# Optimizing Expected Path Lengths with Ant Colony Optimization Using Fitness Proportional Update

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## ABSTRACT

We study the behavior of a Max-Min Ant System (MMAS) on the stochastic single-destination shortest path (SDSP) problem. Two previous papers already analyzed this setting for two slightly different MMAS algorithms, where the pheromone update fitness-independently rewards edges of the best-so-far solution.

The first paper showed that, when the best-so-far solution is not reevaluated and the stochastic nature of the edge weights is due to noise, the MMAS will find a tree of edges successfully and efficiently identify a shortest path tree with minimal noise-free weights. The second paper used reevaluation of the best-so-far solution and showed that the MMAS finds paths which beat any other path in direct comparisons, if existent. For both results, for some random variables, this corresponds to a tree with minimal expected weights.

In this work we analyze a variant of MMAS that works with fitness-proportional update on stochastic-weight graphs with arbitrary random edge weights from $[0, 1]$. For $\delta$ such that any suboptimal path is worse by at least $\delta$ than an optimal path, then, with suitable parameters, the graph will be optimized after $O\left(\frac{n^3 \ln (n/\delta)}{\delta^3}\right)$ iterations (in expectation).

In order to prove the above result, the multiplicative and the variable drift theorem are adapted to continuous search spaces.

## Categories and Subject Descriptors

F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity

## General Terms

Theory, algorithms

## Keywords

Ant colony optimization, stochastic problem, single-destination shortest path, theory


*FOGA'13,* January 16-20, 2013, Adelaide, Australia.


## 1. INTRODUCTION

Ant Colony Optimization (ACO) is a randomized general purpose optimization meta heuristic with a very broad field of application and was first described in Dorigo's Ph.D. thesis [6]. ACO is inspired by swarm intelligence exhibited by ant colonies, where complex behavior emerges from the simple behavior of individual ants, using *pheromones* as an indirect communication mechanism. In particular, the idea is based on the foraging behavior of ant colonies and develops as well as stores its knowledge in the pheromones. The paper [22] presents a way of ensuring exploration of the search space by upper and lower bounds on the pheromones (resulting in so-called Max-Min Ant Systems, MMAS).

Although the single-destination shortest paths problem is one of the most natural applications of ACO, it was also used to solve many other problems, including NP-hard ones such as the Traveling Salesperson Problem (TSP) [6, 7] and more [8]. In general, ACO-algorithms are employed when a solution consists of several components; artificial ants then construct solutions by choosing components. Pheromone is added to components which are often contained in good solutions (as they hopefully carry some responsibility for the good quality of the solution), while pheromone is evaporated from others. The magnitude of the pheromone update is governed by the so-called *evaporation factor* $\rho$. Small $\rho$ leads to a slower but broader search in the search space, usually resulting in better solutions at the cost of a longer running time of the algorithm.

On the theoretical side, there is a good number of analyses regarding the behavior of ACO algorithms for about a decade, starting with early convergence proofs [9, 10] to more recent advances on combinatorial problems like MST [19] and TSP [16], and pseudo-Boolean functions [17]. Of particular interest to this paper is the work in [1] on the single-destination shortest path (SDSP) problem, a problem equivalent to the classical single-*source* shortest path (SSSP) problem. The authors of [1] give an elitist MMAS (which we call MMAS-el, "el" being short for "elitist") for this problem and show a good optimization behavior. We are here interested in a *stochastic* version of SDSP, and the optimization behavior of algorithms similar to MMAS-el.

Experiments have shown that in problems involving uncertainty, ACO algorithms can be particularly successful [2]. The papers [11, 12] give first formal analyses of ACO algorithms in uncertain domains and show convergence to the desired solutions. The paper [14] picks up on MMAS-el and gives a rigorous analysis of its performance on the stochastic

SDSP problem and shows under which conditions optimization is successful.

The general setting of [14] was that the stochastic nature of the path lengths is due to *noise*. The goal then was to find the edge with the best noise-less weight. In particular, from [14] we know that MMAS-el is not necessarily well-suited for finding solutions that are *optimal in expectation*, as the search is guided by the best-so-far solution.

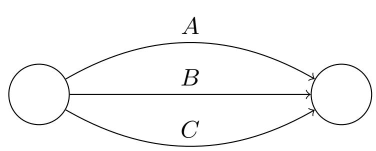
In [4], the MMAS-el algorithm was modified by re-evaluating the best-so-far solution every iteration to avoid permanently being mislead by a single exceptionally good evaluation of a non-optimal solution. The paper shows that, in this case, the pheromones converge to the solution which has a better evaluation against any other solution in a direct comparison more than half of the time; however, such a solution need not exist, and even if it exists, it is not necessarily the solution optimal in expectation.

In this work we analyze a different variant of the MMAS-el, based on fitness proportional pheromone update (we call this algorithm MMAS-fp), on the stochastic SDSP problem; this scheme is also used in practice [22]. In difference to MMAS-el, in each iteration, the newly constructed solution always gets rewarded, but the amount of pheromone added depends on the quality of the solution; no best-so-far solution needs to be stored. This mechanism leads to an implicit averaging and we prove that (a normalizing variant of) MMAS-fp finds all shortest paths that are better than non-optimal paths by at least $\delta$ in

$$O\left(\frac{n^3 \ln (n/\delta)}{\delta^3}\right)$$

iterations.

This gives a qualitative difference to [14] and [4], where the algorithms did not favor paths that are good in expectation, but instead paths either have low weights with reasonable probability (in [14]) or paths which come out better than others in direct comparison with probability higher than 0.5 (in [4]). As an illustration, consider the following (multi-) graph with random variables $A$, $B$ and $C$ as edge weights.




Consider $A$ to be 0 with probability 0.01 and 200 otherwise; $B$ to be 1 with probability 0.6 and 100 otherwise; and $C$ to be always 10. In this setting, MMAS-el as in [14] will eventually find and converge to the edge with random weight $A$, as here a very low weight of 0 is possible. MMAS-el with re-evaluation as in [4] will converge to the edge with weight $B$, as it comes out better than any other edge with probability at least 0.6. Finally, MMAS-fp will converge to edge $C$, the edge best in expectation.

For the mathematical analysis we use the multiplicative and the variable drift theorem (see [5] and [15, 18], respectively), as well as well as recent improvements [4,21]. But as these drift theorems require discrete search spaces, we adapt both formally to continuous domains. This is necessary, as the pheromones are updated with the random outcome of fitness evaluations, so that uncountably many different pheromone values are possible after a single iteration. Our proofs for these adaptations make use of suitable discretizations of the search space. Note that the initial drift theorem [13] did not require finiteness of the search space; similarly, one version of the multiplicative drift theorem does not require this [3].

In Section 2 we give our continuous drift theorems; we define the problem and our algorithms formally in Section 3. Section 4 gives our results on the performance of MMAS-fp and Section 5 concludes.

## 2. DRIFT THEOREM ADAPTATIONS

In this chapter we first adapt the multiplicative drift theorem to continuous search spaces, which we will apply in a first runtime analysis in Section 4. Afterwards we combine the progress made in [4] and [21] on variable drift theorems to new such drift theorem, before we show in a second step that here the restriction to finite state sets is (with a minor restriction) not necessary either.

