Orthogonal Graph Drawing
with Flexibility Constraints

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Abstract. In this work we consider the following problem. Given a planar graph G with maximum degree 4 and a function flex : $E \longrightarrow \mathbb{N}_0$ that gives each edge a *flexibility*. Does G admit a planar embedding on the grid such that each edge e has at most flex(e) bends? Note that in our setting the combinatorial embedding of G is not fixed.

We give a polynomial-time algorithm for this problem when the flexibility of each edge is positive. This includes as a special case the problem of deciding whether G admits a drawing with at most one bend per edge.

1   Introduction

Orthogonal graph drawing is one of the most important techniques for the human-readable visualization of complex data. Its æsthetic appeal derives from its simplicity and straightforwardness. Since edges are required to be straight orthogonal lines—which automatically yields good angular resolution and short links—the human eye may easily adapt to the flow of an edge. The readability of orthogonal drawings can be further enhanced in the absence of crossings, i.e., if the underlying data exhibits planar structure. Unfortunately, not all planar graphs have an orthogonal drawing in which each edge may be represented by a straight horizontal or vertical line. In order to be able to visualize all planar graphs, nonetheless, we allow edges to have bends. Since bends obfuscate the readability of orthogonal drawings, however, we are interested in minimizing the number of bends on the edges. Previous approaches to orthogonal graph drawing in the presence of bends focus on either the minimization of the maximum number of bends per edge or the total number of bends in the drawing.

In typical applications, however, edges have varying importance for the readability depending on their semantic and their importance for the application. Thus, it is convenient to allow some edges to have more bends than others.

We consider the following orthogonal graph drawing problem, which we call FLEXDRAW. Given a *4-planar graph* G, i.e., G is planar and has maximum degree 4, and for each edge e a non-negative integer flex(e), its *flexibility*. Does G admit a planar embedding on the grid such that each edge e has at most flex(e) bends? Such a drawing of G on the grid is called a *flex-drawing*. For a graph with flex(e) > 0 for each edge e in G we shortly say that G has *positive flexibility*.

The problem we consider generalizes a well-studied problem in orthogonal graph drawing, namely the problem of deciding whether a given graph is β-embeddable for some non-negative integer β. A 4-planar graph is β-*embeddable* if it admits an embedding on the grid with at most β bends per edge.

Garg and Tamassia [6] show that it is $\mathcal{NP}$-hard to decide 0-embeddability. The reduction crucially relies on the ability to construct graphs with rigid embeddings. Later, we show that this is impossible if we allow at least one bend per edge. This is a key observation which yields, among others, an efficient algorithm for recognizing 1-embeddable graphs. For special cases, namely planar graphs with maximum degree 3 and series-parallel graphs, Di Battista et al. [1] gave an algorithm that minimizes the total number of bends and hence solves 0-embeddability. On the other hand, Biedl and Kant [2] show that every 4-planar graph admits a drawing with at most two bends per edge with the only exception of the octahedron, which requires an edge with three bends. Similar results are obtained by Liu et al. [9].

Liu et al. [8] claim to have found a characterization of the planar graphs with minimum degree 3 and maximum degree 4 that admit an orthogonal embedding with at most one bend per edge. They also claim that this characterization can be tested in polynomial time. Unfortunately, their paper does not include any proofs and to the best of our knowledge a proof of these results did not appear. Morgana et al. [10] characterize the class of *plane graphs* (i.e., planar graphs with a given embedding) that admit a 1-bend embedding on the grid by forbidden configurations. They also present a quadratic algorithm that either detects a forbidden configuration or computes a 1-bend embedding.

If the combinatorial embedding of a 4-planar graph is given, Tamassia's flow network can be used to minimize the total number of bends [11]. Note that this approach may yield drawings with a linear number of bends for some of the edges. Given a combinatorial embedding that admits a 1-bend embedding, however, the flow network can be modified in a straightforward manner to minimize the total number of bends using at most one bend per edge.

The problem we consider involves considering all embeddings of a planar graph. Many problems of this sort are $\mathcal{NP}$-hard. For instance, 0-embeddability is $\mathcal{NP}$-hard [6], even though it can be decided efficiently if we are given an embedding by minimizing the total number of bends.

Contribution and Outline. In this work we give an efficient algorithm that solves FLEXDRAW for graphs with positive flexibility. Since FLEXDRAW contains the problem of 1-embeddability as a special case this closes the complexity gap between the $\mathcal{NP}$-hardness result for 0-embeddability by Garg and Tamassia [6] and the efficient algorithm for computing 2-embeddings by Biedl and Kant [2].

We present some preliminaries in Section 2. In Section 3 we study orthogonal flex-drawings of graphs with a fixed embedding and introduce the maximum rotation of a graph as a measure of how "flexible" it is. In Section 4 we show that replacing certain subgraphs with graphs that behave similarly does not change the maximum rotation. Based on this fact and the SPQR-tree we give an algorithm that solves FLEXDRAW for biconnected 4-planar graphs with positive

flexibility. We extend our algorithm to arbitrary 4-planar graphs with positive flexibility in Section 5. For full proofs we refer the reader to the long version of this article [3].

