B satisfying adjacency and enclosure properties.*

Notation on ordering: Let G be a bipartite permutation graph with a vertex bipartition, say (A, B), of G. Fix a strong ordering, say π, of (A, B). Let π_A and π_B be the restriction of π on A and B, respectively, that is, π_A and π_B are linear orderings of the vertices of A and B. For $X \in \{A, B\}$ and a pair of vertices $x, y \in X$, we say $x <_{\pi_X} y$ if x appears before y in the ordering π_X. Similarly, for $X \in \{A, B\}$ and $Y, Y' \subseteq X$, we say $Y <_\pi Y'$ if $y <_{\pi_X} y'$ for every $y \in Y$ and $y' \in Y'$. More generally, for $Y, Y' \subseteq A \cup B$, we write $Y <_\pi Y'$ if $Y \cap A <_\pi Y' \cap A$ and $Y \cap B <_\pi Y' \cap B$. For $X \in \{A, B\}$, a set $Y \subseteq X$ and an integer q, where $1 \le q \le |Y|$, we write F_q^Y to denote the first q vertices of Y in the ordering π_X. Similarly, we write L_q^Y to denote the last q vertices of Y in the ordering π_X.

2.2 Complete Bipartite Decomposition

We start by defining the notion of complete bipartite decomposition.

Definition 3 (*Complete Bipartite Decomposition* [33]) Consider a bipartite permutation graph G with vertex bipartition (A, B) and a strong ordering π of (A, B). A sequence of vertex subsets $(Q_1, R_1, Q_2, R_2, \ldots, Q_s, R_s)$, where $Q_i, R_i \subseteq V(G)$ for every $i \in [s]$, is said to be a *complete bipartite decomposition* of G if the following properties hold:

1. $\{Q_1, R_1, Q_2, R_2, \ldots, Q_s, R_s\}$ is a partition of $V(G)$.
2. For every $i \in [s]$, $G[Q_i]$ is a biclique whose edge set is non-empty.
3. For every $i \in [s]$, R_i is an independent set.
4. $Q_1 <_\pi R_1 <_\pi Q_2 <_\pi R_2 <_\pi \cdots <_\pi Q_s <_\pi R_s$.
5. For $i, j \in [s]$, if $E(Q_i, Q_j) \ne \emptyset$, then $|i - j| \le 1$.
6. For $i, j \in [s]$, if $E(Q_i, R_j) \ne \emptyset$, then $i = j$.
7. For $i, j \in [s]$, we have $E(R_i, R_j) = \emptyset$.

Notice that in the above definition, we require that each Q_i induce a non-trivial biclique; that is, for each Q_i, we have $Q_i \cap A = \emptyset$ and $Q_i \cap B = \emptyset$. In particular, isolated vertices are not considered to be bicliques. The next lemma proves that every connected bipartite permutation graph has a complete bipartite decomposition, and further, it can be computed in polynomial time.

Lemma 4 ([33]) *Every connected bipartite permutation graph has a complete bipartite decomposition. Moreover, there is a polynomial time algorithm that takes a connected bipartite permutation graph G with a fixed vertex bipartition (A, B) and a fixed strong ordering π of (A, B) as input, and returns a complete bipartite decomposition of G.*

3 Constructing a Nice Modulator

We classify the set of obstructions for bipartite permutation graphs as follows. Any obstruction of size less than 45 is known as a *small obstruction*, while other obstructions (holes) are said to be large. In this section, we construct a modulator of bounded size, i.e., a set $T \subseteq V(G)$ such that $|T| = k^{\mathcal{O}(1)}$ and $G - T$ is a bipartite permutation graph. The modulator T will have some additional properties as well. For this, we use the following known result.

Theorem 5 ([4]) *There exists a polynomial time, factor 9-approximation algorithm for* BPVD.

Lemma 6 ([13, Lemma 3.2]) *Let $\mathcal{F}$ be a family of sets of cardinality at most d over a universe $\mathcal{U}$ and k be a positive integer. Then there is a $\mathcal{O}(|\mathcal{F}|(k+|\mathcal{F}|))$-time algorithm that finds a non-empty set $\mathcal{F}' \subseteq \mathcal{F}$ such that*

1. *For every $Z \subseteq \mathcal{U}$ of size at most k, Z is a minimal hitting set of $\mathcal{F}$ if and only if Z is a minimal hitting set of $\mathcal{F}'$; and*
2. *$|\mathcal{F}'| \leq d!(k+1)^d$.*

We use Lemma 6 to identify a vertex subset of $V(G)$, which allows us to forget about small induced subgraphs of G and to concentrate on long induced holes for the kernelization.

Lemma 7 *There is an algorithm that takes an instance (G, k) of* BPVD *as input, runs in time polynomial in the input size, and returns a set $T'' \subseteq V(G)$ such that*

1. *every set $Y \subseteq V(G)$ of size at most k is a minimal hitting set of small obstructions in G if and only if it is a minimal hitting set for small obstructions in $G[T'']$, and*
2. *$|T''| \leq (45+1)!(k+1)^{45}$.*

Proof Let $\mathcal{F}$ be the family consisting of all small obstructions of the input graph G. For every small obstruction O, $\mathcal{F}$ contains the set $V(O)$. By definition, each set in $\mathcal{F}$ is of size at most 45. We apply Lemma 6 on $\mathcal{F}$ and, in polynomial time, obtain a set $\mathcal{F}'$ such that (i) Y is a minimal hitting set of $\mathcal{F}$ of size at most k if and only if Y is a minimal hitting set of $\mathcal{F}'$ of size at most k; and (ii) $|\mathcal{F}'| \leq 45!(k+1)^{45}$. We take T'' to be the set of vertices inside any set of $\mathcal{F}'$. Thus $|T''| \leq (45) \cdot 45!(k+1)^{45} \leq (45+1)!(k+1)^{45}$. □

Using Theorem 5, in polynomial time we construct a 9-approximate solution T', and using Lemma 7 in polynomial time we construct a vertex set T''. If $|T'| > 9k$, then we conclude that (G, k) is a no-instance. Otherwise, we have a modulator $T = T' \cup T''$ of size $\mathcal{O}(k^{45})$, such that $G - T$ is a bipartite permutation graph, and every set $Y \subseteq V(G)$ of size at most k is a minimal hitting set of small obstructions in G if and only if it is a minimal hitting set for small obstructions in $G[T]$. We call the modulator constructed above a *nice modulator*. We summarize these discussions in the following lemma.

Lemma 8 (Nice Modulator) *Let (G, k) be an instance to* BPVD. *In polynomial time, we can either construct a nice modulator $T \subseteq V(G)$ of size $\mathcal{O}(k^{45})$, or conclude that (G, k) is a no-instance.*

4 Bounding the Sizes of Bicliques and Independent Sets

This section considers the modulator T of G to the bipartite permutation graph obtained in the previous section. We bound the size of each biclique and independent set in a *complete bipartite decomposition* of $G - T$.

> Throughout this section, we assume that we have fixed a bipartition (A, B) of $G - T$ and a strong ordering π of (A, B). Let $\pi_A = \pi|_A$ and $\pi_B = \pi|_B$. And for $X \subseteq A$ (resp. $Y \subseteq B$), whenever we say that the first or last vertex of X (resp. Y), we mean the first or last vertex of X (resp. Y) in the ordering π_A (resp. π_B). We also assume that $G - T$ is connected. Later, we will remove this requirement. (We assume connectivity so that we can work with a complete bipartite decomposition of $G - T$.) We also fix a complete bipartite decomposition $\mathcal{D} = (Q_1, R_1, \ldots, Q_s, R_s)$ of $G - T$.

4.1 Auxiliary Results

Next, we prove a few simple results that will be used later to bound the size of each biclique and independent set in the complete bipartite decomposition $\mathcal{D}$ of $G - T$.

Lemma 9 *Consider a no-instance (G, k) of BPVD. Let $v \in V(G)$ be such that $v \notin T$ and $(G - v, k)$ is a yes-instance of BPVD. Let $X \subseteq V(G - v)$ be a solution for the instance $(G - v, k)$, and let H be an obstruction in $G - X$. Then, H is a large obstruction.*

Proof Let (G, k) be a no-instance of BPVD and let $X \subseteq V(G - v)$ be a solution of size at most k. That is, $(G - v) - X$ is a bipartite permutation graph. Moreover, by our assumption that (G, k) is a no-instance, $G - X$ is not a bipartite permutation graph. Then, $G - X$ must contain an obstruction, say, H. Note that $v \in V(H)$, as otherwise, H would be an obstruction in $(G - v) - X$, which contradicts the fact that $(G - v) - X$ is a bipartite permutation graph. We first claim that H is a large obstruction. Suppose not. Note that X hits all obstructions in $G - v$. And since $G[T]$ is a subgraph of $G - v$ as $v \notin T$, X hits all obstructions in the subgraph $G[T]$ as well. In particular, X hits all small obstructions in $G[T]$. Let $Y \subseteq X$ be a minimal hitting set for all small obstructions in $G[T]$. Then, by the definitions of T and Y, we can conclude that Y hits all small obstructions in G as well. However, then, as H is an obstruction in $G - X$ and $Y \subseteq X$, we can conclude that H is a small obstruction in $G - Y$, a contradiction. Thus, H is a large obstruction in $G - X$. □

Lemma 10 *Let H be an induced path in G. Consider $v \in V(G) \setminus V(H)$. If v has more than 5 neighbours in $V(H)$, then $G[V(H) \cup \{v\}]$ contains a small obstruction.*

Proof Assume that $|N(v) \cap V(H)| \geq 5$. Let H be a path from x to y for some $x, y \in V(G)$. Let $v_1, v_2, v_3, v_4, v_5 \in V(H)$ be the first 5 neighbours of v that appear as we traverse H from x to y. Note that if $v_i v_{i+1} \in E(G)$ for some $i \in [4]$, then $\{v, v_i, v_{i+1}\}$ induces a triangle, which is an obstruction, and the lemma follows. So,

assume that $v_i v_{i+1} \notin E(H)$ for every $i \in [4]$. This means that no two vertices from $\{v_1, v_2, v_3, v_4, v_5\}$ appear consecutively on H. For $i \in \{1, 3\}$, let u_i be the neighbour of v_i that appears between v_i and v_{i+1} as we traverse H from v_1 to v_5, and let u_5 be the neighbour of v_5 that appears between v_4 and v_5 as we traverse H from v_1 to v_5. Then, notice that $\{v, v_1, u_1, v_3, u_3, v_5, u_5\}$ induces a subdivided claw, which is an obstruction. □

Lemma 11 *Let H' be a graph with a Hamiltonian cycle, and let $C = v_1 v_2 \ldots, v_\ell v_1$ be a Hamiltonian cycle in H', where $\ell \geq 45$. Let $Y \subseteq V(H')$ be such that (i) $1 \leq |Y| \leq 3$, (ii) the vertices of Y appear consecutively in the cycle C (i.e., $Y = \{v_i, v_{i+1}, v_{i+2}\}$ for some $i \in [\ell - 2]$ or $Y = \{v_{\ell-1}, v_\ell, v_1\}$ or $Y = \{v_\ell, v_1, v_2\}$), (iii) $H' - Y$ is an induced path and (iv) $d_{H'}(y) \leq 5$ for every $y \in Y$. Then, H' contains an obstruction.*

Proof Observe first that since $H' - Y$ is an induced path, any chord in the cycle C is incident with Y. First we prove that if a vertex $v \in Y$ has at least 5 neighbors in H', there is an obstruction contained in H'. Such a vertex v has at least 3 neighbors $\{x_1, x_2, x_3\}$ in $H' \setminus Y$ such that vx_i is not an edge in C for $i \in [3]$. Also if any two vertices $\{x_1, x_2, x_3\}$ are consecutive, they form a triangle with v, which is an obstruction. Otherwise $\{x_1, x_2, x_3\}$ along with v induces a subdivided claw, an obstruction. Now, if every vertex in Y has at most 4 neighbors in H', then $|N_{H'}(Y)| \leq 12$. But from pigeonhole principle, there is a maximal induced subpath P_1, of $H' \setminus Y$ of length at least 3 that has no neighbors in Y. This also implies the existence of a minimal path P_2 of length at least 5 with $V(P_1) \subseteq V(P_2)$ where only its endpoints have neighbors in Y. Then any shortest path between the endpoints of P_2 using vertices of Y only, together with the path P_2 induces a cycle of length at least 6, which is an obstruction. □

4.2 Bounding the Size of a Biclique in the Complete Bipartite Decomposition

In this section, we bound the size of each biclique in the complete bipartite decomposition $\mathcal{D} = (Q_1, R_1, \ldots, Q_s, R_s)$ of $G - T$. In particular, we show that if $G - T$ has a sufficiently large biclique, then in polynomial time, we can find and safely delete an "irrelevant vertex" from such a biclique. We start with a marking procedure that marks a set of vertices in a given biclique.