### 2.1 Continuous Multiplicative Drift

If we add the restriction of a finite search space in the theorem just below (i.e., $S$ finite instead of a bounded interval), we get the known multiplicative drift theorem [5]. In a somewhat different setting, this theorem has been proven without use of other drift theorems in [3]; we give another proof here to exemplify the discretization in this simple case of the multiplicative drift theorem, so that our later application in the proof of Theorem 3 is easier to follow.

THEOREM 1. *Let $S = [s_{min}, s_{\max}] \subseteq \mathbb{R}^+$ be a set of positive numbers with minimum $s_{min}$ and maximum $s_{\max}$. Let $(X^t)_{t\in\mathbb{N}}$ be a sequence of random variables over $S \cup \{0\}$. Let $T$ be the random variable denoting the first point in time $t \in \mathbb{N}$ for which $X^t = 0$. Suppose that there exists a constant $\delta > 0$ such that, for all $t$ and closed intervals $I \subseteq S$ with $P(X^t \in I) > 0$,*

$$E(X^t - X^{t+1} \mid X^t \in I) \geq \delta \min(I).$$

*Then, for all closed intervals $I_0 \subseteq S$ with $P(X^0 \in I_0) > 0$, we have*

$$E(T \mid X^0 \in I_0) \leq \frac{1 + \ln (\max(I_0)/s_{min})}{\delta}.$$

PROOF. Let $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon \leq \min(\delta/4, \delta s/8)$. We define a function $f$ discretizing our search space as follows. For all $x$, we let

$$f(x) = \min\left(\left\lceil \frac{x}{\varepsilon} \right\rceil \varepsilon, s_{\max}\right).$$

We let $S' = \{f(x) \mid x \in S\}$ be our discretized search space. As $|S'| \leq s_{\max}/\varepsilon$, $S' \subseteq S$ is a finite set of positive numbers. Note that, for all $x$ with $s_{min} \leq x \leq s_{\max}$, we have $|f(x) - x| \leq \varepsilon$; furthermore, since only 0 is mapped to 0, and 0 is the only pre-image of 0, we have that $T$ is the random variable denoting the first point in time $t \in \mathbb{N}$ for which $f(X^t) = 0$. We want to apply the multiplicative drift theorem from [5] to the sequence of random variables $\big(f(X^t)\big)_{t\in\mathbb{N}}$. For all

$s \in S'$, we have

$$\begin{aligned}
&E\big(f(X^t) - f(X^{t+1}) \mid f(X^t) = s\big)\\
&\quad\geq E\big(f(X^t) - f(X^{t+1}) \mid s - \varepsilon < X^t \leq s\big)\\
&\quad\geq E(X^t - X^{t+1} \mid s - \varepsilon < X^t \leq s) - 2\varepsilon\\
&\quad\geq \delta\,(s - \varepsilon) - 2\varepsilon\\
&\quad\geq s\delta\,(1 - \varepsilon/\delta - 2\varepsilon/(s\delta))\,.
\end{aligned}$$

From the condition on $\varepsilon$ we know that this bound is positive. Let $I_0 \subseteq S$ be a closed interval with $P(X^0 \in I_0) > 0$; thus, there is $s_0 \leq \max(I_0)$ such that $P(f(X^0) = f(s_0)) > 0$. Hence, we get

$$\begin{aligned}
E(T \mid X^0 \in I_0) &\leq E(T \mid f\big(X^0\big) \leq f(\max(I_0)))\\
&\leq \frac{1 + \ln\left((\max(I_0) + \varepsilon)\,/s_{min}\right)}{\delta\,(1 - 1/k - \delta/k)}
\end{aligned}$$

from the multiplicative drift theorem [5]. For every $\varepsilon$ as above, we have that $b(\varepsilon) := E(T|f(X^0))$ is an upper bound on $E(T|X^0)$. Thus, $\lim_{\varepsilon\to 0} b(\varepsilon)$ is also an upper bound on $E(T|X^0)$, because for every $z > \lim_{\varepsilon\to 0} b(\varepsilon)$ an $\varepsilon'$ can be found such that $z > b(\varepsilon')$. Thus,

$$E(T|X^0 \in I_0) \leq \lim_{\varepsilon\to 0} b(\varepsilon) = \frac{1 + \ln\left(\max(I_0)/s_{min}\right)}{\delta}.$$

This shows the claim. $\square$

## 2.2 Continuous Variable Drift

The variable drift theorem presented in [15] was improved in two independent ways, once in [4] (allowing for non-monotone drift functions) and in [21] (allowing for non-differentiable drift functions).

It is easy to see that the proofs of these two drift theorems can be combined to get the combined result as follows.

THEOREM 2. *Let $(X^t)_{t\geq 0}$ be a sequence of random variables over a finite state space $0 \in S \subseteq \mathbb{R}_0^+$ and let $s_{min} := \min\{x \in S | x > 0\}$. Furthermore, let $T$ be the random variable denoting the first point in time $t \in \mathbb{N}$ for which $X^t = 0$. Suppose that there exist $c \geq 1, d > 0$ and a function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that the function $\frac{1}{h(x)}$ is integrable on $[s_{min}, s_{\max}]$ and*

- *for all $t < T$, $E(X^t - X^{t+1} \mid X^t) \geq h(X^t)$;*
- *for all $t < T$, $P(|X^t - X^{t+1}| \leq d) = 1$; and*
- *for all $x < y$ with $y - x \leq d$, we have $h(x) \leq c\,h(y)$.*

*Then*

$$E(T \mid X^0) \leq c\left(\frac{s_{min}}{h(s_{min})} + \int_{s_{min}}^{X^0} \frac{1}{h(x)}dx\right).$$

For $d = s_{\max}$ and $c = 1$, Theorem 2 yields and demands as much as the theorem presented in [21]. A proof of this theorem can be found in the appendix.

We can now adapt this variable drift theorem to one with continuous search spaces much like in Theorem 1 for the case of multiplicative drift, using the same proof idea. However, we will need to add the restriction of $h$ being continuous.

THEOREM 3. *Let $(X^t)_{t\geq 0}$ be a sequence of random variables over a state space $S = [s_{min}, s_{\max}] \cup 0 \subseteq \mathbb{R}_0^+$ and let $s_{min} > 0$. Furthermore, let $T$ be the random variable denoting the first point in time $t \in \mathbb{N}$ for which $X^t = 0$. Suppose that there exist $c \geq 1$, $d, \delta > 0$ and a $h : \mathbb{R}^+ \to \mathbb{R}^+$ continuous such that the function $\frac{1}{h(x)}$ is integrable on $[s_{min}, s_{\max}]$ and, for all $t < T$ and closed intervals $I \subseteq S$ with $P(X^t \in I)$,*

- *$E(X^t - X^{t+1}|X^t \in I) \geq \inf_{x\in I} h(x)$;*
- *$P(|X^t - X^{t+1}| \leq d - \delta) = 1$; and*
- *for all $x < y$ with $y - x \leq d$, we have $h(x) \leq c\,h(y)$.*

*Then, for all intervals $I_0$ with $P(X^0 \in I_0) > 0$,*

$$E(T \mid X^0 \in I_0) \leq c\left(\frac{s_{min}}{h(s_{min})} + \int_{s_{min}}^{\max(I_0)} \frac{1}{h(x)}dx\right).$$