2   Preliminaries

Orthogonal representation. The *orthogonal representation* introduced by Tamassia [11] describes orthogonal drawings of plane graphs, by listing the faces as sequences of bends. The advantage of the orthogonal representation is, that it neglects the lengths of the segments, thus it is possible to apply different operations on the drawing, without the need to worry about the exact geometry. Our orthogonal representation is always normalized, i.e., each edge has only bends in one direction; this slightly differs from the notion introduced by Tamassia.

The orthogonal representation of a plane graph G is defined as a set of lists $\mathcal{R} = \{\mathcal{R}(f_1), \ldots, \mathcal{R}(f_k)\}$ with a list for each face f_i of G. For each face f_i the list $\mathcal{R}(f_i)$ is a circular list of *edge descriptions* containing the edges on the boundary of f_i in clockwise (counter-clockwise if f_i is the external face) order. Each description $r \in \mathcal{R}(f_i)$ contains the following information: edge(r) denotes the edge represented by r, bends(r) is an integer whose absolute value is the number of 90°-bends of edge(r), where positive numbers represent bends to the right and negative numbers bends to the left. For a given edge description $r \in \mathcal{R}(f_i)$ we denote its successor in $\mathcal{R}(f_i)$ by r' and represent the angle α between edge(r) and edge(r') in f_i by their rotation rot$(r, r') = 2 - \alpha/90°$. Every edge has exactly two edge descriptions, if r is one of them, the other is denoted by $\bar{r}$. Since each face forms a rectilinear polygon, every orthogonal representation $\mathcal{R}$ of an orthogonal drawing has the following three properties.

I Each edge description r is consistent with $\bar{r}$, i.e., bends$(\bar{r}) = -$ bends(r).
II The interior bends of any face f_i sum up to 4 and the exterior bends to -4:

$$\sum_{r \in \mathcal{R}(f_i)} (\text{bends}(r) + \text{rot}(r, r')) = \begin{cases} -4, & \text{if } f \text{ is the external face,} \\ +4, & \text{if } f \text{ is an internal face.} \end{cases}$$

III The angles around every node sum up to 360°.

Given an orthogonal representation $\mathcal{R}$ of a graph, a corresponding orthogonal drawing can be computed efficiently [11]. Hence, it is sufficient to work with orthogonal representations. An orthogonal representation is *valid* for a given flexibility function flex if $|\text{bends}(r)| \leq \text{flex}(\text{edge}(r))$ for each edge description r.

For a planar graph $G = (V, E)$ with orthogonal representation $\mathcal{R}$ and two vertices s and t on the outer face f_1, we denote by $\pi_{\mathcal{R}}(s, t)$ the path in $\mathcal{R}(f_1)$ that connects s and t in counter-clockwise direction. Such a path $\pi = \pi(s, t)$ consists of consecutive edge descriptions $r_1, \ldots, r_k$. We define the *rotation* of π as

$$\text{rot}_{\mathcal{R}}(\pi) = \sum_{i=1}^{k} \text{bends}(r_i) + \sum_{i=1}^{k-1} \text{rot}(r_i, r_{i+1}).$$

Moreover, if v is a vertex of G that has exactly one angle in the outer face, we denote by $\text{rot}_{\mathcal{R}}(v)$ the rotation of this angle. Note that, for a single edge description r we have $\text{rot}(r) = \text{bends}(r)$. If it is clear from the context which orthogonal representation is meant we omit the indices of π and rot. The concept of rotation is similar to the spirality defined by Di Battista et al. [1].

The value $\text{rot}(\pi(s,t))$ describes the shape of the path $\pi(s,t)$ in the orthogonal representation in terms of the angle between its start- and its endpoint. Fixing the rotation of $\pi(s,t)$, $\pi(t,s)$ and the outer angles at s and t in a sense determines the shape of the outer face. In Section 4, we will exploit this by replacing certain subgraphs of G with simpler graphs whose outer faces have the same shapes.

Connectivity, st-graphs and the SPQR-tree. A graph is *connected* if there exists a path between any pair of vertices. A *separating k-set* is a set of k vertices whose removal disconnects the graph. Separating 1-sets and 2-sets are *cutvertices* and *separation pairs*. A graph is *biconnected* if it does not have a cut vertex and triconnected if it does not have a separation pair. The maximal biconnected components of a graph are called *blocks*.

The *block-cutvertex tree* of a connected graph is a tree whose nodes are the blocks and cutvertices of the graph. In the block-cutvertex tree a block B and a cutvertex v are joined by an edge if v belongs to B.

A *weak st-graph* is a 4-planar graph $G = (V, E)$ with two designated vertices s and t such that the graph $G + st$ is planar and has maximum degree 4. An *st-graph* is a weak st-graph such that $G + st$ is biconnected. An orthogonal representation $\mathcal{R}$ of a (weak) st-graph with positive flexibility is *valid* if each edge e has at most $\text{flex}(e)$ bends and s and t are embedded on the outer face. A valid orthogonal representation of a (weak) st-graph is *tight* if all angles at s and t in inner faces are $90°$.