The intuition behind the marking procedure is simple. The graph G may contain a large obstruction, say H, such that three vertices u, v, w appear consecutively on H. Suppose that $v \in Q_j$. Then, $u, w \in T \cup Q_{j-1} \cup Q_j \cup R_j \cup Q_{j+1}$. We will argue that v is "irrelevant," which means that in H, we can replace v with another vertex v' such that $uv', v'w \in E(H)$. To do this, we need to set aside sufficiently many common neighbors of u and w in Q_j as "relevant" vertices. For that, we will have to consider different cases depending on which of the sets T, Q_{j-1}, Q_j, R_j and Q_{j+1} the vertices u and w belong to, and identify the "relevant" vertices accordingly. This is precisely what our marking procedure does. The only rather difficult case arises when $u \in Q_{j-1}$ and $w \in R_j \cup Q_{j+1}$. Steps 5A and 5B of the marking procedure below cover this case, and all other cases are covered by Steps 1-4. Informally, in Step 5A, we look for the first possible (first in the ordering π_A) vertex $u' \in Q_{j-1} \cap A$

and the last possible vertex $w' \in (R_j \cup Q_{j+1}) \cap A$ such that u' and w' have at least one common neighbour in $(Q_j \cap B)$. For such a pair (u', w'), we mark at most $\mathcal{O}(k)$ many common neighbours (in $Q_j \cap B$) of u' and w'. We do this only for $\mathcal{O}(k^2)$ many distinct pairs of the form (u', w'). And while doing this, we also ensure that no vertex appears in more than $\mathcal{O}(k)$ many pairs. Step 5B deals with the symmetric case when $u', w' \in B$ and have a common neighbour in $Q_j \cap A$.

Recall that for a set $Y \subseteq A$ and an integer q, where $1 \le q \le |Y|$, by F_q^Y, we mean the set of the first q vertices of Y (in the ordering π_A); similarly, by L_q^Y, we mean the last q vertices of the Y (again, in the ordering π_A). If $Y \subseteq B$, then F_q^Y and L_q^Y respectively denote the sets of the first and last vertices of Y in the ordering π_B.

Now, we formally describe the marking procedure. And towards that end, for each $j \in [s]$, we set $M_j = M'_j = \emptyset$.

Procedure `Mark-1`. The procedure works in 5 steps. For each fixed $j \in [s]$, we do as follows.

Step 1: For each $\{u, v\} \subseteq T$, let $A_j^{\{u,v\}} = N(u) \cap N(v) \cap Q_j \cap A$, and $B_j^{\{u,v\}} = N(u) \cap N(v) \cap Q_j \cap B$. We add the first $\min\left\{k+1, |A_j^{\{u,v\}} \setminus M_j|\right\}$ vertices of $A_j^{\{u,v\}} \setminus M_j$ in the ordering π_A to M_j. Similarly, we add the first $\min\left\{k+1, |B_j^{\{u,v\}} \setminus M_j|\right\}$ vertices of $B_j^{\{u,v\}} \setminus M_j$ in the ordering π_B to M_j.

Step 2: For each $v \in T$, if $|(N(v) \cap Q_j \cap A) \setminus M_j| \le 2(k+1)$, then we add $(N(v) \cap Q_j \cap A) \setminus M_j$ to M_j, otherwise we add the first $k+1$ vertices and the last $k+1$ vertices in $(N(v) \cap Q_j \cap A) \setminus M_j$ in the ordering π_A to M_j. Similarly, if $|(N(v) \cap Q_j \cap B) \setminus M_j| \le 2(k+1)$, then we add $(N(v) \cap Q_j \cap B) \setminus M_j$ to M_j, else we add the first $k+1$ vertices and the last $k+1$ vertices in $(N(v) \cap Q_j \cap B) \setminus M_j$ in the ordering π_B to M_j.

Step 3: For each $u \in F_{k+10}^{Q_j \cap A} \setminus M_j$, we add u to M_j. And for each $u \in F_{k+10}^{Q_j \cap B} \setminus M_j$, we add u to M_j.

Step 4: For each $u \in L_{k+10}^{Q_j \cap A} \setminus M_j$, we add u to M_j and for each $u \in L_{k+10}^{Q_j \cap B} \setminus M_j$, we add u to M_j.

Step 5A: Set $S_j^A = \emptyset$. For every $x \in (Q_{j-1} \cup R_j \cup Q_{j+1}) \cap A$, set $count(x) = 0$. Repeat the following while $|S_j^A| \le (k+10)^2 - 1$, or until it is no longer feasible to do so, whichever happens, earlier. For the first vertex $u \in Q_{j-1} \cap A$ and the last vertex $w \in R_j \cup Q_{j+1} \cap A$ such that $(u, w) \notin S_j^A$, $count(u) \le k+9$, $count(w) \le k+9$ and u and w have a common neighbour in $Q_j \setminus M_j$, add any $k+10$ vertices of $(N(u) \cap N(w) \cap Q_j) \setminus M_j$ to M_j. If $|(N(u) \cap N(w) \cap Q_j) \setminus M_j| \le k+10$, then add all the vertices of $(N(u) \cap N(w) \cap Q_j) \setminus M_j$ to M_j. And add (u, w) to S_j^A, and set $count(u) \leftarrow count(u) + 1$ and $count(w) \leftarrow count(w) + 1$. Finally, we add u to M_{j-1}; and if $w \in R_j$, then we add w to M'_j, otherwise if $w \in Q_{j+1}$, then we add w to M_{j+1}.

Step 5B: Set $S_j^B = \emptyset$. For every $x \in (Q_{j-1} \cup R_j \cup Q_{j+1}) \cap B$, set $count(x) = 0$. Repeat the following while $|S_j^B| \le (k+10)^2 - 1$, or until it is no longer feasible to do so, whichever happens, earlier. For the first vertex $u \in Q_{j-1} \cap B$ and the last vertex $w \in R_j \cup Q_{j+1} \cap B$ such that $(u, w) \notin S_j^B$, $count(u) \le k+9$, $count(w) \le k+9$ and u and w have a common neighbour in $Q_j \setminus M_j$, add any $k+10$ vertices of $(N(u) \cap N(w) \cap Q_j) \setminus M_j$ to M_j. If $|(N(u) \cap N(w) \cap Q_j) \setminus M_j| \le k+10$, then add all the vertices of $(N(u) \cap N(w) \cap Q_j) \setminus M_j$ to M_j. And add (u, w) to S_j^B, and set $count(u) \leftarrow count(u) + 1$ and $count(w) \leftarrow count(w) + 1$. Finally, we add u to M_{j-1}; and if $w \in R_j$, then we add w to M'_j, otherwise if $w \in Q_{j+1}$, then we add w to M_{j+1}.
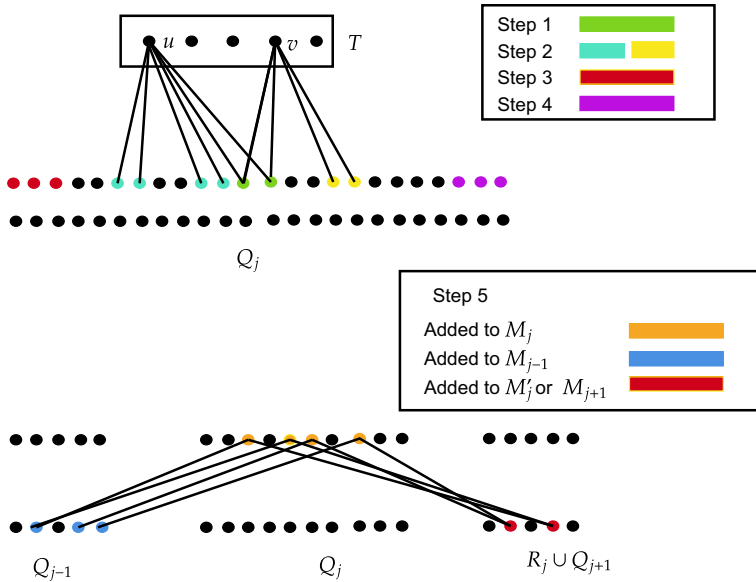

Fig. 3 An illustration of the procedure Mark-1

Remark 12 1. We apply Steps 5A and 5B only for those values of j for which Q_{j-1} and Q_{j+1} exist, i.e., only when $j \notin \{1, s\}$. We also apply Mark-1 in the increasing order of j.

2. Consider $u \in Q_{j-1} \cap A$. Notice that in Step 5A, we add at most $k + 10$ pairs of the form (u, w) to S_j^A, where $w \in R_j \cap Q_{j+1} \cap A$; because each time we add a pair (u, w), we increase $count(u)$ by 1. And we add (u, w) only if $count(u) \leq k + 9$. Thus, every vertex $u \in Q_{j-1} \cap A$ appears in at most $k + 10$ pairs in S_j^A. Similarly, for $w \in (R_j \cup Q_{j+1}) \cap A$, w appears in at most $k + 10$ pairs in S_j^A.

3. At the end of Mark-1, $|S_j^A| \leq (k + 10)^2$.

4. The properties analogous to the ones in items 2 and 3 holds for Step 5B.

We now bound the size of the set M_j at the end of the procedure Mark-1.

Remark 13 Observe that the Procedure Mark-1 can be executed in polynomial time. Also note that $|M_j| = \mathcal{O}(k \cdot |T|^2)$. In Step 1, we add $\mathcal{O}(k \cdot \binom{|T|}{2})$ vertices to M_j. In Step 2, we add $\mathcal{O}(k \cdot |T|)$ vertices. In Steps 3 and 4, we add $\mathcal{O}(k)$ vertices. Furthermore, in Steps 5A and 5B, we consider $\mathcal{O}(k^2)$ pairs, and for each pair, we add $\mathcal{O}(k)$ vertices to M_j. Also, in Steps 5A and 5B, we add $\mathcal{O}(k^2)$ vertices to M_{j-1}, M_j' and M_{j+1}. We thus add a total of $\mathcal{O}(k \cdot |T|^2)$ vertices to M_j.