PROOF. Let $h_{min} = \inf_{s\in[s_{min},s_{\max}]} h(s)$; elementary analysis shows $h_{min} \neq 0$. For all $\varepsilon$ such that

$$0 < \varepsilon < \min\{h_{min}/4, d/2, \delta/2\}$$

we define $f$ such that, for all $x \geq 0$,

$$f(x) = \max\left(\left\lfloor\frac{x}{\varepsilon}\right\rfloor\varepsilon, s_{min}\right),$$

just as in the proof of Theorem 1. Let $S' = \{0\} \cup \{f(s) \mid s \in S\}$. Clearly, $S'$ is a finite set of positive numbers. For all $x > s_{min}$ we have $|f(x) - x| \leq \varepsilon$ and $f(s_{min}) = s_{min}$. We define a function $h' : \mathbb{R}^+ \to \mathbb{R}^+$, such that, for all $x$, $h'(x) = \inf_{x\leq s<x+\varepsilon} h(s) - 2\varepsilon$, and $c' := c(1 + \frac{2\varepsilon}{h_{min}-2\varepsilon})$. We have, for all $t < T$ and $x \in S'$,

$$\begin{aligned}
&E(f(X^t) - f(X^{t+1}) \mid f(X^t) = x)\\
&\quad\geq E(X^t - X^{t+1} - 2\varepsilon \mid x \leq X^t < x + \varepsilon)\\
&\quad\geq E(X^t - X^{t+1} \mid x \leq X^t < x + \varepsilon) - 2\varepsilon\\
&\quad\geq \inf_{x\leq s<x+\varepsilon} h(s) - 2\varepsilon = h'(x)
\end{aligned}$$

and

$$\begin{aligned}
P(f(X^t) - f(X^{t+1}) \leq d) &\geq P(X^t - X^{t+1} + 2\varepsilon \leq d)\\
&= P(X^t - X^{t+1} \leq d - 2\varepsilon)\\
&\geq P(X^t - X^{t+1} \leq d - \delta) = 1.
\end{aligned}$$

For all $z$, let $\bar{h}(z) = \inf_{z\leq s<z+\varepsilon} h(s)$. We have, for all $x < y$ with $y - x \leq d$, using $h_{min} \leq \bar{h}(y)$, we have

$$\begin{aligned}
c'h'(y) &= c\left(1 + \frac{2\varepsilon}{h_{min} - 2\varepsilon}\right)\left(\bar{h}(y) - 2\varepsilon\right)\\
&\geq c\left(1 + \frac{2\varepsilon}{\bar{h}(y) - 2\varepsilon}\right)\left(\bar{h}(y) - 2\varepsilon\right)\\
&= c\bar{h}(y)\\
&\geq c\bar{h}(y) - 2\varepsilon\\
&\geq \bar{h}(x) - 2\varepsilon\\
&= h'(x).
\end{aligned}$$

Thus, we can apply Theorem 2 to the sequence of random variables $(f(X^t))_{t\in\mathbb{N}}$. Thus, for all closed intervals $I_0 \subseteq S$ with $P(X^t \in I_0)$ we have

$$\begin{aligned}
&E(T \mid X^0 \in I_0) \leq E(T \mid f(X^0) \leq \max(I_0))\\
&\leq c'\left(\frac{f(s_{min})}{h'(f(s_{min}))} + \int_{f(s_{min})}^{\max(I_0)} \frac{1}{h'(x)}dx\right)\\
&\leq c'\left(\frac{s_{min}}{\bar{h}(s_{min}) - 2\varepsilon} + \int_{s_{min}}^{\max(I_0)} \frac{1}{\bar{h}(x) - 2\varepsilon}dx\right).
\end{aligned}$$

As above, we can choose $\varepsilon$ arbitrarily small, so that $c'$ converges to 1 and, for all $x$, $h'(x)$ converges to $h(x)$. As in

Theorem 1, we can take the infimum of all upper bounds as an upper bound, leading to

$$E(T \mid X^0 \in I_0) \leq c\left(\frac{s_{min}}{h(s_{min})} + \int_{s_{min}}^{\max(I_0)} \frac{1}{h(x)}dx\right).$$

This shows the claim. □

Finally, we will use a simplified continuous version of the drift theorem concerned with negative drift from [20]; note that the given proof makes no particular use of the requirement of finiteness of the search space, so that we state the following theorem without this condition.

THEOREM 4. *Let $(X^t)_{t\geq 0}$ be a sequence of random variables over a state space $S = [s_{\min}, s_{\max}] \subseteq \mathbb{R}$. Furthermore, let $T$ be the random variable denoting the first point in time $t \in \mathbb{N}$ for which $X^t = s_{\min}$. Suppose that there is a minimal drift $\varepsilon > 0$ and a maximal step size $\delta$ such that, for all $t < T$ and closed intervals $I \subseteq S$ with $P(X^t \in I)$,*

- *$E(X^t - X^{t+1} | X^t \in I) \geq \varepsilon$; and*
- *$P(X^t - X^{t+1} \leq \delta) = 1$.*

*Let $\ell = (s_{\max} - s_{\min})/\delta$. Then there is a constant $c > 0$ such that*

$$P(T \leq 2^{c\ell}) = 2^{-\Omega(\ell)}.$$

This follows from [20, Theorem 4] by rescaling the search space by a factor of $1/\delta$. The second condition is here simplified: instead of requiring the probability to jump a distance of $d$ to be inverse exponentially related to $d$, we just require that the probability of large jumps is 0.

# 3. THE ACO ALGORITHMS

In this section we first formally introduce the single-destination shortest path (SDSP) problem. Afterwards the two algorithms which we will analyze in Section 4 are presented in detail.

## 3.1 Problem Definition

The single-source shortest path problem is one of the most-studied problems in computer science. Given a weighted graph $(V, E, w)$, the goal is to find a shortest path from a given source-vertex to every other vertex in the graph. In this work, we analyze the single-*destination* shortest path problem. Here, from every vertex in the graph, a shortest path has to be found to a single destination vertex. Both problems are equivalent, because one provides the optimal solution for the other if the direction of all edges are reverted. For the sake of simplicity we only deal with weakly connected directed acyclic graphs (DAGs) with a unique sink. This ensures that there is a path from every vertex to the sink, which we regard as the destination vertex.

To model the *stochastic* SDSP, we exchange the deterministic weights $w$ of a graph with random variables $X$. Each edge $e$ now carries a random variable $X_e \in [0, 1]$ that serves as stochastic weight. The deterministic version of SDSP is a special case of the stochastic SDSP, in which all random variables have variance zero.

DEFINITION 5. *Let $(V, E)$ be a DAG (we allow for multiple parallel edges). Assume that there is a unique sink (a vertex without outgoing edges). For each edge $e \in E$, let $X_e \in [0, 1]$ be a random variable describing the stochastic length of $e$. We denote by $X = (X_e)_{e\in E}$ the family of all these. For any (directed) path $p$ consisting of the edges $E_p \subseteq E$, we let $X_p = \sum_{e\in E_p} X_e$ be the* (random) length *of the path $p$.*

*If, for each path $p$ in $G$, $X_p$ is a random variable in $[0, 1]$, then the triple $G = (V, E, X)$ is called* graph with (bounded) stochastic edge weights *or simply a* stochastic-weight graph.