We distinguish st-graphs with $\deg(s), \deg(t) \leq 2$ by the degrees of s and t. An st-graph is of Type (1,1) if $\deg(s) = \deg(t) = 1$, it is of Type (1,2) if one of them has degree 1 and the other one has degree 2 and it is of Type (2,2) if $\deg(s) = \deg(t) = 2$.

To handle the decomposition of biconnected graphs into triconnected components we use the SPQR-tree, which was introduced by Di Battista and Tamassia [4,5]. A detailed description of the SPQR-tree can be found in the literature [4,5,7]. Here we just give a sketch and some notation.

The SPQR-tree $\mathcal{T}$ of a graph G is a rooted tree that is determined by the *split pairs* of G. A split pair is a pair of vertices that are either connected by an edge or that is a separation pair. In the latter case the corresponding connected components are called the *split components* of the split pair.

The SPQR-tree $\mathcal{T}$ has four different types of nodes, namely S-,P-,Q- and R-nodes. Each node μ of $\mathcal{T}$ has an associated biconnected multigraph, its *skeleton*, denoted by $\text{skel}(\mu)$, which can be seen as a simplified version of the original graph. An edge uv in $\text{skel}(\mu)$ indicates that $\{u,v\}$ is a split pair and the edge uv represents one or more split components of $\{u,v\}$. The *pertinent graph* of a node μ, denoted by $\text{pert}(\mu)$ is the graph that is represented by the subtree of $\mathcal{T}$ with root μ. Note that in particular each pertinent graph is an st-graph.
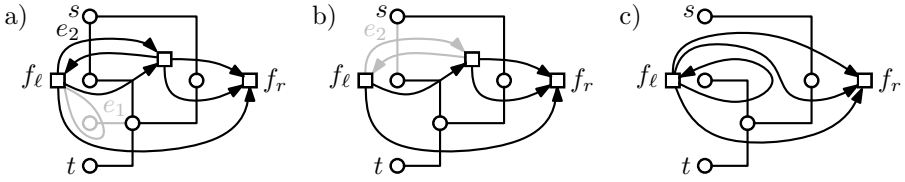
Fig. 1. An st-graph with flexibility 1 for all edges with $\mathrm{rot}(\pi(s,t)) = 1$ and its flex graph $G^{\times}$ (a), after removal of bridge e_1 (b), and removal of edge e_2 (c)

The SPQR-tree of a graph G represents all planar embeddings of G in the sense that choosing planar embeddings for all skeletons of $\mathcal{T}$ corresponds to a choosing a planar embedding of G and vice versa.

Our approach. We start out with an observation. Let G be a 4-planar graph with positive flexibility and let $\{s,t\}$ be a split pair of G that splits G into two subgraphs G_1, G_2 and let e_{ref} be an edge of G_1. Let ρ be the maximum rotation of $\pi(s,t)$ over all embeddings of G_2 where s and t are on the outer face.

If G_2 is of Type (1,1) then obviously the following holds. If G admits a valid orthogonal drawing with the given flexibility such that e_{ref} is embedded on the outer face then also the graph G' where G_2 is replaced by an edge f with flexibility ρ admits such a drawing. Graphs of Type (1,2) and (2,2) allow for similar substitutions.

Thus we can substitute st-graphs of each type with a small gadget graph to obtain a new graph G' such that if G has a valid drawing then also G' has one. We show that the converse is also true, i.e., if the graph G' admits such an embedding then also G does. We then exploit this characterization algorithmically using the SPQR-tree of G to successively replace subgraphs of G by simpler graphs.

3 The Maximum Rotation with a Fixed Embedding

The goal of this section is to derive a description of the valid orthogonal representations of a given (weak) st-graph with positive flexibility and a fixed embedding. Namely, we prove that the values that can be obtained for $\mathrm{rot}(\pi(s,t))$ form an interval for these graphs. We show that if there exists a valid orthogonal representation $\mathcal{R}$ with $\mathrm{rot}_{\mathcal{R}}(\pi(s,t)) \geq 0$ then there exists an orthogonal representation $\mathcal{R}'$ with $\mathrm{rot}_{\mathcal{R}'}(\pi(s,t)) = \mathrm{rot}_{\mathcal{R}}(\pi(s,t)) - 1$, which can be obtained from $\mathcal{R}$ by only altering the number of bends on certain edges.

To model the possible changes of an orthogonal representation $\mathcal{R}$ of a (weak) st-graph G that can be performed by only changing the number of bends on edges we introduce the *flex graph* $G^{\times}$ of G with respect to $\mathcal{R}$, which is based on the bidirected dual graph of G. Thus, the flex graph is a directed multigraph. See Fig. 1a for an illustration. We start out by adding to G the edge st and embed it into the outer face of G thus splitting the outer face into two faces f_ℓ and f_r, where f_ℓ is bounded by $\pi(s,t)$ and the new edge $\{s,t\}$ and f_r is