Reduction Rule 1 *If there exists a vertex $v \in Q_j \setminus M_j$ for some $j \in [s]$, then delete v.*

Lemma 14 *Reduction Rule 1 is safe.*

Proof Consider an application of Reduction Rule 1 in which a vertex, say $v \in Q_j \setminus M_j$ was deleted for some $j \in [s]$. We show that (G, k) is a yes-instance of BPVD if and only if $(G - v, k)$ is a yes-instance of BPVD. Observe first that if (G, k) is a yes-instance, so is $(G - v, k)$, as $G - v$ is an induced subgraph of G. Assume now for a contradiction that $(G - v, k)$ is a yes-instance, but (G, k) is not. And let $X \subseteq V(G - v)$ be a solution of size at most k. That is, $(G - v) - X$ is a bipartite permutation graph. Moreover, by our assumption that (G, k) is a no-instance, $G - X$ is not a bipartite permutation graph. Then, $G - X$ must contain an obstruction, say, H. By Lemma 9 it follows that H is a large obstruction in $G - X$. That is, H is a hole of length at least 45.

Let u and w be the neighbours of v in H, i.e., $H = uvw \ldots u$. And thus $H - v$ is an induced path from w to u in G. Without loss of generality, assume that $v \in A$. Then, $u, w \in B$. We show that we can construct another hole H' in $(G - v) - X$, which will contradict the fact that $(G - v) - X$ is a bipartite permutation graph. For this, we consider different cases depending on which $Q_i \cup R_i$ or T each of the two vertices u and w belong. And in each case (except Cases 6.2.1.3 and 6.2.3.2.2), we will show that there exists $v' \in Q_j \cap A$ such that $v' \notin X \cup \{v\}$ and $uv', v'w \in E(G)$. (Notice that the fact that $uv', v'w \in E(G)$ also implies that $v' \notin V(H)$, as H is a hole and u and w are the only two neighbours of v in H.) So, we replace v with v' to get a new graph H' from H; and H' is completely contained in $(G - v) - X$. Then, we will use Lemmas 10 and 11 to argue that H' contains an obstruction, which will lead to a contradiction. In Cases 6.2.1.3 and 6.2.3.2.2, we replace v with a 3-vertex path abc such that $abc \notin X \cup V(H)$ and $ua, cw \in E(G)$. Again, by replacing v with the path abc, we will get a contradiction.

Recall that $v \in Q_j \setminus M_j$. Notice that for $x \in \{u, w\}$, if $x \notin T$, then, by the definition of a complete bipartite decomposition, $x \in Q_{j-1} \cup Q_j \cup Q_{j+1} \cup R_j$.

1. Case 1: $u, w \in T$. Notice that in Step 1 of the Procedure Mark-1, we must have marked $k + 1$ common neighbors of u and v in $Q_j \cap A$, as otherwise, we would have marked v as well. That is, $|A_j^{\{u,w\}} \cap M_j| = k + 1$. Since, $|X| \le k$, we have $(A_j^{\{u,w\}} \cap M_j) \setminus X \ne \emptyset$. Let $v' \in (A_j^{\{u,w\}} \cap M_j) \setminus X$. Then, $uv', v'w \in E(G)$.

2. Case 2: $u \in T$, $w \in Q_j \cup R_j \cup Q_{j+1}$. In Step 2 of the Procedure Mark-1, we must have marked the $k + 1$ last neighbors of u in $Q_j \cap A$, as otherwise, we would have marked v as well. That is, $|L_{k+1}^{N(u) \cap Q_j \cap A \cap M_j}| = k + 1$. Since $|X| \le k$, we have $L_{k+1}^{N(u) \cap Q_j \cap A \cap M_j} \setminus X \ne \emptyset$. Let $v' \in L_{k+1}^{N(u) \cap Q_j \cap A \cap M_j} \setminus X$. Then, $v <_\pi v'$. Observe now that $v'w \in E(G)$. If $w \in Q_j$, then, clearly $v'w \in E(G)$ as Q_j is a biclique. Suppose that $w \in R_j \cup Q_{j+1}$. Let $w' \in Q_j \cap B$. Then, $w' <_\pi w$. And we have $v'w' \in E(G)$, as Q_j is a biclique. We thus have $v <_\pi v'$ and $w' <_\pi w$; and $vw, v'w' \in E(G)$. Then, by the definition of the strong ordering, we have $v'w \in E(G)$.

3. Case 3: $u \in T$, $w \in Q_{j-1}$. This case is symmetric to Case 2. In Step 2 of the Procedure Mark-1, we must have marked the first $k + 1$ neighbors of u in $Q_j \cap A$, as otherwise, we would have marked v as well. That is, $|F_{k+1}^{N(u) \cap Q_j \cap A \cap M_j}| = k + 1$. Again, there exists $v' \in (F_{k+1}^{N(u) \cap Q_j \cap A \cap M_j}) \setminus X$. The rest of the arguments proceed as in Case 2.

We have thus covered all the cases in which at least one neighbor of v in H belongs to T. Assume now that $u, w \notin T$. Then, by the definition of complete bipartite decomposition, $u, w \in Q_{j-1} \cup Q_j \cup R_j \cup Q_{j+1}$.

4. Case 4: $u \in Q_{j-1} \cup Q_j \cup Q_{j+1}$ and $w \in Q_{j-1} \cup Q_j \cup Q_{j+1}$.

   a. Case 4.1: $u, w \in Q_j$. In Step 3 of the procedure Mark-1, we must have marked the first $k+10$ vertices of $Q_j \cap A$, as otherwise, we would have marked v as well. That is, $|F_{k+10}^{Q_j \cap A \cap M_j}| = k + 10$. Since, $|X| \le k$, we have $F_{k+10}^{Q_j \cap A \cap M_j} \setminus X \ne \emptyset$. Let $v' \in F_{k+10}^{Q_j \cap A \cap M_j} \setminus X$. Since $v' \in Q_j$, we have $uv', v'w \in E(G)$.

   b. Case 4.2: $u, w \in Q_{j-1}$. Just like in Case 4.1, there exists $v' \in F_{k+10}^{Q_j \cap A \cap M_j} \setminus X$. Then, $v' <_\pi v$. Consider $w' \in Q_j \cap B$. Since Q_j is a biclique, such a vertex w' exists. Then, $v'w' \in E(G)$, and $u <_\pi w'$ and $w <_\pi w'$. Then, by the definition of strong ordering, since $uv, v'w' \in E(G)$, we have $uv' \in E(G)$; and since $wv, v'w' \in E(G)$, we have $v'w \in E(G)$.

   c. Case 4.3: $u, w \in Q_{j+1}$. Symmetric to Case 4.2. In this case, there exists $v' \in L_{k+10}^{Q_j \cap A \cap M_j} \setminus X$. And using symmetric arguments, we can show that $uv', v'w \in E(G)$.

   d. Case 4.4: $u \in Q_j, w \in Q_{j-1}$. In Step 3 of the procedure Mark-1, we must have marked the first $k + 10$ vertices of $Q_j \cap A$, as otherwise, we would have marked v as well. That is, $|F_{k+10}^{Q_j \cap A \cap M_j}| = k + 10$. Since, $|X| \le k$, we have $F_{k+10}^{Q_j \cap A \cap M_j} \setminus X \ne \emptyset$. Let $v' \in F_{k+10}^{Q_j \cap A \cap M_j} \setminus X$. Since $v' \in Q_j$, we have $uv' \in E(G)$. Notice that $v' <_\pi v$. Now, since $w \in Q_{j-1}$ and $u \in Q_j$, we have $w <_\pi u$. Therefore, by the definition of strong ordering, we have $v'w \in E(G)$.

   e. Case 4.5: $u \in Q_j, w \in Q_{j+1}$. Symmetric to Case 4.4. In this case, there exists $v' \in L_{k+10}^{Q_j \cap A \cap M_j} \setminus X$. Hence, $v <_\pi v'$. And we have $u <_\pi w$. Now, using symmetric arguments, we can show that $uv', v'w \in E(G)$.

   The only remaining possibility in Case 4 is when $u \in Q_{j-1}$ and $w \in Q_{j+1}$. We will consider this possibility in Case 6.

5. Case 5: $u \in Q_j \cup Q_{j+1} \cup R_j, w \in R_j$. In Step 4 of the procedure Mark-1, we must have marked the last $k + 10$ of vertices of $Q_j \cap A$, as otherwise, we would have marked v as well. That is, $|L_{k+10}^{Q_j \cap A \cap M_j}| = k + 10$. Since $|X| \le k$, we have $(Q_j \cap A \cap M_j) \setminus X \ne \emptyset$. Let $v' \in (Q_j \cap A \cap M_j) \setminus X$. Then, $v <_\pi v'$. We now claim that $uv', v'w \in E(G)$. (1) If $u \in Q_j$, then $uv' \in E(G)$. Also, we have $u <_\pi w$, and since $vw, uv' \in E(G)$, by the definition of strong ordering, we get $v'w \in E(G)$. (2) Suppose that $u \in R_j \cup Q_{j+1}$. Consider $w' \in Q_j \cap B$. Such a vertex w' exists as Q_j is a biclique. Then, $w' <_\pi u$ and $w' <_\pi w$. Again, since Q_j is a biclique, we have $v'w' \in E(G)$. Hence, by the definition of strong ordering, we have $uv' \in E(G)$, (as $v <_\pi v'$, $w' <_\pi u$ and $uv, v'w' \in E(G)$) and $v'w \in E(G)$, (as $v <_\pi v'$, $w' <_\pi w$ and $vw, v'w' \in E(G)$).

6. Case 6: $u \in Q_{j-1}, w \in R_j \cup Q_{j+1}$. There are two possibilities. Either we considered the pair (u, w) (i.e., $(u, w) \in S_j^B$) in Step 5B and added common neighbours of u and w to M_j, or we did not.

a. Case 6.1: $(u, w) \in S_j^B$. In this case, when we considered (u, w), we must have added $k + 10$ common neighbours of u and w in $Q_j \setminus M_j$ to M_j, for otherwise we would have added v as well. That is, $|N(u) \cap N(w) \cap Q_j \cap M_j| \geq k + 10$. Therefore, $(N(u) \cap N(w) \cap Q_j \cap M_j) \setminus X \neq \emptyset$. Let v' be such a common neighbour of u and w in $Q_j \setminus X$.

b. Case 6.2: $(u, w) \notin S_j^B$. Then, we must have either $|S_j^B| = (k + 10)^2$ or $count(u) = k + 10$ or $count(w) = k + 10$, for otherwise we would have considered the pair (u, w) in Step 5B. Notice that for each pair $(u', w') \in S_j^B$, we added at least one vertex to M_j. We added at least $|S_j^B|$ vertices to M_j in Step 5B.

   i. Case 6.2.1: Suppose that $count(u) = k + 10$. This means that there exist $k + 10$ pairs of the form $(u, w') \in S_j^B$; and corresponding to each such pair, we added a common neighbour of u and w', say $v' \in Q_j \cap A$, to M_j. Since $|X| \leq k$, there exists $(u, w') \in S_j^B$ such that corresponding to (u, w'), we added $v' \in N(u) \cap N(w') \cap Q_j \cap A$ to M_j, and $w', v' \notin X$. There are two further possibilities: $w <_\pi w'$ or $w' <_\pi w$.