Note that, in a DAG with a unique sink, the sink is reachable from every other vertex. Furthermore, every graph where edge weights are random according to bounded distributions can be scaled and shifted to be a stochastic-weight graph in the sense defined above without changing the (expected) shortest path tree of this "normalized" instance (but with changes to the behavior of the algorithms in this paper); that is, if all random weights are in the interval $[a, b]$ (with $a < b$), then mapping all weights with

$$x \mapsto \frac{x - a}{b - a}$$

will lead to such normalized weights.

## 3.2 MMAS-fp

MMAS-fp is very similar to the ACO-algorithms for SDSP given in [1,4,14]; the key difference is the fitness proportional pheromone update, instead of pheromone update based on the best-so-far solution. The MMAS starts with a homogeneous pheromone distribution on the edges of the graph, and then iteratively updates this distribution. Every iteration, from each vertex of the graph (other than the sink) an artificial ant performs a random walk over the graph until it hits the sink, and then updates the pheromones on the edges outgoing from its start vertex. Note that, in contrast to many applications, only one ant (per vertex) is used, as opposed to sending out several ants and then (for example) choosing only the best to make an update.

The complete MMAS-fp algorithm is described in Algorithm 1 and uses other definitions from this section.

**Algorithm 1** MMAS-fp

**Parameters:** $\rho$, $\tau_{min}$;
**Input:** DAG $G = (V, E)$;
initialize pheromones $\tau$
**while** termination criterion not met **do**
    **for** $u \in V$ in parallel **do**
        construct simple path $p_u$ from $u$ to sink w.r.t. $\tau$;
        $w \leftarrow evaluate(p_u)$;
        $E_u \leftarrow \{(u, v) \in E \mid v \in V\}$;
        **for** $e \in E_u$ **do**
            **if** $e$ first edge in $p_u$ **then**
                $\tau(e) \leftarrow \max((1 - \rho)\tau(e) + \rho(1 - w)), \tau_{min})$;
            **else**
                $\tau(e) \leftarrow \max((1 - \rho)\tau(e), \tau_{min})$;
**return** $\tau$;

### *Path construction.*

An ant constructs a path as follows. If the ant is currently in vertex $v$ after walking the path $p$ and $v$ is not the sink, it randomly chooses one of the edges leaving $v$. The pheromones are stored in a function $\tau : E \to \mathbb{R}^+$. We let $E_v$

be the set of edges leaving $v$, and we let the probability of choosing edge $e \in E_v$ be exactly

$$P(\text{“choose edge } e\text{ ”}) = \frac{\tau(e)}{\sum_{e' \in E_v} \tau(e')}.$$

Afterwards the ant traverses the chosen edge, adds its new position to the path $p$ and further builds its path from there. Algorithm 2 specifies the path construction. At the beginning, the pheromone values of all edges coming from a vertex of out-degree $m$ are initialized with $1/m$ to make every choice equally probable. Note that, since we will only be dealing with DAGs, the path construction does not have to consider loops.

**Algorithm 2** Path Construction

**Input:** DAG G = (V,E), start vertex $u$, pheromones $\tau$;
$i \leftarrow 0$, $v_0 \leftarrow u$;
$V_1 \leftarrow \{p \in V | (v_0, v) \in E\}$;
**while** $V_{i+1} \neq \emptyset$ **do**
  $i \leftarrow i + 1$
  choose $v_i \in V_i$ proportional to $\tau(v_{i-1}, v_i)$;
  $V_{i+1} \leftarrow \{v \in V | (v_i, v) \in E\}$;
**return** $(v_0, \ldots, v_i)$

*Pheromone Update.*

After all ants have finished constructing their paths, the pheromones are updated; each ant starting from a vertex $v$, updates all and only the edges leaving $v$. To get a Max-Min Ant System, we ensure that no pheromone value drops below a predefined threshold $\tau_{min}$. An explicit upper bound for the pheromones is not needed, because every edge loses a factor of $\rho$ of its pheromones and can gain at most $\rho$, such that the highest value the pheromones can reach is 1.

For a given vertex $v$, let $e_v$ be the edge that the ant starting at $v$ chose as the first edge, and let $w$ be the (randomly evaluated) length of its path; then, for each edge $e$ outgoing from $v$, the new pheromone on $e$ is

$$\tau'(e) = \begin{cases} \max((1-\rho)\tau(e) + \rho(1-w), \tau_{min}), & \text{if } e = e_v; \\ \max((1-\rho)\tau(e), \tau_{min}), & \text{otherwise.} \end{cases}$$

In particular, all edges evaporate some pheromone, and only the chosen edge gets rewarded, with a higher amount of pheromone the shorter the path is.

## 3.3 MMAS-fp with Normalization

A problem with MMAS-fp is that, for low average values of the random variables, *all* pheromone values more or less approach the lower pheromone border, which makes the influence of the value of $\tau_{min}$ too strong.

Thus, we introduce the a variant of MMAS-fp, called MMAS-fp-norm, which performs a normalization of the pheromone values at the end of each iteration (see Algorithm 3).

As we will see, for this variant of MMAS-fp we get tighter bounds on the optimization time.

# 4. RUNTIME ANALYSIS

In this section we analyze the algorithms from Section 3 on the stochastic SDSP. We start with an analysis of $m$-parallel links, multigraphs representing simple decision points for the algorithm. Afterwards, we use the results of the analysis

**Algorithm 3** MMAS-fp-norm

**Parameters:** $\rho$, $\tau_{min}$;
**Input:** DAG $G = (V, E)$;
initialize pheromones $\tau$
**while** termination criterion not met **do**
  **for** $u \in V$ in parallel **do**
    construct simple path $p_u$ from $u$ to sink w.r.t. $\tau$;
    $w \leftarrow evaluate(p_u)$;
    $E_u \leftarrow \{(u, v) \in E \mid v \in V\}$;
    **for** $e \in E_u$ **do**
      **if** $e$ first edge in $p_u$ **then**
        $\tau(e) \leftarrow \max((1-\rho)\tau(e) + \rho(1-w)), \tau_{min})$;
      **else**
        $\tau(e) \leftarrow \max((1-\rho)\tau(e), \tau_{min})$;
    $r = \sum_{e \in E_u} \tau(e)$;
    **for** $e \in E_u$ **do**
      $\tau(e) \leftarrow \tau(e)/r$;
**return** $\tau$;

for parallel links to give an upper bound on the expected running time of the algorithm on arbitrary graphs. But first we give some definitions.

If an edge $e$ was constantly reinforced, its pheromone value $\tau$ would still not increase arbitrarily high, but converge to a point where the reinforcement and the loss of the evaporation cancel out. The closer $\tau$ gets to this point, the smaller is its expected gain. At the same time, other edges can be at $\tau_{\min}$ and will therefore not lose any more pheromone. This motivates the following definition, which we will use as our optimization goal.