bounded by $\pi(t, s)$ and $\{s, t\}$. We denote this graph by $\bar{G}$ and its dual graph by $\bar{G}^*$. We set $V^\times = V(\bar{G}^*)$ and we define $E^\times$ as follows. For each edge e of G denote its incident faces in $\bar{G}$ by f_u and f_v and let r_u and r_v be the edge descriptions of e in $\mathcal{R}(f_u)$ and $\mathcal{R}(f_v)$, respectively. We add the edge (f_u, f_v) if $-\text{flex}(e) < \text{bends}(r_u)$ and, analogously, we add (f_v, f_u) if $-\text{flex}(e) < \text{bends}(r_v)$. Consider an edge (f_u, f_v) of $G^\times$ and let r_u and r_v be the edge descriptions of the corresponding edge e in G. The fact that $(f_u, f_v) \in E^\times$ indicates that it is possible to decrease $\text{bends}(r_u)$ (and thus increase $\text{bends}(r_v)$) by at least 1 without violating the flexibility of e.

Assume that there exists a simple directed path from f_ℓ to f_r in $G^\times$. Let $f_\ell = f_1, f_2, \ldots, f_k = f_r$ be this path. We construct a new orthogonal representation $\mathcal{R}'$ from $\mathcal{R}$ as follows. For each edge $f_i f_{i+1}$, $i = 1, \ldots, k-1$, let e_i be the corresponding edge of G and let $r_i \in \mathcal{R}(f_i), \bar{r}_i \in \mathcal{R}(f_{i+1})$ be its edge descriptions. We obtain $\mathcal{R}'$ from $\mathcal{R}$ by decreasing $\text{bends}(r_i)$ by 1 and increasing $\text{bends}(\bar{r}_i)$ by 1 for $i = 1, \ldots, k-1$. First, it is clear that $\mathcal{R}'$ satisfies Properties I and III since we increase and decrease the number of bends consistently and we do not change any angles at vertices. Property II holds since each face of G has either none of its edge descriptions changed or exactly one of them is increased by 1 and exactly one of them is decreased by 1. Moreover, since the path starts at f_ℓ and ends at f_r we have that $\text{rot}_{\mathcal{R}'}(\pi(s, t)) = \text{rot}_{\mathcal{R}}(\pi(s, t)) - 1$. We now show that such a path exists if $\text{rot}(\pi(s, t)) \geq 0$.

Lemma 1. *Let G be a weak st-graph with positive flexibility and let $\mathcal{R}$ be a valid orthogonal representation of G with $\text{rot}_{\mathcal{R}}(\pi(s, t)) \geq 0$. Then the flex graph $G^\times$ contains a directed path from f_ℓ to f_r.*

Proof. Assume that G is a minimal counter example such that $G^\times$ does not contain such a path. First, we show that in $G^\times$ there exists at least one edge starting from f_ℓ. Let $\pi(s, t)$ be composed of the edge descriptions $r_1, \ldots, r_k$ in $\mathcal{R}(f)$, where f is the outer face of G. Then, by assumption we have $\text{rot}(\pi(s, t)) = \sum_{i=1}^{k} \text{bends}(r_i) + \sum_{i=1}^{k-1} \text{rot}(r_i, r_{i+1}) \geq 0$. Since $\text{rot}(r_i, r_{i+1}) \leq 1$ for $i = 1, \ldots, k-1$ we have that $\sum_{i=1}^{k} \text{bends}(r_i) \geq -k+1$ and hence there is at least one r_j with $\text{bends}(r_j) \geq 0$. Hence, $G^\times$ contains an edge corresponding to $\text{edge}(r_j)$ that starts at f_ℓ. This shows that there always exists an edge (f_ℓ, f_u) in $G^\times$. We distinguish three types of edges (f_ℓ, f_u). If $f_u = f_r$ then (f_ℓ, f_u) is the desired path.

If $f_u = f_\ell$ the corresponding edge e of G is a bridge whose removal does not disconnect s and t, see Fig. 1b, then let H be the connected component of $G - e$ containing s and t and let $\mathcal{S}$ be the restriction of $\mathcal{R}$ to H. For the outer face of H we have that $\text{rot}_{\mathcal{S}}(\pi(s, t)) + \text{rot}_{\mathcal{S}}(s) + \text{rot}_{\mathcal{S}}(\pi(t, s)) + \text{rot}_{\mathcal{S}}(t) = -4$. Since $\pi_{\mathcal{R}}(t, s) = \pi_{\mathcal{S}}(t, s)$ we have that $\text{rot}_{\mathcal{S}}(\pi(t, s)) = \text{rot}_{\mathcal{R}}(\pi(t, s))$. Moreover, since we only remove edges the angles at s and t (and thus their rotations) do not decrease, i.e., we have $\text{rot}_{\mathcal{S}}(t) \leq \text{rot}_{\mathcal{R}}(t)$ and $\text{rot}_{\mathcal{S}}(s) \leq \text{rot}_{\mathcal{R}}(s)$. Hence, we have that $\text{rot}_{\mathcal{S}}(\pi(s, t)) \geq -4 - \text{rot}_{\mathcal{R}}(\pi(t, s)) - \text{rot}_{\mathcal{R}}(s) - \text{rot}_{\mathcal{R}}(t) = \text{rot}_{\mathcal{R}}(\pi(s, t)) \geq 0$. Since H has fewer edges than G it is not a counter example and its flex graph $H^\times$ contains a path from f_ℓ to f_r. Since $H^\times$ is a subgraph of $G^\times$ this contradicts the assumption that G is a counter example.