      A. Case 6.2.1.1: $w <_\pi w'$. Now, if $v <_\pi v'$, then, since $u <_\pi w$ and $uv', vw \in E(G)$, by the definition of strong ordering, we get $v'w \in E(G)$. On the other hand, if $v' <_\pi v$, then, since $w <_{w'}$ and $vw, v'w' \in E(G)$, by the definition of the strong ordering, we get $v'w \in E(G)$.

      B. Case 6.2.1.2: $w' <_\pi w$ and $v <_\pi v'$. Then, since $vw, v'w' \in E(G)$, we get $v'w \in E(G)$.

      C. Case 6.2.1.3: $w' <_\pi w$ and $v' <_\pi v$. Note that we must have marked the $k + 10$ last vertices of $Q_j \cap A$ (i.e., the vertices of $L_{k+10}^{Q_j \cap A}$), for otherwise, we would have marked v as well. So, let $z \in L_{k+10}^{Q_j \cap A} \setminus X$. Since Q_j is a biclique, there exists $y \in Q_j \cap B$, and we have $zy \in E(G)$. Notice that we have $v', v, z \in A$ with $v' <_\pi v <_\pi z$ and $u, y, w', w \in B$ with $u <_\pi y <_\pi w' <_\pi w$. Now, since $zy, v'w' \in E(G)$, we get $zw' \in E(G)$. Then, since $zw', vw \in E(G)$, we get $vw', zw \in E(G)$. We thus have v', w', z with $uv', v'w', w'z, zw \in E(G)$.

      In addition, we can safely assume that $v', w', z' \notin V(H)$. First, since $vw' \in E(G)$ and H is a hole such that u and w are the only neighbours of v in H, we can conclude that $w' \notin V(H)$. Now, suppose that $z \in V(H)$. Then, since H is an induced cycle and $zw, wv \in E(G)$, we can conclude that z and v are the only neighbors of w in H. Recall that we chose z arbitrarily from $L_{k+10}^{Q_j \cap A} \setminus X$. In other words, there exists at most one such $z \in L_{k+10}^{Q_j \cap A} \setminus X$ with $z \in V(H)$. So, we can always choose an z from $L_{k+10}^{Q_j \cap A} \setminus X$ such that $z \notin V(H)$. Similarly, since $uv', uv \in E(G)$, if $v' \in V(H)$, then v and v' are the only neighbours of u in H. In other words, there exists at most one such v' in H. Again, we chose the pair (u, w') with a common neighbour v' arbitrarily from S_j^B such that $w', v' \notin X$. Since $count(u) = k + 10$,

we have sufficiently many pairs (u, w') with the additional property that $v' \notin V(H)$.

ii. Case 6.2.2: Suppose that $count(w) = k + 10$. This case is symmetric to Case 6.2.1.

iii. Case 6.2.3: Suppose that $count(u) < k + 10$ and $count(w) < k + 10$. Then, $|S_j^B| = (k + 10)^2$. Since $|X| \le k$ and $count(x) \le k + 10$ for every $x \in X \cap (Q_{j-1} \cup R_j \cup Q_{j+1})$, there exists a pair $(u', w') \in S_j^B$ such that in Step 5B, corresponding to (u', w'), we marked a common neighbour, say $v' \in Q_j \cap A$, of u' and w', and $u', v', w' \notin X$. Now, in Step 5, since we did not consider the pair (u, w), (which has an unmarked common neighbour v, and $count(u) < k + 10$ and $count(w) < k + 10$), and since we considered the pair (u', w'), it cannot be the case that we simultaneously have $u <_\pi u'$ and $w' <_\pi w$. So, we either have $u' \le_\pi u$ or $w \le_\pi w'$. In fact, there are three possible scenarios: (i) $u' \le_\pi u <_\pi w \le_\pi w'$ or (ii) $u <_\pi u' <_\pi w \le_\pi w'$ or (iii) $u' \le_\pi u <_\pi w' \le_\pi w$. We consider each of these scenarios separately.

A. Case 6.2.3.1: $u' \le_\pi u <_\pi w \le_\pi w'$. Suppose first that $v <_\pi v'$. Then, since $vu, v'u' \in E(G)$, we also have $v'u \in E(G)$. But then, since $vw, v'u \in E(G)$, we also have $v'w \in E(G)$. That is, we have $uv', v'w \in E(G)$. Suppose now that $v' <_\pi v$. Then, since $v'w', vu \in E(G)$, we also have $v'u \in E(G)$. Then, since $vw, v'w' \in E(G)$, we also have $v'w \in E(G)$. That is, we have $uv', v'w \in E(G)$.

B. Case 6.2.3.2: $u <_\pi u' <_\pi w \le_\pi w'$. We divide this case into two, depending on whether or not $v <_\pi v'$.

B1. Case 6.2.3.2.1: $v' <_\pi v$. Since $v'u', vu \in E(G)$, we also have $v'u \in E(G)$. And since $vw, v'w' \in E(G)$, we also have $v'w \in E(G)$. We thus have $uv', v'w \in E(G)$.

B2. Case 6.2.3.2.2: $v <_\pi v'$: Note that we must have marked the first $k + 10$ vertices of $Q_j \cap A$ (i.e., the vertices of $F_{k+10}^{Q_j \cap A}$), for otherwise we would have marked v as well. So, let $z \in F_{k+10}^{Q_j \cap A} \setminus X$. Since Q_j is a biclique, there exists $y \in Q_j \cap B$, and we have $zy, vy, v'y \in E(G)$. Also, notice that $z <_\pi v$ and $u' <_\pi y <_\pi w$. Now, since $zy, v'u' \in E(G)$, we also have $zu' \in E(G)$. And since $v'y, vw \in E(G)$, we also have $v'w \in E(G)$. We thus have $z, u', v'(\notin X)$ such that $uz, zu', u'v', v'w \in E(G)$.

Finally, using the same arguments as in Case 6.2.1.3, we conclude that $z, v' \notin V(H)$. And since $vw, v'u' \in E(G)$, we have $vu' \in E(G)$ as well, which implies that $u' \notin V(H)$.

C. Case 6.2.3.3: $u' \le_\pi u <_\pi w' \le_\pi w$. This case is symmetric to Case 6.2.3.2.

In all the Cases discussed above, except in Cases 6.2.1.3 and 6.2.3.2.2 (and their symmetric counterparts), let H' be the graph obtained from H by replacing the vertex v with the vertex v' and the edges uv, vw by the edges $uv', v'w$. Then, $v' \notin V(H)$, as v is the only common neighbour of u and w in H. Notice also that

no vertex of H' belongs to $X \cup \{v\}$. Thus, the graph H' is contained in $(G-v)-X$, and $H'-v'$ is an induced path in G. By Lemma 10, v' has at most 4 neighbours in $V(H') \setminus \{v'\}$, for otherwise $G[V(H')]$ contains a small obstruction, (which is an obstruction in $(G-v)-X$), a contradiction to the assumption that X is a solution to $G-v$ of BPVD. However, then, by Lemma 11, $G[V(H')]$ contains an obstruction, which is again a contradiction to the assumption that X is a solution to $G-v$ of BPVD. Similarly, in Case 6.2.1.3, let H' be the graph obtained from H by replacing v with the vertices v', w', z and the edges uv, vw with $uv', v'w', w'z, zw$. In Case 6.2.3.2.2, let H' be the graph obtained from H by replacing the vertex v with vertices z, u', v' and the edges uv, vw with the edges $uz, zu', u'v', v'w$. In Cases 6.2.1.3 and 6.2.3.2.2, let $I = V(H') \setminus V(H)$. Note that in both these cases, H' is a cycle as we have already argued that $V(H) \cap I = \emptyset$, and $H'-I$ is an induced path in G. Also, note that no vertex of H' belongs to $X \cup \{v\}$. By Lemma 10, each of three vertices in I has at most 4 neighbours in $V(H) \setminus I$. However, then, by Lemma 11, we conclude that H' contains an obstruction, which is an obstruction in $(G-v)-X$ as well, a contradiction to the assumption that X is a solution to $G-v$ of BPVD.

This concludes the proof. □

4.3 Bounding the Size of an Independent Set in the Complete Bipartite Decomposition

In this section, we bound the number of vertices in each independent set R_i for each $i \in [s]$ in the complete bipartite decomposition $\mathcal{D}$ of $G-T$. First, we describe the construction of a set M'_j with respect to an independent set R_j, $j \in [s]$ in the complete bipartite decomposition $\mathcal{D}$ of $G-T$. Recall that while applying the procedure Mark-1, we already added some vertices of R_j to M'_j.

Procedure Mark-2. The procedure works in 4 steps as follows.

Step 1: For each $v \in T$, if $|(N(v) \cap R_j \cap A) \setminus M'_j| \leq k+1$, then we add $(N(v) \cap R_j \cap A) \setminus M'_j$ to M'_j, and otherwise we add the first $k+1$ vertices in $(N(v) \cap R_j \cap A) \setminus M'_j$ in the ordering π_A to M'_j. Similarly, if $|(N(v) \cap R_j \cap B) \setminus M'_j| \leq k+1$, then add $(N(v) \cap R_j \cap B) \setminus M'_j$ to M'_j, and else we add the first $k+1$ vertices in $(N(v) \cap R_j \cap B) \setminus M'_j$ in the ordering π_B to M'_j.

Step 2: For each pair $x, y \in T$, if $|(N(x) \cap N(y) \cap R_j \cap A) \setminus M'_j| \leq k+1$, then we add $(N(x) \cap N(y) \cap R_j \cap A) \setminus M'_j$ to M'_j, else we add the first $k+1$ vertices in $(N(x) \cap N(y) \cap R_j \cap A) \setminus M'_j$ in the sequence π to M'_j. Similarly, for each pair $x, y \in T$, if $|(N(x) \cap N(y) \cap R_j \cap B) \setminus M'_j| \leq k+1$, then we add $(N(x) \cap N(y) \cap R_j \cap B) \setminus M'_j$ to M'_j, else we add first $k+1$ vertices in $(N(x) \cap N(y) \cap R_j \cap B) \setminus M'_j$ in the sequence π to M'_j.

Step 3: for each $u \in F_{k+1}^{R_j \cap A} \setminus M'_j$, we add u to M'_j. And for each $u \in F_{k+1}^{R_j \cap B} \setminus M'_j$, we add u to M'_j.

Step 4: for each $u \in L_{k+1}^{R_j \cap A} \setminus M'_j$, we add u to M'_j and for each $u \in L_{k+1}^{R_j \cap B} \setminus M'_j$, we add u to M'_j.

We now bound the size of the set M'_j at the end of the procedure Mark-2.