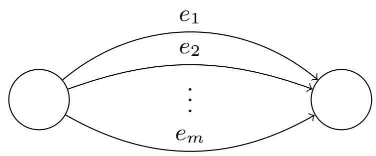
DEFINITION 6. *Let $v$ be a vertex in a stochastic-weight graph $(V, E, X)$. We call $v$ $\beta$-*optimized *iff the probability of choosing the first edge on an optimal path from $v$ to the sink is at least $(1 - \beta)$. We call a graph $\beta$-*optimized *iff all its vertices are $\beta$-*optimized.

Of course two edges with random variables that have almost identical expected value will be hard to tell apart and the relation between the pheromones on those edges will change only very slowly. The difference in the expected value of the random variables between the optimal edge(s) and the others will be an important measurement for the difficulty of the problem, which motivates the following definition.

DEFINITION 7. *Let $v$ be a vertex of degree $m$ in a stochastic-weight graph $(V, E, X)$. For every outgoing edge $e_i$ of $v$ let $\ell_i$ denote the expected length of the shortest path from $v$ to the sink using $e_i$. Without loss of generality we assume that $\ell_i \leq \ell_{i+1}$ for all $1 \leq i \leq n-1$. We call the vertex $v$ $\delta$-*different *iff, for all $i$ with $2 \leq i \leq n$, we have either $\ell_i = \ell_1$ or $\ell_i - \ell_1 \geq \delta$. We call the vertex $v$ strictly $\delta$-*different *iff, for all $i$ with $2 \leq i \leq n$, we have that $\ell_i - \ell_1 \geq \delta$. We call a graph (strictly) $\delta$-*different *iff all its vertices are (strictly) $\delta$-*different.

## 4.1 MMAS-fp on Parallel Links

We start our analysis with a simple case in which the ants will make only a single decision with $m$ alternatives. A simple mathematical model for this is a graph with only two vertices, one of them the sink, and multiple parallel links towards the sink. An *$m$-parallel link* is a directed multigraph with two vertices and $m$ edges $e_1, e_2, \ldots e_m$ from one vertex to the sink; every edge represents one alternative. The following diagram illustrates this graph.




In this case, MMAS-fp simplifies a lot, as only one ant constructs a solution in every iteration, and each path constructions consists only of a single choice. This will allow us to make an easier mathematical analysis; later, we will reduce the case of general graphs to multiple applications of our findings for parallel links.

The crucial step in analyzing an algorithm with a drift theorem lies in finding a suitable potential function. At all times the potential has to decrease in expectation, for the Multiplicative Drift Theorem by at least a constant factor. Also, all states of potential 0 must be optimal states.

In every iteration, we denote with $\tau$ the pheromone of the optimal edge $e_{\max}$, and with $r$ the sum of the pheromone on the edges. For further analysis, we choose $\phi = 1 - \tau/r$ as the potential. As $\tau \leq r$, the potential lies always between zero and one. When the potential is very small, then we know that the probability of choosing the optimal edge is very high, namely $1 - \phi = \tau/r$.

As in many other analyses of ACO systems, it is interesting to see that the total pheromone $r$ behaves very similar to the pheromones on a single edge. In particular, it loses a factor of $\rho$ of its pheromones in every iteration as well and gets reinforced with a random variable between 0 and 1 times $\rho$, that is built out of the random variables of the single edges and their probability to be chosen. But as it can never get reinforced with a value grater than $\rho$ and also $\rho$ of its pheromones evaporate, $r$ is bounded from above, as the next lemma proves.

Lemma 8. *Let $(V, E, X)$ $m$-parallel link; let the evaporation factor $\rho$ be $0 < \rho < 0.5$, and $\tau_{\min} \leq 1/m$. Then, after every iteration of MMAS-fp, we have $r \leq 2$ and $r(1-\rho) + \tau_{\min} m\rho + \rho \leq 2$.*

Proof. We prove the statement by induction over the number of the iterations. We denote with $r^t$ the total pheromone after $t$ iterations. For the induction base, $r^0 \leq 2$ comes directly from the choice of $1/m$ for the initial pheromone values. For the induction step we have that

$$r^{t+1} \leq r^t (1-\rho) + \tau_{\min} m\rho + \rho \leq r^t - r^t\rho + \rho + \rho,$$

as every $X_e$ can evaluate only to numbers smaller 1 and $m\tau_{\min} \leq 1$. Now we use the induction hypothesis and get

$$r^{t+1} \leq r^t(1-\rho) + \rho + \rho \leq 2(1-\rho) + \rho + \rho \leq 2$$

as desired. $\square$

Equipped with these tools we can now formally prove an upper bound on the expected number of iterations that are needed to $\beta$-optimize a strictly $\delta$-different $m$-parallel link.

Theorem 9. *Let $G = (V, E, X)$ be a strictly $\delta$-different $m$-parallel link. Let $\beta, \rho$ and $\tau_{\min}$ be such that $0 < \beta < 0.5$, $0 < \rho \leq \frac{\beta\delta}{8}$ and $\tau_{\min} \leq \rho/m$. Then, after in expectation*

$$O\left(\frac{\ln(1/\beta)}{\delta\rho\tau_{\min}}\right)$$

*iterations of MMAS-fp, $G$ is $\beta$-optimized.*

Proof. We let $e_{\max}$ be the (unique) optimal edge and let $x = 1 - E(X_{e_{\max}})$. For all $t \in \mathbb{N}$, let $\tau^t$ be the pheromone on $e_{\max}$ and $r^t$ be the total pheromone in iteration $t$; furthermore, let $Y^t = \left(1 - \frac{\tau^t}{r^t}\right)$ for $1 - \frac{\tau^t}{r^t} \geq \beta$ and 0 else. Let $T$ be the random variable describing the first point in time $t$ such that $Y^t < \beta$. Then, for all $t < T$, we have

$$\begin{aligned} E\left(\Delta^t Y\right) &:= E\left(Y^t - Y^{t+1} \mid Y^t = 1 - \frac{\tau^t}{r^t}\right) \\ &= \left(1 - \frac{\tau^t}{r^t}\right) - E\left(1 - \frac{\tau^{t+1}}{r^{t+1}}\right) \\ &= E\left(\frac{\tau^{t+1}}{r^{t+1}}\right) - \frac{\tau^t}{r^t} \end{aligned}$$

Fix any $t < T$ and let $\tau = \tau^t$, $r = r^t$ and $\Delta Y = \Delta^t Y$. Because in the worst case $m$ edges can be at $\tau_{\min}$ and thus not lose any pheromones and with $P$ ("$e_{\max}$ gets selected") $= \frac{\tau}{r}$ we have

$$\begin{aligned} E\left(\frac{\tau^{t+1}}{r^{t+1}}\right) \geq{}& \frac{\tau}{r}\frac{\tau(1-\rho) + \rho x}{(r - m\tau_{\min})(1-\rho) + m\tau_{\min} + \rho x} \\ &+ \left(1 - \frac{\tau}{r}\right)\frac{\tau(1-\rho)}{(r - m\tau_{\min})(1-\rho) + m\tau_{\min} + \rho(x-\delta)}. \end{aligned}$$