Otherwise, f_u is an internal face of G, see Fig. 1c. Let e be the corresponding edge of G. Let $H := G-e$ and let $\mathcal{S}$ be the orthogonal representation $\mathcal{R}$ restricted to H. Note that the flex graph of $H^\times$ of H can be obtained from $G^\times$ by removing all edges between f_ℓ and f_u and merging f_ℓ and f_u into a single node f'_ℓ. As above we obtain that $\mathrm{rot}_\mathcal{S}(\pi(s,t)) \geq 0$ and hence in $H^\times$ there exists a path from f'_ℓ to f_r. The corresponding path in $G^\times$ (after undoing the contraction of f_ℓ and f_u) either starts at f_ℓ or at f_u and ends at f_r. In the former case we have found our path, in the latter case the path together with the edge (f_ℓ, f_u) forms the desired path. Again this contradicts the assumption that G is a counter example.   □

Recall that a valid orthogonal representation of a (weak) st-graph is tight if the inner angles at s and t are 90°. We show that a valid orthogonal representation can be made tight without decreasing $\mathrm{rot}(\pi(s,t))$.

Lemma 2. *Let G be a weak st-graph with positive flexibility and let $\mathcal{R}$ be a valid orthogonal representation. Then there exists a valid orthogonal representation $\mathcal{R}'$ of G with the same planar embedding such that $\mathcal{R}'$ is tight, $\mathrm{rot}_{\mathcal{R}'}(\pi(s,t)) \geq \mathrm{rot}_\mathcal{R}(\pi(s,t))$ and $\mathrm{rot}_{\mathcal{R}'}(\pi(t,s)) \geq \mathrm{rot}_\mathcal{R}(\pi(t,s))$.*

Let G be an st-graph with positive flexibility and a fixed planar embedding $\mathcal{E}$. Lemma 1 shows that the attainable values of $\mathrm{rot}(\pi(s,t))$ for a given st-graph with a fixed embedding form an interval. Hence, the set of possible rotations can be described by the boundaries of this interval and we define the *maximum rotation* of G with respect to $\mathcal{E}$ as $\mathrm{maxrot}_\mathcal{E}(G) = \max_{\mathcal{R} \in \Omega} \mathrm{rot}_\mathcal{R}(\pi(s,t))$ where Ω contains all valid orthogonal representations of G whose embedding is $\mathcal{E}$.

The following theorem states that indeed the maximum rotation essentially describes the orthogonal representations of st-graphs with fixed embedding and positive flexibility.

Theorem 1. *Let G be an st-graph with positive flexibility and fixed embedding $\mathcal{E}$. Then for each $\rho \in \{-1, \ldots, \mathrm{maxrot}_\mathcal{E}(G)\}$ there exists a valid and tight orthogonal representation $\mathcal{R}$ of G with planar embedding $\mathcal{E}$ such that $\mathrm{rot}_\mathcal{R}(\pi(s,t)) = \rho$.*

Using a variant of Tamassia's flow network [11] the maximum rotation can be computed efficiently for st-graphs with a fixed embedding.

Theorem 2. *Given an st-graph $G = (V, E)$ with fixed embedding $\mathcal{E}$ with s and t on the outer face we can compute $\mathrm{maxrot}_\mathcal{E}(G)$ in $O(n^{3/2})$ time or decide that G does not admit a valid orthogonal representation with this embedding.*

4   Biconnected Graphs

Until now the planar embedding of our input graph was fixed. Now, we assume that this embedding is variable. Following the approach of the previous section we define the maximum rotation of a (weak) st-graph G as $\mathrm{maxrot}(G) = \max_{\mathcal{E} \in \Psi} \mathrm{maxrot}_\mathcal{E}(G)$ where Ψ contains all planar embeddings of G such that s and t are embedded on the outer face.

In this section we show that $\mathrm{maxrot}(G)$ essentially describes all valid orthogonal representations of G in the sense that substituting a subgraph H of G with a different graph H' with $\mathrm{maxrot}(H) = \mathrm{maxrot}(H')$ does not change $\mathrm{maxrot}(G)$. We further use this substitution to give an algorithm that computes maxrot by successively reducing the size of the graph. To handle the different possible planar embeddings we use the SPQR-tree and we substitute subgraphs with small graphs that have only one embedding. We need the following technical lemma.

Lemma 3. *Let G be an st-graph with $\deg(s), \deg(t) \leq 2$ and let $\mathcal{R}$ be a tight orthogonal representation of G. Then $\mathrm{rot}(\pi(s,t)) + \mathrm{rot}(\pi(t,s)) = -x$ where x is 0,1 and 2 for graphs of Type (1,1), (1,2) and (2,2), respectively.*

The following theorem shows that indeed the maximum rotation describes all possible rotation values of an st-graph.