Remark 15 Observe that the Procedure Mark-2 can be executed in polynomial time. Observe also that $|M'_j| \leq (k + 1)(|T| + |T|^2 + 1)$ for every $j \in [s]$. To see this, fix $j \in [s]$. Note that for each $v \in T$, we added at most $2(k + 1)$ neighbours to v to M'_j, i.e., at most $2(k + 1)$ vertices from $(N(v) \cap R_j) \setminus M'_j$. Therefore the number of vertices we added to M'_j in Step 1 is at most $2(k + 1)|T|$. And in Step 2, for each pair $x, y \in T$, we added at most $2(k + 1)$ common neighbours of x and y to M'_j, and therefore the number of vertices we added to M'_j in Step 2 is at most $2(k + 1)|T|^2$. In each of Steps 3 and 4, we added at most $2(k + 1)$ vertices to M'_j. Also, during the procedure Mark-1, we added $\mathcal{O}(k^2)$ vertices to M'_j. Thus, $|M'_j| = \mathcal{O}(k \cdot |T|^2)$.

Using the set M'_j, we get the following reduction rule.

Reduction Rule 2 *If there exists $v \in R_j \setminus M'_j$ for some $j \in [s]$, then delete v.*

Lemma 16 *Reduction Rule 2 is safe.*

Proof Consider an application of Reduction Rule 2 in which a vertex, say $v \in R_j \setminus M'_j$ was deleted for some $j \in [s]$. We show that (G, k) is a yes-instance of BPVD if and only if $(G - v, k)$ is a yes-instance of BPVD. Observe first that if (G, k) is a yes-instance, so is $(G - v, k)$, as $G - v$ is an induced subgraph of G. Assume now for a contradiction that $(G - v, k)$ is a yes-instance, but (G, k) is not. And let $X \subseteq V(G - v)$ be a solution of size at most k. That is, $(G - v) - X$ is a bipartite permutation graph. Moreover, by our assumption that (G, k) is a no-instance, $G - X$ is not a bipartite permutation graph. Then, $G - X$ must contain an obstruction, say, H. By Lemma 9 it follows that H is a large obstruction in $G - X$. That is, H is a hole of length at least 45.

Let u and w be the neighbours of v in H, i.e., $H = uvw \ldots u$. Moreover, $H - v$ is an induced path from w to u. Without loss of generality, assume that $v \in A$. Then, $u, w \in B$. We show that we can construct another hole H' in $(G - v) - X$, which will contradict the fact that $(G - v) - X$ is a bipartite permutation graph. For this, we consider different cases depending on which $Q_i \cup R_i$ or T each of the two vertices u and w belong. Recall that $v \in R_j \setminus M'_j$. Notice that for $x \in \{u, w\}$, if $x \notin T$, then, by the definition of a complete bipartite decomposition, $x \in Q_j$.

1. $u, w \in T$. Notice that as $v \in (N(u) \cap N(w) \cap R_j \cap A) \setminus M'_j$, by Step 2 of the Procedure Mark-2, we must have marked $k + 1$ common neighbours of u, w in $R_j \cap A$, i.e., we have added $k + 1$ vertices in $(N(u) \cap N(w) \cap R_j \cap A)$ to M'_j as otherwise, we would have added v to M'_j as well. That is, we have $|M'_j \cap N(u) \cap N(w) \cap A| \geq k + 1$. Since, $|X| \leq k$, we have $(M'_j \cap N(u) \cap N(w) \cap A) \setminus X \neq \emptyset$. Also notice that $N(u) \cap N(w) \cap V(H) = \{v\}$, as H is a hole. Let $v' \in (M'_j \cap N(u) \cap N(w) \cap A) \setminus X$ and H' be the graph obtained from H by replacing the vertex v by v' and by replacing edges uv, vw by $uv', v'w$. Notice that no vertex of H' belongs to $X \cup \{v\}$ and the graph $H' - v$ is an induced path in G. Moreover, H' is a cycle of length at least 45 in G. By Lemma 10, v' have at most 4 neighbours in $H' - v'$. By Lemma 11 we conclude that H' contains an obstruction, which is also an obstruction in $(G - v) - X$, contradicts that X is a solution to $G - v$ of BPVD.

2. $u \in Q_j$, $w \in T$. (analogous arguments follows for the case $u \in T$, $w \in Q_j$) In Step 1 of the Procedure Mark-2, we have added $k + 1$ neighbours of w in $R_j \cap A$ to M'_j which are before v in sequence π, as otherwise, we would have added v as well to M'_j. Thus, $|N(w) \cap M'_j \cap A| = k + 1$. Let $v' \in N(w) \cap R_j \cap A \setminus X$. As $v' <_\pi v$, we have $v'u \in E(G)$, by the definition of the strong ordering, as Q_j is a non-trivial biclique and hence u must have a neighbour u' in $Q_j \cap A$ and hence all the vertices between u' to v in π are neighbours of u, which implies $v' \in N(u)$. Let H' be the graph obtained from H by replacing the vertex v with vertex v' and edge uv, vw by edges uv', $v'w$. And by Lemma 10, each of the vertices u', v' and w' has at most 4 neighbours in $V(H') \setminus \{u', v', w'\}$. By Lemma 11 we conclude that H' contains an obstruction, which is also an obstruction in $(G - v) - X$, contradicts that X is a solution to $G - v$ of BPVD.

3. $u, w \in Q_j$. Note that $v \notin F_{k+1}^{R_j \cap A}$, as in Step 3 of the Procedure Mark-2, we have added all the vertices in the set $F_{k+1}^{R_j \cap A}$ to M'_j. Since $|X| \leq k$, $F_{k+1}^{R_j \cap A} \setminus X \neq \emptyset$. Let $v' \in F_{k+1}^{R_j \cap A} \setminus X$. As $v' <_\pi v$, we have $v'u$, $v'w \in E(G)$, by the definition of the strong ordering, as Q_j is a non-trivial biclique and hence u, w must have a neighbour u' in $Q_j \cap A$ and hence all the vertices between u' to v in π are neighbours of u, w, which implies $v' \in N(u) \cap N(w)$. Let H' be the graph obtained from H by replacing the vertex v with vertex v' and edge uv, vw by edges uv', $v'w$. And by Lemma 10, each of the vertices u', v' and w' has at most 4 neighbours in $V(H') \setminus \{u', v', w'\}$. By Lemma 11 we conclude that H' contains an obstruction, which is also an obstruction in $(G - v) - X$, contradicts that X is a solution to $G - v$ of BPVD.

□

Observation 17 *After an exhaustive application of Reduction Rule 1, note that for every $j \in [s]$, $Q_j \setminus M_j = \emptyset$. Thus, by Remark 13, $|Q_j| = |M_j| = \mathcal{O}(k \cdot |T|^2)$.*

Observation 18 *After an exhaustive application of Reduction Rule 2, note that for every $j \in [s]$, $R_j \setminus M'_j = \emptyset$. Thus, by Remark 15, $|R_j| = |M'_j| = \mathcal{O}(k \cdot |T|^2)$.*

Observation 17 and 18 together imply the following result.

Lemma 19 *Given an instance (G, k) of BPVD and a nice modulator $T \subseteq V(G)$ of size $k^{\mathcal{O}(1)}$, in polynomial time, we can construct an equivalent instance (G', k) such that G' is an induced subgraph of G, $T \subseteq V(G')$, and for each connected component of $G' - T$ with a complete bipartite decomposition $(Q_1, R_1, \ldots, Q_s, R_s)$, we have $|Q_j \cup R_j| = \mathcal{O}(k \cdot |T|^2)$.*

5 Bounding the Size of a Connected Component

In this section, we bound the size of each connected component in $G - T$. Consider a connected component of $G - T$ with a complete bipartite decomposition $(Q_1, R_1, Q_2, R_2, \ldots, Q_s, R_s)$. Recall that in previous sections, we bounded the size of each biclique Q_i and independent set R_i. This section aims to bound the number

of Q_is and R_is in each connected component of $G - T$. Without loss of generality, let $\mathcal{C} = \bigcup_{i \in [s]}(Q_i \cup R_i)$. For a pair of biclique Q_i and independent set R_i, the set $Q_i \cup R_i$ is called a *block*.

Reduction Rule 3 *Let v be a vertex in T. If v is contained in at least $k + 1$ triangles (v, a_i, b_i, v) intersecting exactly at $\{v\}$, where $a_i, b_i \in V(G) \setminus T$, (that is, for $i \in [k + 1]$, there exist distinct vertices $a_i, b_i \in V(G) \setminus T$ such that $\{v, a_i, b_i\}$ induces a triangle and for distinct i, $j \in [k + 1]$, $\{a_i, b_i\} \cap \{a_j, b_j\} = \emptyset$), then delete v from G, and reduce k by 1. The resultant instance is $(G - v, k - 1)$.*

The safeness of the above reduction rule is easy to see as every solution to (G, k) of BPVD must contain v. From now on, we assume that Reduction Rule 3 is not applicable.

Reduction Rule 4 *Let v be a vertex in T. If there exist more than $6(k + 1)$ (distinct) indices i such that there exist $a_i \in Q_i \cup R_i$, $b_i \in N(a_i) \cap Q_i$ and $vb_i \notin E(G)$, then delete v and reduce k by 1. The resultant instance is $(G - v, k - 1)$.*

Lemma 20 *Reduction Rule 4 is safe.*

Proof Notice that (v, a_i, b_i) is an induced P_3. By the pigeonhole principle, there are at least $3(k + 1)$ non-consecutive blocks $Q_i \cup R_i$ which contain a pair (a_i, b_i) such that (v, a_i, b_i) is an induced P_3. Let $\mathcal{P}$ be the set of such induced P_3's. That is, $\mathcal{P}$ is a set of distinct induced P_3's (v, a_i, b_i), intersecting exactly at $\{v\}$ and for every pair of P_3's, (v, a_i, b_i) and (v, a_j, b_j), where $a_i, b_i \in Q_i \cup R_i$ and $a_j, b_j \in Q_j \cup R_j$, the blocks $Q_i \cup R_i$ and $Q_j \cup R_j$ are not consecutive. Notice that the vertices of any three P_3s in $\mathcal{P}$ induce a subdivided claw (the forbidden subgraph T_2 in Fig. 1). Thus the set of vertices of the $3(k + 1)$ induced P_3s in $\mathcal{P}$ induce $k + 1$ subdivided claws that intersect only at v, which implies that any solution to the instance (G, k) of BPVD must contain v. □

From now on, we assume that Reduction Rules 3 and 4 are not applicable.

Lemma 21 *Let $\mathcal{C}$ be a connected component in $G - T$. Then there are at most $7|T|(k + 1)$ many disjoint blocks $(Q_i \cup R_i)$ in the complete bipartite decomposition of $\mathcal{C}$ such that $N(T) \cap (Q_i \cup R_i) \neq \emptyset$.*

Proof Let $v \in T$ has a neighbour $a_i \in Q_i \cup R_i$, for some $i \in [j]$. As R_i is an independent set, and by the properties of the complete bipartite decomposition, a vertex in R_i can have neighbors only in Q_i. Since $\mathcal{C}$ is a connected component, it implies that a_i must have a neighbor in Q_i. Let $b_i \in Q_i$ be a neighbour of a_i in G. Consider the following cases:

Case 1: $b_i \in N(v)$. Notice that we obtain a triangle (v, a_i, b_i, v) in this case. If there are more than $k + 1$ such disjoint (a_i, b_i) pairs such that both a_i, b_i are adjacent to v, then there are $k + 1$ triangles of the form (v, a_i, b_i, v) intersecting exactly at $\{v\}$ and $a_i, b_i \in T$. By non-applicability of Reduction Rule 3, such a case cannot occur. Hence, for any vertex $v \in T$, v has neighbours a_i's in at most $k + 1$ different $Q_i \cup R_i$ blocks such that there is a vertex $b_i \in N(a_i) \cap N(v) \cap Q_i$.