To enhance readability let $A := r(1-\rho) + \rho x + \rho\tau_{\min} m - \rho\delta$; with some straightforward manipulations, we get

$$\begin{aligned} E(\Delta Y) &\geq -\frac{\tau}{r} + \frac{\tau}{r}\frac{\tau(1-\rho) + \rho x}{A + \rho\delta} + \left(1 - \frac{\tau}{r}\right)\frac{\tau(1-\rho)}{A} \\ &= \tau\frac{\rho\delta\left(1 - \frac{\tau}{r}\right)(1-\rho) - \frac{A\rho\tau_{\min} m}{r}}{A(A + \rho\delta)}. \end{aligned}$$

Because of Lemma 8 we know $A + \rho\delta \leq 2$ and with $\rho x - \rho\delta < \rho$ and $r \geq \tau_{\min} m$ we have

$$E(\Delta Y) \geq \frac{\tau}{4}\left(\rho\delta\left(1 - \frac{\tau}{r}\right)(1-\rho) - 2\rho\right).$$

As $\rho < 0.5$:

$$E(\Delta Y) \geq \frac{\tau\rho}{4}\left(\left(1 - \frac{\tau}{r}\right)0.5\delta - 2\rho\right).$$

From $\rho \leq \frac{\beta\delta}{8}$ and $\left(1 - \frac{\tau}{r}\right) \geq \beta$ we get $\rho < \frac{\left(1 - \frac{\tau}{r}\right)\delta}{8}$, and we get $2\rho < \left(1 - \frac{\tau}{r}\right)\delta/4$ and thus

$$\left(1 - \frac{\tau}{r}\right)0.5\delta - 2\rho \geq 0.25\left(1 - \frac{\tau}{r}\right)\delta.$$

So we have

$$E(\Delta Y) \geq \frac{\tau}{32}\left(1 - \frac{\tau}{r}\right)\rho\delta \geq \left(1 - \frac{\tau}{r}\right)\frac{\tau_{\min}\rho\delta}{32} \geq Y^t\frac{\tau_{\min}\rho\delta}{32}.$$

We have $\tau^0 = \frac{1}{m}$, so $Y^0 = \left(1 - \frac{1}{m}\right)$ and once $Y^t < \beta$, the optimization process is finished. Now we can apply the continuous multiplicative drift theorem on the the random variables $Y^t$, which gives

$$E(T) = O\left(\frac{\ln(1/\beta)}{\delta\rho\tau_{\min}}\right).$$

This finishes the proof. $\square$

If we choose $\rho = \Theta(\beta\delta)$ and $\tau_{\min} = \Theta(\rho/m)$ we get an optimization time of $O\left(\frac{m\ln(1/\beta)}{\beta^2\delta^3}\right)$.

## 4.2 MMAS-fp-norm on Parallel Links

Now we turn to the analysis of MMAS-fp with normalization. In this case, the total pheromone is always 1, so we use as potential $1-\tau$, with $\tau$ the pheromone on the optimal edge.

Let $G = (V, E, X)$ be a strictly $\delta$-different $m$-parallel link. For all $t \in \mathbb{N}$, let $\tau^t$ be the pheromones in the $t$-th iteration of the MMAS-fp-norm Algorithm and let $Y^t = (1-\tau^t)$ and $\Delta^t Y = Y^t - Y^{t+1}$. Let $T$ be the random variable describing the first time when $Y^t \leq \beta$.

As the drift function we use

$$h : \mathbb{R}^+ \to \mathbb{R}^+, x \to min(x, (1-x))\frac{\delta\rho}{9}.$$

Intuitively, we can show a small drift if only very little pheromone is on the desired edge (as this edge is chosen infrequently) and similarly, only a small edge when already a lot of pheromone is present (as the update rule, in this case, makes only very small updates); for medium amounts of pheromone we can expect the best drift. In the following we will prove all three conditions that are needed for the Continuous Variable Drift Theorem separately. Afterwards, we put everything together in the proof of Theorem 13.

We start with condition one, which requires that for every possible potential, the drift-function $h$ gives a lower bound on the drift expected if given a state with that potential.

LEMMA 10. *Let $\beta, \tau_{\min}$ and $\rho$ be such that $0 < \beta < 0.5$, $0 < \tau_{\min} \leq \frac{\beta\delta}{2m}$ and $0 < \rho \leq 0.5$. Then, for all $t < T$,*

$$E\left(\Delta^t Y \mid Y^t = 1-\tau\right) \geq h\left(Y^t\right).$$

PROOF. Let $e_{\max}$ be the optimal edge. With $x$ we denote $1 - E(X_{e_{\max}})$. Let $t \in \mathbb{N}$ with $Y^t > \beta$, let $\tau := \tau^t$ and $\Delta Y = \Delta^t Y$. Then

$$\begin{aligned}E\left(\Delta Y\right) &= (1-\tau) - E\left(1-\tau^{t+1} \mid Y^t = 1-\tau\right)\\ &= E\left(\tau^{t+1} \mid Y^t = 1-\tau\right) - \tau.\end{aligned}$$

In the following we denote $\tau'$ as the pheromone on $e_{\max}$ before the normalization and $r'$ as the sum of all pheromones at the same time, such that $\tau^{t+1} = \frac{\tau'}{r'}$. As, in the worst case, $m$ edges can be at $\tau_{\min}$ and thus not lose any pheromones and with $P$ ("$e_{\max}$ gets selected") $= \tau$ we have

$$\begin{aligned}E(\tau^{t+1}) \geq{}& \tau\frac{\tau\left(1-\rho\right)+\rho x}{\left(1-m\tau_{\min}\right)\left(1-\rho\right)+m\tau_{\min}+\rho x}\\ &+\left(1-\tau\right)\frac{\tau\left(1-\rho\right)}{\left(r-m\tau_{\min}\right)\left(1-\rho\right)+m\tau_{\min}+\left(\rho x-\rho\delta\right)}\end{aligned}$$

To enhance readability, we let $A = (1-\rho)+\rho x+\rho\tau_{\min}m-\rho\delta$. Then we have

$$\begin{aligned}E\left(\Delta Y\right) &\geq -\tau+\tau\frac{\tau\left(1-\rho\right)+\rho x}{A+\rho\delta}+\left(1-\tau\right)\frac{\tau\left(1-\rho\right)}{A}\\ &= \frac{\rho\delta\left(1-\tau\right)\tau\left(1-\rho\right)-\tau A\left(A+\rho\delta\right)}{A\left(A+\rho\delta\right)}\\ &\quad+\frac{A\tau\left(\tau\left(1-\rho\right)+\rho x\right)+A\left(1-\tau\right)\tau\left(1-\rho\right)}{A\left(A+\rho\delta\right)}.\end{aligned}$$

Because in every iteration $r'$ can only gain at most $\rho$, we have that $A+\rho\delta \leq r'_{\max} \leq 1+\rho$ and thus $A(A+\rho\delta) \leq (1+\rho)^2$.

$$\begin{aligned}E\left(\Delta Y\right) \geq{}& \frac{\tau}{(1+\rho)^2}\big(\rho\delta\left(1-\tau\right)\left(1-\rho\right)+\\ &A\left(-A-\rho\delta+\tau\left(1-\rho\right)+\rho x+\left(1-\tau\right)\left(1-\rho\right)\right)\big)\\ ={}& \frac{\tau}{(1+\rho)^2}\left(\rho\delta\left(1-\tau\right)\left(1-\rho\right)-A\left(\rho\tau_{\min}m\right)\right)\end{aligned}$$