Theorem 3. *Let G be an st-graph with positive flexibility and let ρ be an integer. Then there exists a tight orthogonal representation $\mathcal{R}$ of G with $\mathrm{rot}(\pi(s,t)) = \rho$ if and only if $-\mathrm{maxrot}(G) - x \leq \rho \leq \mathrm{maxrot}(G)$ where x depends on the Type of G and $x = 0, 1, 2$ for Types (1,1), (1,2) and (2,2), respectively.*

Proof. We first show the only if part. Let $\mathcal{R}$ be any embedding of G. By the definition of $\mathrm{maxrot}(G)$ we clearly have that $\mathrm{rot}_{\mathcal{R}}(\pi(s,t)) \leq \mathrm{maxrot}(G)$. By definition we also have that $\mathrm{rot}_{\mathcal{R}}(\pi(t,s)) \leq \mathrm{maxrot}(G)$ (otherwise by mirroring we could obtain an orthogonal representation $\mathcal{R}'$ with $\mathrm{rot}_{\mathcal{R}'}(\pi(s,t)) > \mathrm{maxrot}(G)$) and hence with Lemma 3 we obtain $-\mathrm{rot}(\pi(s,t)) - x \leq \mathrm{maxrot}(G)$.

It remains to show that for any given ρ in the range we can find a valid orthogonal representation. If $-1 \leq \rho \leq \mathrm{maxrot}(G)$ we find an orthogonal representation as follows. Let $\mathcal{R}$ be a valid orthogonal embedding of G with $\mathrm{rot}(\pi(s,t)) = \mathrm{maxrot}(G)$. By Lemma 2 we can reduce the inner angles at s and t to $90°$ without decreasing $\mathrm{rot}(\pi(s,t))$. By Theorem 1 we thus find the desired orthogonal representation.

If $\rho \leq -2$ by Lemma 3 we need to find a valid orthogonal representation $\mathcal{R}$ with $\mathrm{rot}_{\mathcal{R}}(\pi(t,s)) = -\rho - x =: \rho'$. Note that by the definitions of ρ and x we have that $0 \leq \rho' \leq \mathrm{maxrot}(G)$. As above we obtain a valid orthogonal embedding $\mathcal{R}'$ of G with $\mathrm{rot}_{\mathcal{R}'}(\pi(s,t)) = \rho'$. We obtain $\mathcal{R}$ by mirroring $\mathcal{R}'$.   □

Note that if s (or t) has degree 1 then its incident edge allows for three different rotations and hence the range of valid rotations contains at least three integers. This observation together with the theorem yields the following.
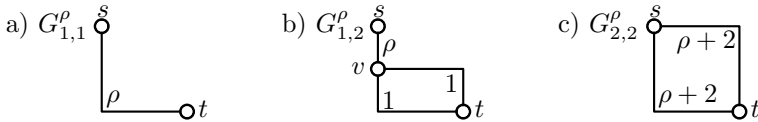
Corollary 1. *Let G be an st-graph with positive flexibility. If G admits a valid drawing then $\mathrm{maxrot}(G) \geq 1$ if G is of Type (1,1) or (1,2) and $\mathrm{maxrot}(G) \geq -1$ if G is of Type (2,2).*

In particular, Theorem 3 shows that an st-graph G with $\deg(s) = \deg(t) = 1$ essentially behaves like a single edge st with flexibility $\mathrm{maxrot}(G)$. The following lemma shows that we can replace any st-graph with $\deg(s), \deg(t) \leq 2$ in a graph G by a different st-graph of the same type and with the same maximum rotation without changing $\mathrm{maxrot}(G)$. Fig. 2 illustrates the lemma and its proof.
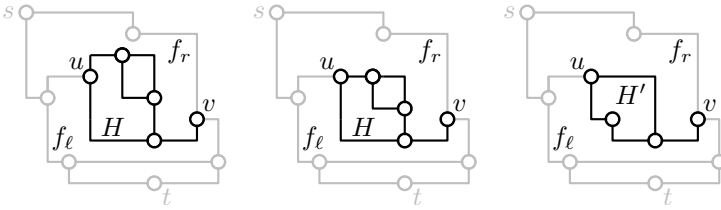
Fig. 2. Illustration of Lemma 4, st-graph G with split pair $\{u, v\}$ splitting off H (left), replacement of H with a tight orthogonal representation (middle) and replacement of H with a graph H' with $\mathrm{maxrot}(H) = \mathrm{maxrot}(H') = 3$ (right)

Fig. 3. Gadgets for st-graphs with maximum rotation ρ depending on the Type

Lemma 4. *Let $G = (V, E)$ be an st-graph with positive flexibility and let $\{u, v\}$ be a split pair of G that splits G into two components G^- and H such that G^- contains s and t and H is an st-graph of Type (1,1), Type (1,2) or Type (2,2) (with respect to vertices u and v). Let H' be an st-graph with designated vertices u', v' of the same type as H with $\mathrm{maxrot}(H') = \mathrm{maxrot}(H)$.*

Then G admits a valid orthogonal representation $\mathcal{R}$ with $\mathrm{rot}_{\mathcal{R}}(\pi(s, t)) = \rho$ if and only if the graph G', which is obtained from G by replacing H with H' admits a valid orthogonal representation $\mathcal{R}'$ with $\mathrm{rot}_{\mathcal{R}'}(\pi(s, t)) = \rho$.