Case 2: $b_i \notin N(v)$. Notice that (v, a_i, b_i) is an induced P_3. If v has more than $6(k+1)$ neighbours a_i's in different $Q_i \cup R_i$ such that there exists $b_i \in N(a_i) \cap Q_i \setminus N(v)$, then Reduction Rule 4 would be applicable. By non applicability of Reduction Rule 4, we obtain that for any vertex $v \in T$, v has neighbours a_i's in at most $6(k+1)$ different $Q_i \cup R_i$'s such that there is a vertex $b_i \in N(a_i) \cap Q_i \setminus N(v)$.

Thus, every vertex $v \in T$ has neighbors in at most $7(k+1)$ different $Q_i \cup R_i$'s. Hence, there are at most $7|T|(k+1)$ many disjoint blocks $(Q_i \cup R_i)$ such that $N(T) \cap (Q_i \cup R_i) \neq \emptyset$. □

If $\mathcal{C}$ has $3500|T|k(k+1)$ disjoint blocks, then by the pigeon hole principle and Lemma 21, there are at least $500k$ consecutive blocks in $\mathcal{C}$ that do not contain any vertex from $N(T)$. Let $Q_1 \cup R_1, \ldots, Q_{500k} \cup R_{500k}$ be the set of $500k$ such consecutive blocks in $\mathcal{C}$ that are disjoint from $N(T)$. Let $j = 500k/2$. Consider $\mathcal{D}_L = \{Q_i \cup R_i | i \in [j - 2k, j - 3]\} \setminus R_{j-3}$ and $\mathcal{D}_R = \{Q_i \cup R_i | i \in [j + 3, j + 2k]\}$. Let $F = \{R_{j-3}\} \cup \{Q_i \cup R_i | i \in [j - 2, j + 2]\}$ and $Z = \{Q_i | i \in [j - 2k, j + 2k]\}$. Observe that, for a vertex $v \in \mathcal{D}_L \cup \mathcal{D}_R$ and a vertex $u \in T$, $\text{dist}_G(u, v) \geq 240k$. This observation will be used to prove further results. Let $Q = Q_{j-3}$ and $Q' = Q_{j+3}$. Let Y be a $Q_i, Q_{i'}$ cut in $G - T$, where $i \in [j - 2k, j - 3]$ and $i' \in [j + 3, j + 2k]$, where Y must contain vertices from only block $Q_a \cup R_a$, $a \in [i + 1, i' - 1]$. Let τ be the size of minimum $Q_i, Q_{i'}$ cut in $G - T$ over all pairs i, i', $i \in [j - 2k, j - 3]$ and $i' \in [j + 3, j + 2k]$.

Reduction Rule 5 *Let F be as defined above. Delete all the vertices of F from G. Introduce three new bicliques $S_1 = K_{k^2, k^2}$, $S_2 = K_{\lceil \tau/2 \rceil, \lfloor \tau/2 \rfloor}$, $S_3 = K_{k^2, k^2}$. Also add edges such that $G[V(Q) \cup S_1]$ and $G[S_1 \cup S_2]$, $G[S_2 \cup S_3]$ and $G[V(Q') \cup S_3]$ are complete bipartite graphs. The bicliques appear in the order Q, S_1, S_2, S_3, Q'.*

Let G' be the reduced graph after applying the Reduction Rule 5. Let $S = S_1 \cup S_2 \cup S_3$. Notice that $G' - T$ is a bipartite permutation graph by construction.

Observation 22 *There are no small obstructions containing any vertices from $F \cup \mathcal{D}_L \cup \mathcal{D}_R$ or $S \cup \mathcal{D}_L \cup \mathcal{D}_R$ in G'.*

Proof Let $\mathcal{D}$ be a set such that $\mathcal{D} \in \{F, S\}$. Suppose that there is a small obstruction O in G such that $V(O) \cap (\mathcal{D}_L \cup \mathcal{D}_R \cup \mathcal{D}) \neq \emptyset$. Since for any vertex $v \in \mathcal{D}_L \cup \mathcal{D}_R \cup \mathcal{D}$ and a vertex $u \in T$, $\text{dist}_G(u, v) \geq 240k$ and $|O| \leq 45$, for both choices of $\mathcal{D}$. Hence, $V(O) \cap T = \emptyset$. However, this is a contradiction since $G \setminus T$ has no obstructions. So there are no *small* obstructions containing any vertices from $\mathcal{D}_L \cup \mathcal{R}_R \cup \mathcal{D}$ in G. □

Observation 23 *Any hole H in G which contains a vertex from $F \cup \mathcal{D}_L \cup \mathcal{D}_R$, intersects all bicliques in $F \cup \mathcal{D}_L \cup \mathcal{D}_R$. Furthermore, such H is of length at least $500k$.*

Proof Since there are no *large* holes in $G - T$, $V(H) \cap T \neq \emptyset$. Without loss of generality, suppose that H intersects a block $Q_i \cup R_i$, for $i \in [s]$ but does not intersect some $Q_{i+1} \in Z$. Then any biclique $Q_{i'}$ where $i' < i$ contains at least two vertices from the hole H. Let a_1 and a_2 be two such vertices with an induced path in H. Let $H = (s, v_1, v_2, \ldots a_1, \ldots a_2, \ldots, s)$. Notice that a_1 and a_2 can not belong to different partitions of Q_{i-21} since H is a hole. But Q_{i-21} has some vertex v in its other
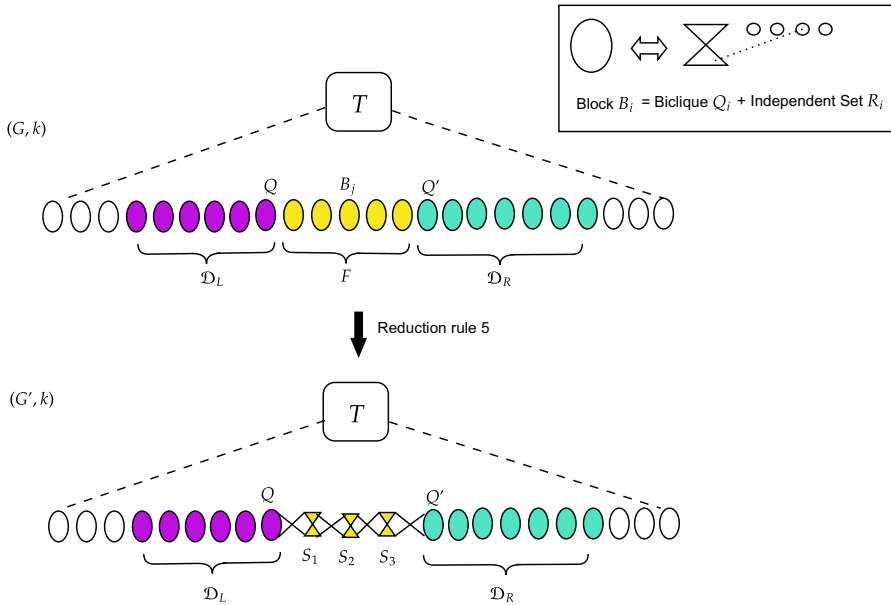
Fig. 4 An illustration of Reduction Rule 5

partition. But then we get a cycle $C = (s, ..., a_1, v, a_2, ..., s)$. However, v can have at most 5 neighbors on the induced path of the hole $(a_1, ..., a_2)$; otherwise, there is a small obstruction containing v which is completely contained in $G - T$, which is not possible. Since the length of the cycle C is at least 40, we can construct a new hole H_1 such that $V(H_1) \subseteq V(C)$ which is completely contained in $G - T$, which is a contradiction. Notice that any such hole must have one vertex from each of the $500k$ consecutive bicliques. Hence the length of the hole is more than $500k$. $\qquad\square$

The following claim can be argued similarly.

Observation 24 *Any hole H in G' which contains a vertex from $S \cup \mathcal{D}_L \cup \mathcal{D}_R$, intersects all the bicliques in $S \cup \mathcal{D}_L \cup \mathcal{D}_R$.*

Lemma 25 *Reduction Rule 5 is safe.*

Proof We show that (G, k) is a yes instance of BPVD if and only if (G', k) is a yes instance of BPVD. In the forward direction, suppose that (G, k) is a yes instance of BPVD and let X be its minimal solution of size at most k. Recall that $\mathcal{D}_L$ and $\mathcal{D}_R$ contain $2k - 3$ blocks. Therefore there must exist bicliques Q_ℓ, Q_r in blocks in $\mathcal{D}_L$ and $\mathcal{D}_R$, respectively such that $Q_\ell \cap X = Q_r \cap X = \emptyset$, as $|X| \leq k$. Let $W = R_\ell \cup (\bigcup_{i=\ell+1}^{r-1} Q_i \cup R_i)$ that is W is the set of all the vertices which are after Q_ℓ and before Q_r in the ordering π. Recall that τ is the size of minimum $Q_i, Q_{i'}$ cut in $G - T$ over all pairs $i, i', i \in [j - 2k, j - 3]$ and $i' \in [j + 3, j + 2k]$ where the cut has only vertices in between Q_i and Q'_i.

Claim 26 *Either $X \cap W = \emptyset$ or $|X \cap W| = \tau$.*

Proof Suppose that $X \cap W \neq \emptyset$ and $|X \cap W| < \tau$. Let $v \in X \cap W$. As X is a minimal solution, then for every vertex $u \in X$, there exists an obstruction that does not contain any vertex of $X \setminus \{u\}$. This implies that there exists a hole H containing v and not containing any vertex in $X \setminus \{v\}$. Consider the smallest such hole. By Observation 23, H is a hole of length at least $500k$, and it intersects with every biclique in Z and therefore intersects with Q_ℓ and Q_r as well. Let $a \in Q_\ell$ be the last and $b \in Q_r$ be the first vertex in ordering π of H. Observe that v must be contained on an a to b subpath P_{ab} of H which has all its internal vertices only from W.

Recall that τ is the size of a minimum $Q_i, Q_{i'}$ cut in $G - T$ over all pairs i, i', $i \in [j - 2k, j - 3]$ and $i' \in [j + 3, j + 2k]$. Therefore, the minimum cut for Q_ℓ, Q_r is of size at least τ, and there are at least τ many paths from Q_ℓ to Q_r that are (internally) disjoint in F. We have that $|X \cap W| < \tau$. As $Q_\ell \cup Q_r \cap X = \emptyset$ there exists a path P from a vertex in Q_ℓ to a vertex in Q_r which is completely disjoint from X. This implies that there is a path from $a \in Q_\ell$ to $b \in Q_r$ which is completely disjoint from X and contains vertices only from $W \cup Q_\ell \cup Q_r$. Next, using this a to b path P, we construct another a to b path P' such that length of P' is at most length of P_{ab} and we can construct an obstruction by replacing P_{ab} in H by P' which does not contain any vertex of X and is also a hole in $G - X$, which contradicts our assumption that X is a solution to (G, k) of BPVD. Towards this, consider the following cases:

- If $|V(P) \cap R_i| > 1$, for some $i \in [\ell, r - 1]$ and let v_1 and v_2 be two vertices of P in R_i with $\pi(v_1) \leq \pi(v_2)$. By the properties of strong ordering $N(v_2) \subseteq N(v_1)$. Let u, u' be the neighbours of v_2 in P such that u' is closer to v_1 in P. In this case we construct a shorter path P' from P by replacing subpath v_1 to u by edge $v_1 u$. From now onwards we assume that $|V(P) \cap R_i| \leq 1$.
- If $|V(P) \cap Q_i| > 3$, for some $i \in [\ell, r]$. If P contains vertices from only one partition say A of Q_i and let $v_1, v_2 \in V(P) \cap Q_i$ such that distance between v_1, v_2 in P is the largest among all pairs in $V(P) \cap Q_i$. Let $u \in Q_i \cap B$. Then we construct a shorter path P' from P by replacing subpath v_1 to v_2 by path $v_1 u, u v_2$. If P contains vertices both partition A, B of Q_i and let $v_1 \in A \cap V(P) \cap Q_i$, $v_2 \in B \cap V(P) \cap Q_i$. Then we construct a shorter path P' from P by replacing subpath v_1 to v_2 by path $v_1 v_2$. From now onwards, we assume that $|V(P) \cap Q_i| \leq 3$ and P do not contain vertices from the partition A and B of Q_i.