Because in every iteration $r'$ can only lose at most $\rho$, we have that $A \geq r'_{\min} \geq 1-\rho$ and thus

$$E\left(\Delta Y\right) \geq \frac{\tau\rho(1-\rho)}{(1+\rho)^2}\left(\delta(1-\tau)-\tau_{\min}m\right)$$

From $\tau_{\min} \leq \frac{\beta\delta}{2m}$ we get that $\tau_{\min}m \leq \frac{\beta\delta}{2}$. As $(1-\tau) \geq \beta$ we have $\delta(1-\tau)-\tau_{\min}m \geq \frac{\delta(1-\tau)}{2}$ and

$$E\left(\Delta Y\right) \geq \tau(1-\tau)\delta\rho\frac{(1-\rho)}{2(1+\rho)^2}.$$

As $0 < \rho < 0.5$ we have

$$E\left(\Delta Y\right) \geq \tau(1-\tau)\frac{\delta\rho}{9} \geq h(1-\tau) = h(Y^t)$$

as desired. □

The following lemma deals with the second condition of the Theorem 1. It gives an upper bound on how much the potential function can change in one step. The main argument is that $\tau$ can only change by $\rho$ at most, because it either loses a factor $\rho$ of its current value, which is never bigger than 1, or it gains $\rho X_{e_{\max}}$ with $X_{e_{\max}}$ being at most 1.

LEMMA 11. *Let $0 < \rho < 0.5$. For all $t < T$ we have*

$$P(|Y^t - Y^{t+1}| < 4\rho) = 1.$$

PROOF. Let $t < T$, $\tau_{\min} \leq \tau^t \leq (1-\beta)$ and $\tau := \tau^t$. In the following we denote $\tau'$ as the pheromone on the optimal edge before the normalization and $r'$ as the sum of all pheromones at the same time such that $\tau^{t+1} = \frac{\tau'}{r'}$. If $\tau^{t+1} > \tau^t$ then

$$|(1-\tau)-(1-\tau^{t+1})| \leq \frac{\tau'}{r'}-\tau \leq \frac{\tau+\rho}{1-\rho}-\tau < 4\rho$$

and if $\tau^{t+1} \leq \tau^t$

$$|(1-\tau)-(1-\tau^{t+1})| \leq \tau-\frac{\tau'}{r'} \leq \tau-\frac{\tau(1-\rho)}{1+\rho} < 4\rho.$$

This finishes the proof. □

With the following lemma we give a factor by which the drift function may decrease at most in one step for increasing potential.

LEMMA 12. *For all $x, y \in \mathbb{R}^+$ with $\tau_{\min} \leq x < y \leq 1-\tau_{\min}$ and $y - x \leq 4\rho$, we have $h(x) \leq \left(1+\frac{4\rho}{\tau_{\min}}\right)h(y)$.*

PROOF. For $y \leq 0.5$ we have $h(y) = y\frac{\rho\delta}{9}$, which is monotonous and therefore we have $h(x) \leq ch(y)$ for every $c \geq 1$. For $y \geq 0.5$ we have $h(y) = (1-y)k$, with $k = \delta\rho/9$. We have

$$\frac{h(x)}{h(y)} = \frac{1-x}{1-y} \leq \frac{1-y+4\rho}{1-y} \leq 1+\frac{4\rho}{\tau_{\min}}$$

as desired. □

Now we put the pieces together and apply the variable drift theorem, which will give us the desired upper bound on the runtime of MMAS-fp-norm on parallel links.

THEOREM 13. *Let $\beta, \tau_{\min}$ and $\rho$ be such that $0 < \beta < 0.5$, $0 < \tau_{\min} \leq \frac{\beta\delta}{2m}$ and $0 < \rho \leq 0.5$. Then, after in expectation*

$$O\left(\frac{\ln(1/(\tau_{\min}\beta))}{\delta \min(\rho, \tau_{\min})}\right)$$

*iterations of MMAS-fp-norm, the strictly $\delta$-different $m$-parallel link $G$ is $\beta$-optimized.*

PROOF. For all $t \in \mathbb{N}$, we change $Y^t$ to 0 if $\tau < \beta$ (this can only increase drift). Lemmas 10 to 12 show that with $d = 4\rho$ and $c = 1 + \frac{4\rho}{\tau_{\min}}$ for $h$ all conditions of the continuous variable drift theorem are fulfilled. Thus, we get

$$\begin{aligned} E(T \mid Y_0) &\leq c\left(\frac{s_{\min}}{h(s_{\min})} + \int_{s_{\min}}^{Y_0} \frac{1}{h(x)} dx\right) \\ &= c\left(\frac{\beta}{\beta\rho\delta/9} + \int_{\beta}^{1-\tau_{\min}} \frac{1}{h(x)} dx\right) \\ &= c\frac{9}{\rho\delta}\left(1 + \int_{\beta}^{0.5} \frac{1}{x} dx + \int_{0.5}^{1-\tau_{\min}} \frac{1}{1-x} dx\right). \end{aligned}$$

This equals

$$\begin{aligned} & c\frac{9}{\rho\delta}\left(1 + \int_{\beta}^{0.5} \frac{1}{x} dx + \int_{\tau_{\min}}^{0.5} \frac{1}{y} dy\right) \\ =& \left(1 + \frac{4\rho}{\tau_{\min}}\right) \frac{9}{\rho\delta}\left(1 + 2\ln 0.5 - \ln(\tau_{\min}) - \ln\beta\right) \\ =& O\left(\frac{\ln(1/(\tau_{\min}\beta))}{\min(\tau_{\min}, \rho)\delta}\right). \end{aligned}$$

This finishes the proof. □

Recall that, for MMAS-fp, Theorem 9 gives a bound of

$$O\left(\frac{\ln(1/\beta)}{\delta\rho\tau_{\min}}\right).$$

The bound of Theorem 13 improves on the denominator, as it now only included the smaller of $\tau_{\min}$ and $\rho$; however, the numerator as an addition $\log(1/\tau_{\min})$, which is comparatively small.

Especially for the next section, we are also interested in how long an edge stays optimized. As we have drift in the right direction and a small step size, an application of the negative drift theorem [20] gives us an answer.

THEOREM 14. *Let $\beta, \tau_{\min}$ and $\rho$ be such that $0 < \beta < 0.5$, $0 < \tau_{\min} \leq \frac{\beta\delta}{2m}$ and $0 < \rho \leq 0.5$. Then, once $G$ is $\beta$-optimized, there is a constant $c$ such that, for all $s$, $G$ stays $(\beta + s\rho)$-optimized for $2^{cs}$ iterations with probability $1 - 2^{-\Omega(s)}$.*

PROOF. In Lemma 10 we proved a strictly positive drift towards being $\beta$-optimized. From Lemma 11 we know that the step-size of the potential is in $O(\rho)$. Now the drift theorem concerned with negative drift from Oliveto and Witt [20] in the simplified form stated in Theorem 4 gives the desired result. □

Theorem 14 allows for the following interpretation of small pheromone update factor $\rho$: the MMAS does not quickly adapt to unlucky evaluations of random variables, but conservatively stays with the medium-term best option.