We now present three especially simple families of replacement graphs, called *gadgets*, for st-graphs of Types (1,1), (1,2) and (2,2), respectively; see Fig. 3. Let ρ be an integer. The graph $G_{1,1}^{\rho}$ is simply an edge st with $\mathrm{flex}(st) = \rho$. The graph $G_{1,2}^{\rho}$ has three vertices s, v, t and two edges between s and v, both with flexibility 1, and the edge vt with flexibility ρ. The gadget $G_{2,2}^{\rho}$ consists of two parallel edges between s and t, both with flexibility $\rho + 2$. Note that by Corollary 1 all edges of our gadgets have again positive flexibility and that $\mathrm{maxrot}(G_{1,1}^{\rho}) = \mathrm{maxrot}(G_{1,2}^{\rho}) = \mathrm{maxrot}(G_{2,2}^{\rho}) = \rho$. Moreover, each of these graphs has a unique embedding with s and t on the outer face.

We now describe an algorithm that computes $\mathrm{maxrot}(G)$ for a given st-graph G with positive flexibility or decides that G does not admit a valid orthogonal representation. We use the SPQR-tree $\mathcal{T}$ of $G + st$, rooted at the Q-node corresponding to st to represent all planar embeddings of G with s and t on the outer face. Our algorithm processes the nodes of the SPQR-tree in a bottom-up fashion and computes the maximum rotation of each pertinent graph from the maximum rotations of the pertinent graphs of its children. For each node μ we have a variable $\mathrm{maxrot}(\mu)$. We will prove later that after processing a node we have that $\mathrm{maxrot}(\mu) = \mathrm{maxrot}(\mathrm{pert}(\mu))$. For each Q-node μ we initialize

maxrot(μ) to be the flexibility of the corresponding edge. We now show how to compute maxrot(μ) from the maximum rotations of its children. We make a case distinction based on the type of μ.

If μ **is an R-node** let $\mu_1, \ldots, \mu_k$ be the children of μ and let $H_1, \ldots, H_k$ be their pertinent graphs. Each virtual edge in skel(μ) represents at least one incidence of an edge of G to its poles. Since skel(μ) is 3-connected each node has at least degree 3 and hence no virtual edge can represent more than two incidences, i.e., the nodes of skel(μ) have degree at most 2 in the subgraphs of G that are represented by the virtual edges of μ. As we already know their maximum rotations we can simply replace each of the graphs by a corresponding gadget; we call the resulting graph G_μ. Since the embeddings of all gadgets are completely symmetric it is enough to compute the maximum rotations of G_μ for the only two embeddings $\mathcal{E}_1$ and $\mathcal{E}_2$ induced by the embeddings of skel(μ). We set maxrot(μ) = max{maxrot$_{\mathcal{E}_1}(G_\mu)$, maxrot$_{\mathcal{E}_2}(G_\mu)$} if one of them admits a valid representation. Otherwise we stop and return "infeasible".

If μ **is a P-node** we treat μ similar as in the case where μ is an R-node. Again, we have that each pole has degree at least 3 in skel(μ) and hence no virtual edge can represent more than two edge incidences. We replace each virtual edge with the corresponding gadget and try all possible embeddings of skel(μ), which are at most six and store the maximum rotation or stop if none of the embeddings admits a valid representation.

If μ **is an S-node** let $\mu_1, \ldots, \mu_k$ be the children of μ. We set maxrot(μ) = $\sum_{i=1}^{k}$ maxrot(μ_i) + k − 1.

Theorem 4. *Given an st-graph $G = (V, E)$ with positive flexibility it can be checked in $O(n^{3/2})$ time whether G admits a valid orthogonal representation. In the positive case* maxrot(G) *can be computed within the same time complexity.*

Proof. We prove the invariant that after processing node μ we have maxrot(μ) = maxrot(pert(μ)). The proof is by induction on the height h of the SPQR-tree $\mathcal{T}$ of $G + st$. Let μ be the node of $\mathcal{T}$ whose parent corresponds to st.

If $h = 1$ then G is a single edge e and μ its corresponding Q-node. Since maxrot(G) = flex(e) the claim holds. For $h > 1$ let $\mu_1, \ldots, \mu_k$ be the children of μ. By induction we have that maxrot(μ_i) = maxrot(pert(μ_i)) for $i = 1, \ldots, k$. We make a case distinction based on the type of μ.

If μ is an R- or a P-node then by Lemma 4 we have that maxrot(G_μ) = maxrot(pert(μ)) and since the gadgets have a unique embedding we consider all relevant embeddings of G_μ. If none of the embeddings admits a valid orthogonal representation then obviously also pert(μ) and thus G do not admit valid orthogonal representations.