By above arguments we obtain a path P' that contains at most 4 vertices from each block between Q_ℓ and Q_r, for $\ell, r \in [s]$. As the number of blocks between Q_ℓ and Q_r are bounded by $4k + 1$ and P' contains vertices only from blocks between Q_ℓ and Q_r length of P' is bounded by $4(4k + 1) \leq 17k$.

Next, we construct a cycle H' from H by replacing path P_{ab}, by P'. Notice that no vertex of H' belongs to $X \cup \{v\}$ and the graph $H' - V(P')$ is an induced path in G. By Lemma 10, each of the vertices in P' has at most 4 neighbours in $V(H') \setminus V(P')$. As the length of H is at least $500k$, the size of the graph $V(H') - V(P')$ is at least $483k$.

Claim 27 H' contains an obstruction.

Proof Observe first that since $H' - V(P')$ is an induced path, all the chords in the cycle H' are incident with $V(P')$. Consider $y \in V(P')$. Since $d_H(y) \leq 5$ by Lemma 10, we

can conclude that the H' has at most 5 chords that are incident with y. Note that if y is adjacent to two vertices that appear consecutively on H', i.e., if $yv_i, yv_{i+1} \in E(H')$ for some $i \in [\ell - 1]$ or $yv_\ell, yv_1 \in E(H')$, then H' contains a triangle, which is an obstruction. So, assume that there does not exist $y \in V(P')$ such that y is adjacent to two vertices that appear consecutively on H'. Suppose that H' does not contain a hole of length at least 5, then for every vertex $v_i \in H'$, vertex v_{i+2} is adjacent to a vertex in $V(P')$. Intuitively every alternate vertex must have a neighbor in $V(P')$ so that every cycle of length at least 5 has a chord. However, $|N(V(P')) \cap V(H')| \le 85k$ implies that there is an induced path of length at least 5 such that it does not contain any neighbor of $V(P')$. Let $P^\star$ be longest induced path in H' such that endpoints of $P^\star$ have neighbours in $V(P')$ and no internal vertex of $P^\star$ is adjacent to any vertex of $V(P')$. Then as there is no triangle in H', we obtain that $V(P^\star)$ together with $V(P')$ induces a hole of length at least 5, a contradiction. Hence H' contains an obstruction. □

By the above claim, H' contains an obstruction, which is also an obstruction in $(G - v) - X$, which contradicts that X is a solution to $G - v$ of BPVD. Therefore, we have proved that either $X \cap W = \emptyset$ or $|X \cap W| = \tau$. □

Using the above claim, we consider the following cases:

Case 1: $X \cap W = \emptyset$

In this case, we show that X is a solution to (G', k) of BPVD. Suppose not. Then there is an obstruction H in $G' - X$, which must contain a vertex $v \in W$. By arguments similar to the above, we can find a new obstruction contained in $G - X$, which will be a contradiction.

Case 2: $|X \cap W| = \tau$

In this case, we show that $X' = (X \setminus W) \cup S_2$ is a solution to (G', k) of BPVD. Suppose that there is an obstruction H in $G' - X'$. Then H must contain a vertex from S. By Observation 22, H must be a hole containing a vertex from S_2, which contradicts that $S_2 \subseteq X'$. This completes the proof in the forward direction.

Next, for the reverse direction, consider that (G', k) is also a yes instance of BPVD and let X be its minimal solution of size at most k. Recall that $\mathcal{D}_L$ and $\mathcal{D}_R$ contain $2k - 3$ blocks. Therefore there must exist bicliques Q_ℓ, Q_r in blocks in $\mathcal{D}_L$ and $\mathcal{D}_R$, respectively such that $Q_\ell \cap X = Q_r \cap X = \emptyset$, as $|X| \le k$. Let $W = R_\ell \cup (\bigcup_{i=\ell+1}^{j-3} Q_i \cup R_i) \cup S \cup (\bigcup_{i=j+3}^{r-1} Q_i \cup R_i)$, that is W is the set of all the vertices which are after Q_ℓ and before Q_r.

Claim 28 Either $X \cap W = \emptyset$ or $|X \cap W| = \tau$.

Proof Suppose that $X \cap W \ne \emptyset$ and $|X \cap W| < \tau$. Let $v \in X \cap W$. As X is a minimal solution, then for every vertex $u \in X$, there exists an obstruction that does not contain any vertex of $X \setminus \{u\}$. This implies that there exists a hole H containing v and not containing any vertex in $X \setminus \{v\}$. Consider the smallest such hole. Such a hole H has a length of at least 500k. By Observation 24, we have that H, intersects with every biclique in $S \cup \mathcal{D}_L \cup \mathcal{D}_R$ and therefore intersects with Q_ℓ and Q_r as well. Let $a \in Q_\ell$ be the last and $b \in Q_r$ be the first vertex in ordering π of H. Observe that v must be contained on a to b subpath P_{ab} of H which has all its internal vertices only from W.

In graph $G - T$, there are at least τ many paths between Q_d and Q_e who are vertex disjoint in F as the minimum cut size is at least τ. Let these paths be $P_1, \ldots P_\tau$. Let their intersections with Q and Q' in G be $\{a_1, \ldots a_\tau\}$ and $\{b_1, \ldots b_\tau\}$. As $|X \cap W| < \tau$ then there exists a path P_i from a to b such that $V(P_i) \cap X = \emptyset$. But then we have a subpaths P_i^1 from a to a_i and P_i^2 from b_i to b of P_i which are disjoint from X. Also, we have that a vertex $u \in S_2 \setminus X$ and at least $k^2 - k$ many vertices from both partitions of S_1 and S_3 that are not in X. By our construction if $a_i \in A$ ($\in B$) then a_i is adjacent to every vertex in $S_1 \cap B$ ($S_1 \cap A$). Similar arguments hold for b_i and S_3. Hence we have a path P from a to b containing path P_i^1 from a to a_i, a vertex in $S_1 \setminus X$, u and a vertex in $S_2 \setminus X$, b_i to b path P_i^2. Notice that P is disjoint from X. Using the arguments similar to the forward direction, we can obtain another path P' that contains at most 4 vertices from each block between Q_ℓ and Q_r. As the number of blocks between Q_ℓ and Q_r are bounded by $4k$ and P' contains vertices only from blocks between Q_ℓ and Q_r length of P' is bounded by $4.(4k + 1) \leq 17k$. Analogous to arguments in the previous section, we can obtain a cycle H' using P' from H that contains an obstruction, which is also an obstruction in $(G' - v) - X$, which contradicts that X is a solution to $G' - v$ of BPVD. Therefore, we have proved that either $X \cap W = \emptyset$ or $|X \cap W| = \tau$. □

Case 1: $X \cap W = \emptyset$

Here we claim that X is a solution to G. Suppose it is not true. Then there is an obstruction in H in $G - X$. This obstruction can not be contained in $G - (\mathcal{D}_L \cup F \cup \mathcal{D}_R)$ as X hits all such obstructions. Hence it must intersect $(\mathcal{D}_L \cup F \cup \mathcal{D}_R)$ and must be a *large* hole passing through F. From Observation 23, H intersects both Q_ℓ and Q_r. However, in G', we can find a path disjoint from X between a and b in W of length at most $17k$ and get a cycle H' by replacing the segment between a and b in H by this path in G'. By arguments similar to Claim 27, we can show that H' contains an obstruction which is also obstruction in $G' - X$.

Case 2: $|X \cap W| = \tau$

Let $X' = X \setminus W$, $X'' = X' \cup mincut(Q_i, Q_{i'})$, where $mincut(Q_i, Q_{i'})$ be a Q_i, $Q_{i'}$ cut in $G - T$, which is minimum overall $i, i', i \in [j - 2k, j - 3]$ and $i' \in [j + 3, j + 2k]$. We claim that X'' is a solution to (G, k) of BPVD size at most k. If not, there is an obstruction in $G - X''$. This obstruction can not be contained in $G - (\mathcal{D}_L \cup F \cup \mathcal{D}_R)$ as $X - X''$ hits all such obstructions. Hence this obstruction must contain vertices from $\mathcal{D}_L \cup F \cup \mathcal{D}_R$. But then by Observation 22 it must be a *large* hole passing through all the bicliques in Z. This is not possible since $mincut(Q_i, Q_{i'})$ disconnects all paths between Q_i and $Q_{i'}$ and hence must intersect this hole. Hence X'' is a solution to G. This completes the proof in the reverse direction. □

With the above reduction rule, we obtain the following result.

Lemma 29 *Given an instance (G, k) of BPVD and a nice modulator $T \subseteq V(G)$ of size $k^{\mathcal{O}(1)}$, in polynomial time, we can construct an equivalent instance (G', k) such that, $T \subseteq V(G')$, T is a nice modulator for G' and for each connected component $\mathcal{C}$ of $G' - T$ with a complete bipartite decomposition $(Q_1, R_1, \ldots, Q_s, R_s)$, the number of blocks $(Q_i \cup R_i)$s in the connected component $\mathcal{C}$ is at most $3500|T|k^2 = \mathcal{O}(k^2 \cdot |T|)$.*

6 Bounding the Number of Connected Components

In Sect. 5, we showed that the size of any connected component of $G - T$ is upper bounded by $k^{\mathcal{O}(1)}$. In this section, we show that the number of connected components of $G - T$ is also upper bounded by $k^{\mathcal{O}(1)}$. This, together with the fact that $|T| \leq k^{\mathcal{O}(1)}$, result in a polynomial kernel for BPVD.

A bipartite permutation graph is a connected component with no neighbor in T. Hence, we can safely remove it from our instance.

Reduction Rule 6 *If there is a connected component $\mathcal{C}$ in $G - T$ such that $N(T) \cap V(\mathcal{C}) = \emptyset$, then reduce (G, k) to $(G - V(\mathcal{C}), k)$.*

From now onwards, we assume that the Reduction Rules 3, 4 and 6 are not applicable. We partition the set of all the connected components in $G - T$ into two sets $\mathbb{C}_{\geq 2}$ and $\mathbb{C}_{=1}$, where $\mathbb{C}_{\geq 2}$ contains all the connected components of size at least 2 whereas $\mathbb{C}_{=1}$ contains all the connected components of size exactly 1. First, we bound the size of $\mathbb{C}_{\geq 2}$.