## 4.3 MMAS-fp-norm on SDSP

In this section we extend our analysis to arbitrary graphs. We suppose our graphs to be $\delta$-different and give an upper bound on the expected runtime on the algorithm to $\beta$-optimize the graph. Note that every graph is $\delta$-different for some $\delta > 0$, but usually this $\delta$ is not known beforehand, which makes it hard to set the parameters right. To avoid confusion we first introduce vocabulary that will help us talk about the length of paths more clearly.

DEFINITION 15. *Let $v$ be a vertex in the directed stochastic-weight graph. We define* OEPL$(v)$ *(optimal expected path length) as the expected length of the in path from $v$ to the sink which is shortest in expectation.*

The main idea in the following proofs is to start with the sink, which is of course always optimized, and then gradually widen the circle of optimized vertices until the whole graph is optimized, as was done previously in many other papers.

Whenever a decision making process in a vertex has to be analyzed, a parallel link is constructed to simulate this process such that Theorem 13 can be applied. This step is performed in Lemma 16.

LEMMA 16. *Let $v$ be a vertex of degree $m$ in a $\delta$-different stochastic-weight graph $G = (V, E, X)$ without parallel edges. Suppose that all vertices on a shortest path from $v$ to the sink except for $v$ be $\delta/(2n)$-optimized and stay so optimized for any polynomial number of rounds. Then, with parameters $\rho$ and $\tau_{\min}$ such that $\frac{\delta^2}{16n^2} = \tau_{\min} \leq \rho \leq 0.5$, after in expectation $O\left(\frac{\ln(n/\delta)}{\delta\tau_{\min}}\right)$ iterations of MMAS-fp-norm, $v$ is $\delta/(4n)$-optimized.*

PROOF. Let $p$ be a shortest path from $v$ to the sink such that every vertex other than $v$ is $\delta/(2n)$-optimized; let $v'$ be the second vertex on $p$ (right after $v$). Thus, an ant starting from $v$ using the edge to $v'$ and then walking randomly according to the construction procedure construct a path of expected length $x$ with

$$\begin{aligned} x &\leq \mathrm{OEPL}(v) + 1 - (1 - \delta/(2n))^n \\ &\leq \mathrm{OEPL}(v) + \delta/2 \end{aligned}$$

as we deviate from the optimal path with probability at most $1 - (1 - \delta/(2n))^n$, and this gives a path of length at most 1 (this uses the definition of stochastic-weight graph, which requires all path lengths to be random variables in $[0, 1]$); the second inequality uses Bernoulli's inequality. All paths not using the edge $(v, v')$ can in the worst case have an expected length $y$ of at least

$$y \geq \mathrm{OEPL}(v) + \delta$$

as $G$ is $\delta$-different. Let $m$ be the out-degree of $v$ in $G$. Now we can simulate the optimization of $v$ by constructing a parallel link $G'$ consisting of $v$ and the sink. We use $m$ edges from $v$ to the sink with random variables $X_1, \ldots X_m \in [0, 1]$ with expected values $E(X_1) = \mathrm{OEPL}(v) + \delta/2$ and $E(X_i) = \mathrm{OEPL}(v) + \delta$ for all $i$ with $2 \leq i \leq m$; thus, we have

$$E(X_i) - E(X_1) \geq \frac{\delta}{2}.$$

Thus, the constructed parallel link $G'$ is a strictly $\delta' := \delta/2$-different stochastic-weight graph. As $G$ has no parallel edges, the out-degree $m$ of $v$ is bounded by $n$. The optimization time of $G'$ gives an upper bound on the time until $v$ in $G$ is optimized: $X_1$ represents all the (in expectation) shortest paths, the other $X_i$ represent the other choices. We apply Theorem 13 with $\beta = \delta/(2n)$ such that the crucial inequality

$$\tau_{\min} = \frac{\delta^2}{16n^2} = \frac{\delta}{4n}\frac{\delta}{2}\frac{1}{2n} \leq \frac{\beta\delta'}{2m}$$

is fulfilled. The theorem yields that after

$$O\left(\frac{\ln(1/(\tau_{\min}\beta))}{\delta'\tau_{\min}}\right) = O\left(\frac{\ln(1/(\tau_{\min}\delta))}{\delta\tau_{\min}}\right)$$

iterations of MMAS-fp-norm $G'$ is $\delta/(4n)$-optimized. □

Now all that is left to do is applying Lemma 16 to every vertex in an order such that the lemma is always applied to vertices the shortest expected paths of which use only vertices that are already optimized. Note that this Theorem applies to a limited range of $\rho$, for which a good bound is derivable.

THEOREM 17. *Let $G = (V, E, X)$ be a $\delta$-different stochastic-weight graph. Let $\tau_{\min}$ and $\rho$ be such that*

$$\tau_{\min} \leq \frac{\delta^2}{16n^2} \text{ and } \tau_{\min} \leq \rho \leq \frac{\delta}{4n(\log n)^2}.$$

*Then, after in expectation*

$$O\left(\frac{n\ln(1/(\tau_{\min}\delta))}{\delta\tau_{\min}}\right)$$

*iterations of MMAS-fp-norm the graph is $\delta/2$-optimized.*

PROOF. We apply Theorem 14 with $s = (\log n)^2$ and Lemma 16 once for each vertex going backwards in a topological sorting of all vertices. This gives a high probability of success within the stated time bound; using a standard restart argument, we get the desired bound on the expectation. □

If we plug $\tau_{\min}$ into the runtime formula, we get an expression dependent only on the graph-size $n$ and the difference $\delta$:

$$O\left(\frac{n^3\ln(n/\delta)}{\delta^3}\right).$$

Note that a very similar proof would also work with the MMAS-fp algorithm.

# 5. SUMMARY

In this work we saw that MMAS-fp (with or without normalization) is well suited for solving the stochastic SDSP problem for stochastic-weight graphs: in contrast to other algorithms, it optimizes the expected path length, as opposed to "winning paths" (paths which come out shorter than any other path with a probability of at least 50%) like in [4] for an elitist MMAS.

On the downside, a parameter $\delta$ has to be estimated upfront in order to set the parameters right. An upper bound for the expected optimization time of the algorithm then depends on this constant $\delta$ and the size of the graph $n$: $O\left(\frac{n^3\ln(n/\delta)}{\delta^3}\right)$. In future works, one could try to make the parameters dependent of an approximation factor instead of the not very practical value $\delta$. It would also be interesting to see how MMAS-fp behaves for $\delta$ too small; we conjecture that, in this case, the algorithm would converge to some "robust" solution, a solution which has very good expected value even when deviating from it randomly.

For the analysis we used drift theorems on the continuous pheromone-values; in particular, for the algorithms we considered, the state of the algorithm is a random element of an uncountable set, if the random variables used have uncountable support. Thus, we cannot restrict ourselves to a finite state space, which drift theorems usually demand. In order to be able to use drift analysis, we showed that the finiteness-condition is mostly superfluous: we proved that a multiplicative and a variable drift theorem are true for an interval in $\mathbb{R}$ as the search space. It will be interesting to see if the conditions of the theorem can be further weakened, making drift analysis an even more universal and handy tool.


## *Acknowledgments*

The authors are thankful to the anonymous reviewers, which helped improve this version a lot. Also, we are indebted to Benjamin Doerr for several fruitful discussions.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