If μ is an S-node and the pertinent graphs of its children admit a valid orthogonal representation then there always exists a valid orthogonal representation of pert(μ). Let $H_1, \ldots, H_k$ be the pertinent graphs of the children of μ and let $v_1, \ldots, v_{k+1}$ be the vertices in skel(μ) such that v_i and v_{i+1} are the poles of H_i. By Theorem 3 there exist tight orthogonal representations $\mathcal{R}_1, \ldots, \mathcal{R}_k$ of $H_1, \ldots, H_k$ with rot($\pi(v_i, v_{i+1})$) = maxrot(μ_i). We put these orthogonal representations together such that the angles at the nodes $v_2, \ldots, v_k$ on

$\pi(v_1, v_{k+1})$ are $90°$. Hence we get an orthogonal representation of $\text{pert}(\mu)$ with $\text{rot}(\pi(v_1, v_{k+1})) = \sum_{i=1}^{k} \text{maxrot}(\mu_i) + k - 1$. On the other hand if we had an orthogonal representation of $\text{pert}(\mu)$ with a higher rotation then at least one of its children μ_i would need to have a rotation that is bigger than $\text{maxrot}(\mu_i)$.

This proves the correctness of the algorithm. For the running time note that the SPQR-tree can be computed in linear time [7]. The time for computing $\text{maxrot}(\mu)$ for a given node μ from the maximum rotations of its children can be done in $O(|\text{skel}(\mu)|^{3/2})$ time by Theorem 4 since $\text{skel}(\mu)$ has only a constant number of embeddings. The total running-time follows from the fact that the total size of all skeletons is in $O(n)$.   □

This theorem can be used to solve FLEXDRAW for biconnected 4-planar graphs with positive flexibility. Such a graph G admits a valid orthogonal representation if and only if one of the graphs $G - e$, $e \in E(G)$ (which is an st-graph with respect to the endpoints of e) admits a valid orthogonal representation such that e can be added to this representation. This can be done if and only if $\text{maxrot}(G - e) + \text{flex}(e) \geq 2$. This can be seen as follows. Let s and t be the endpoints of e. Adding e to $G - e$ creates a new interior face and the total rotation of this new face needs to be 4. We can have at most two $90°$ angles at s and t, hence $\text{maxrot}(G - e) + \text{flex}(e) \geq 2$ is a necessary condition. On the other hand, it is not hard to see that it is possible to add e to a tight orthogonal representation of $G - e$. If $\text{flex}(e) \geq 3$ then we can add e to a tight orthogonal representation of $G - e$ with $\text{rot}(\pi(s, t)) = -1$. Otherwise, we add e to a tight orthogonal representation of $G - e$ with $\text{rot}(\pi(s, t)) = 2 - \text{flex}(e)$, which is possible since $2 - \text{flex}(e) \geq -1$ holds in this case. We obtain the following theorem; the running time is due to $O(n)$ applications of the algorithm for st-graphs.

Theorem 5. FLEXDRAW *can be solved in time* $O(n^{5/2})$ *for biconnected 4-planar graphs with positive flexibility.*

5   Connected Graphs

In this section we generalize our results to connected 4-planar graphs that are not necessarily biconnected. We analyze the conditions under which orthogonal representations sharing a cut vertex can be combined and use the block-cutvertex tree to derive an algorithm that decides whether a connected 4-planar graph with positive flexibility admits a valid orthogonal drawing.

Lemma 5. *Let G be a connected 4-planar graph with cutvertex v and corresponding cut components $H_1, \ldots, H_k$. Then G admits a valid orthogonal representation if and only if all cut components H_i have valid orthogonal representations such that at most one of them has v not on the outer face.*

Now let G be a connected 4-planar graph with positive flexibility and $\mathcal{B}$ its block-cutvertex tree. Let further B be a block of G that is a leaf in $\mathcal{B}$ and let v be the unique cutvertex of B.

If B is the whole graph G we return "true" if and only if G admits any valid orthogonal representation. This can be checked with the algorithm from the previous Section.

If B is not the whole graph G we check whether B admits a valid orthogonal representation having v on its outer face. This can be done with the algorithm from the previous section by rooting the SPQR-tree of B at all edges incident to v. If it does admit such an embedding then by Lemma 5 G admits a valid orthogonal embedding if and only if the graph G', which is obtained from G by removing the block B, admits a valid orthogonal embedding. We check G' recursively. If B does not admit such an embedding we mark B and proceed with another unmarked leaf. If we ever encounter another block B' that has to be marked we return "infeasible". This is correct as in this case B has to be embedded in the interior of B' and vice versa, which is obviously impossible. Checking a single block B can be done in $O(|B|^{5/2})$ time by Theorem 5. Since the total size of all blocks is in $O(n)$ the total running-time is $O(n^{5/2})$. This proves the following theorem.

Theorem 6. FLEXDRAW *can be solved in* $O(n^{5/2})$ *time for 4-planar graphs with positive flexibility.*

Conclusion. We have shown that FLEXDRAW can be solved efficiently for graphs with positive flexibility. Moreover, it is straightforward to generalize our algorithm to positive flexibility functions flex : $E \longrightarrow \mathbb{N} \cup \{\infty\}$, i.e., some edges may be bent arbitrarily often. An interesting open question is whether FLEXDRAW can still be handled if few edges are required to have no bends.

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