Lemma 30 $|\mathbb{C}_{\geq 2}| \leq 7|T|(k + 1)$.

Proof Consider any vertex $v \in T$ such that v has a neighbor, say a_i, in a connected component, say $\mathcal{C}_i$, where $\mathcal{C}_i \in \mathbb{C}_{\geq 2}$. Note that for vertex a_i, there exists a neighbor $b_i \in \mathcal{C}_i$ since $\mathcal{C}_i$ has size at least 2.
Case 1: (The vertex b_i is adjacent to v.) Therefore, we have a triangle (v, a_i, b_i, v). If v has more than $k + 1$ such different pairs of (a_i, b_i) such that b_i is adjacent to v, then there are $k + 1$ triangles of the form (v, a_i, b_i, v) having a common vertex v. It implies that any solution of size k must contain v. By non-applicability of Reduction Rule 3, such a case cannot occur. Hence, for any vertex $v \in T$, v has neighbors (a_i's) in at most $k + 1$ different components $\mathcal{C}_i \in \mathbb{C}_{\geq 2}$ such that there is a vertex $b_i \in \mathcal{C}_i \cap N(v)$.
Case 2: (The vertex b_i is not adjacent to v.) Therefore, (v, a_i, b_i) is an induced P_3. Let v has more than $6(k + 1)$ neighbors (a_i's) in different $\mathcal{C}_i$ such that there exists $b_i \in \mathcal{C}_i \setminus N(v)$. Therefore, there exists some $Q_i \cup R_i$ in component $\mathcal{C}_i$ such that $a_i \in Q_i \cup R_i$ and $b_i \in N(a_i) \cap Q_i \setminus N(v)$. Since vertex v has more than $6(k + 1)$ such neighbors a_i, Reduction Rule 4 would be applicable. By non-applicability of Reduction Rule 4, such a case cannot occur. Hence, for any vertex $v \in T$, v has neighbors (a_i's) in at most $6(k + 1)$ different components $\mathcal{C}_i$ such that there is a vertex $b_i \in N(a_i) \cap Q_i \setminus N(v)$.

Thus, every vertex $v \in T$ has neighbors at most in $(k + 1) + 6(k + 1)$, that is, $7(k + 1)$ different components $\mathcal{C}_i$'s. Hence, $|\mathbb{C}_{\geq 2}| \leq 7|T|(k + 1)$. □

Next, we proceed to bound the size of the set $\mathbb{C}_{=1}$. Towards that, we will utilize the following marking scheme.

Procedure Mark-3. We initialise $M = \emptyset$ and for each $\{x, y\} \subseteq T$, we initialise $M(x, y) = \emptyset$, and do as follows: For each $\{x, y\} \subseteq T$, if $|M(x, y)| \leq k + 1$ and if there exists $u \in \mathbb{C}_{=1}$ such that $u \in (N(x) \cap N(y)) \setminus M$, then we add u to $M(x, y)$ and M, i.e., we set $M(x, y) \leftarrow M(x, y) \cup \{u\}$ and $M \leftarrow M \cup \{u\}$.

Remark 31 Observe first that $M = \bigcup_{\{x,y\} \subseteq T} M(x, y)$. And in the procedure `Mark-3`, corresponding to each $\{x, y\} \subseteq T$, we add at most $k + 1$ vertices to $M(x, y)$. Thus, $|M(x, y)| \leq k + 1$, and therefore, $|M| \leq (k + 1)\binom{|T|}{2}$, as there are $\binom{|T|}{2}$ many distinct sets $\{x, y\} \subseteq T$.

Reduction Rule 7 *If there exists $v \in \mathbb{C}_{=1} \setminus M$, then delete v.*

Lemma 32 *Reduction Rule 7 is safe.*

Proof Consider an application of Reduction Rule 7 in which a vertex $v \in \mathbb{C}_{=1}$ was deleted. We can assume that $d_G(v) \geq 1$, as otherwise, $d_G(v) = 0$, and v would have been deleted by Reduction Rule 6. Note that $N_G(v) \subseteq T$ as $v \in \mathbb{C}_{=1}$.

We show that (G, k) is a yes-instance if and only if $(G - v, k)$ is a yes-instance. First, if (G, k) is a yes-instance, so is $(G - v, k)$, as $G - v$ is an induced subgraph of G. Now, assume that $(G - v, k)$ is a yes-instance. And let $X \subseteq V(G - v)$ be a solution of size at most k for the instance $(G - v, k)$. That is, $(G - v) - X$ is a bipartite permutation graph. Notice that if $G - X$ is also a bipartite permutation graph, then (G, k) is a yes-instance. So, assume for a contradiction that $G - X$ is not a bipartite permutation graph. Moreover, let H be an obstruction in $G - X$. Then, $v \in V(H)$, as otherwise H would be an obstruction in $(G - v) - X$, a contradiction.

We first claim that H is a large obstruction. Suppose not. Notice that as $v \notin T$, $G[T]$ is a subgraph of $G - v$, and therefore, X hits all the obstructions in $G[T]$. Let $Y \subseteq X$ be a minimal set that hits all the small obstructions in $G[T]$. Then, by the definition of T, Y hits all the small obstructions in G. In particular, Y hits H. But this is not possible, as $Y \subseteq X$ and H is an obstruction in $G - X$. Thus, H is a hole of length at least 45. In particular, H is a cycle that contains v. Then, we must have $d_G(v) \geq 2$, as otherwise, $d_G(v) = 1$, and therefore there does not exist any cycle in G that contains v.

Let x and y be the neighbors of v in the hole H. Note that we have $|M(x, y)| = k+1$, as otherwise we would have added v to $M(x, y)$ during the procedure `Mark-3`. Since $|X| \leq k$, we have $M(x, y) \setminus X \neq \emptyset$. Let $v' \in M(x, y) \setminus X$. Observe that $v' \notin V(H)$, as otherwise, the edges $v'x$ and $v'y$ would be chords of the cycle H, which would contradict the fact that H is a hole. And let H' be the graph obtained from H by replacing v with v'. That is, $V(H') = (V(H) \setminus \{v\}) \cup \{v'\}$ and $E(H') = (E(H) \setminus \{vx, vy\}) \cup \{v'x, v'y\}$. Notice that no vertex of H' belongs to $X \cup \{v\}$ and that the graph $H' - v$ is an induced path. Notice that as $V(H') \cap (X \cup \{v\}) = \emptyset$, any obstruction in $G[V(H')]$ is an obstruction in $(G - v) - X$. We now show that $G[V(H')]$ contains an obstruction, which will contradict the assumption that $(G - v) - X$ is a complete bipartite graph. If v' has at least 5 neighbours in $V(H' - v)$, then, by Lemma 10, $G[V(H')]$ contains a subdivided claw (T_2), which is an obstruction. Otherwise, by Lemma 11, $G[V(H')]$ contains an obstruction. $\square$

Observe that by Remark 31 and by applying the Reduction Rule 7 repeatedly, we can reduce the graph such that in the reduced instance, $|\mathbb{C}_{=1}| \leq (k + 1)\binom{|T|}{2}$. This reduction and Lemma 30 implies the following result:

Lemma 33 *Given an instance (G, k) and a nice modulator $T \subseteq V(G)$ of size $k^{\mathcal{O}(1)}$, in polynomial time, we can construct an equivalent instance (G', k) such that the number of connected components in $G' - T$ is $\mathcal{O}(k \cdot |T|^2)$.*

7 Kernel Size Analysis

Now we are ready to prove the main result of our paper, that is, Theorem 1. Before proceeding with the proof, let us state all the bounds that contribute to the kernel size.

> Size of nice modulator T: $\mathcal{O}(k^{45})$
> Number of connected components in $G - T$: $\mathcal{O}(k \cdot |T|^2)$.
> Number of blocks in any connected component in $G - T$: $\mathcal{O}(k^2 \cdot |T|)$
> Size of any block $(Q_i \cup R_i)$ in $G - T$: $\mathcal{O}(k \cdot |T|^2)$.

Proof of Theorem 1 Let (G, k) be an instance of the BPVD problem. First, we show that if G is not connected, we can reduce it to the connected case. If there is a connected component $\mathcal{C}$ that is a bipartite permutation graph, we delete it. Clearly, (G, k) is a yes instance if and only if $(G \setminus \mathcal{C}, k)$ is a yes instance. We repeat this process until every connected component of G is not a bipartite permutation graph. At this stage, if the number of connected components is at least $k + 1$, then we conclude that G can not be made into a bipartite permutation graph by deleting at most k vertices. Thus, we assume that G has at most k connected components. Now we show how to obtain a kernel for the case when G is connected, and for the disconnected case, we just run this algorithm on each connected component. This only increases the kernel size by a factor of k. From now onwards, we assume that G is connected.

From Lemma 8, in polynomial time, we can obtain a nice modulator $T \subseteq V(G)$ of size $\mathcal{O}(k^{45})$ or conclude that (G, k) is a no-instance.

Note that $G - T$ is a bipartite permutation graph. Next, we take the complete bipartite decomposition of each component in $G - T$. Now by Theorem 33, in polynomial time we return a graph G such that $G - T$ has $\mathcal{O}(k \cdot |T|^2)$ components.

We now bound the size of each connected component of $G - T$. Let G' be a connected component of $G - T$. By Lemma 29, in polynomial time we can reduce the graph G' such that G' has at most $\mathcal{O}(k^2 \cdot |T|)$ blocks.

Next, we bound the size of each block $Q_i \cup R_i$ in G'. By Lemma 19, in polynomial time we can reduce the graph G' such that for each block $Q_i \cup R_i$, $|Q_j \cup R_j| = \mathcal{O}(k \cdot |T|^2)$. Therefore the total number of vertices in any connected component G' is at most $\mathcal{O}(k \cdot |T|^2) \cdot \mathcal{O}(k^2 \cdot |T|)$, that is, $\mathcal{O}(k^3 \cdot |T|^3)$.

As the graph $G - T$ has at most $\mathcal{O}(k \cdot |T|^2)$ number of components, the total size of the graph $G - T$ is at most $\mathcal{O}(k \cdot |T|^2) \cdot \mathcal{O}(k^3 \cdot |T|^3)$, that is, $\mathcal{O}(k^4 \cdot |T|^5)$. It follows that $|V(G)| = \mathcal{O}(k^4 \cdot |T|^5) + |T|$, that is, $\mathcal{O}(k^4 \cdot |T|^5)$. Recall that $|T| = \mathcal{O}(k^{45})$. Therefore, the size of the obtained kernel is $\mathcal{O}(k^4 \cdot |T|^5)$, that is, $\mathcal{O}(k^{229})$.   □

8 Conclusion

In this paper, we studied BIPARTITE PERMUTATION VERTEX DELETION from the perspective of kernelization complexity and designed a polynomial kernel of size $\mathcal{O}(k^{229})$. This answers an open question posed by Bożyk et al. [4]. We remark that the size of the kernel can be brought closer to $\mathcal{O}(k^{100})$ by doing a more careful case

analysis. However, getting a kernel of size $\mathcal{O}(k^{20})$ would require significantly new ideas, and we leave that as an open problem. Indeed, showing whether PERMUTATION VERTEX DELETION is FPT remains a challenging open problem.

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