11

Simultaneous Embedding of Planar Graphs

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11.1 Introduction

Traditional problems in graph drawing involve the layout of a single graph, whereas in simultaneous graph drawing we are concerned with the layout of multiple related graphs. In particular, consider the problem of drawing a series of graphs that share all, or parts of the same vertex set. The graphs may represent different relations between the same set of objects, or alternatively, the graphs may be the result of a single relation that changes through time.

In this chapter we survey efforts to address the following problem: Given a series of graphs that share all, or parts of the same vertex set, what is a natural way to layout and display them? The layout and display of the graphs are different aspects of the problem, but also closely related, as a particular layout algorithm is likely to be matched best with a specific visualization technique. As stated above, however, the problem is too general and it is unlikely that one particular layout algorithm will be best for all possible scenarios. Consider the case where we only have a pair of graphs in the series, and the case where we have hundreds of related graphs. The "best" way to layout and display the two series is likely going to be different. Similarly, if the graphs in the sequence are very closely related or not related at all, different layout and display techniques may be more appropriate.

For the layout of the graphs, there are two important criteria to consider: the *readability* of the individual layouts and the *mental map preservation* in the series of drawings. The readability of individual drawings depends on aesthetic criteria such as display of symmetries, uniform edge lengths, and minimal number of crossings. Preservation of the mental map can be achieved by ensuring that vertices that appear in consecutive graphs in the series, remain in the same positions. These two criteria are often contradictory. If we individually layout each graph, without regard to other graphs in the series, we may optimize readability at the expense of mental map preservation. Conversely, if we fix the vertex positions in all graphs, we are optimizing the mental map preservation but the individual layouts may be far from readable. In simultaneous graph embedding, vertices are placed in the exact same locations in all the graphs, while the layout of the edges may differ.

Visualization of related graphs, that is, graphs that are defined on the same set of vertices, arise in many different settings. Software engineering, databases, and social network analysis are all examples of areas where multiple relationships on the same set of objects are often studied. In evolutionary biology, phylogenetic trees are used to visualize the ancestral relationship among groups of species. Depending on the assumptions made, different algorithms produce different phylogenetic trees. Comparing the outputs and determining the most likely evolutionary hypothesis can be difficult if the drawings of the trees are laid out independently of each other.

While in some of the above examples the graphs are not necessarily planar, solving the planar case can provide intuition and ideas for the more general case. With this in mind, here we concentrate on the problem of simultaneous embedding of planar graphs. Simultaneous embedding of planar graphs generalizes the notion of traditional graph planarity and is motivated by its relationship with problems of graph thickness, geometric thickness, and applications such as the visualization of graphs that evolve through time.

The thickness of a graph is the minimum number of planar subgraphs into which the edges of the graph can be partitioned; see [MOS98] for a survey. Thickness is an important concept in VLSI design, since a graph of thickness k can be embedded in k *layers*, with any two edges drawn in the same layer intersecting only at a common vertex and vertices placed in the same location in all layers. A related graph property is *geometric thickness*, defined to be the minimum number of layers for which a drawing of G exists having all edges drawn as straight-line segments [DEH00]. Finally, the *book thickness* of a graph G is the minimum number of layers for which a drawing of G exists, in which edges are drawn as straight-line segments and vertices are in convex position [BK79]. It has been shown that the book thickness of planar graphs is no greater than four [Yan89].

11.1.1   Problem Definitions

This chapter is structured along three basic simultaneous embedding results for planar graphs, SIMULTANEOUS GEOMETRIC EMBEDDING (SGE), SIMULTANEOUS EMBEDDING WITH FIXED EDGES (SEFE), and SIMULTANEOUS EMBEDDING (SE), Figure 11.1 illustrates the three cases. For all three problems the input always consists of two planar graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ sharing a common subgraph $G = (V, E) = (V_1 \cap V_2, E_1 \cap E_2)$.

The most strict variant is SIMULTANEOUS GEOMETRIC EMBEDDING (SGE), which asks for planar straight-line drawings of G_1 and G_2 such that common vertices have the same coordinates in both drawings. The requirements of SGE are very strict, and as we will see in Section 11.2, there exist a lot of examples that do not admit such an embedding. While the problem SIMULTANEOUS EMBEDDING WITH FIXED EDGES still requires common vertices to have the same coordinates, it relaxes the straight-line requirement by allowing arbitrary curves for representing edges. To maintain the mental map, common edges are
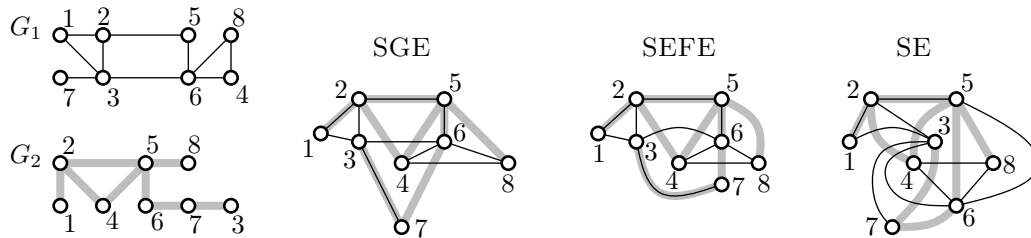
Figure 11.1 Two graphs G_1 and G_2 together with an SGE, a SEFE and an SE. In the SGE all edges are straight line segments while some edges in the SEFE are not. The SE contains common edges ($\{3,7\}$ and $\{5,6\}$) that are drawn differently with respect to G_1 and G_2.

still required to be represented by the same curves. Finally, SIMULTANEOUS EMBEDDING drops the constraints on the curves altogether and just requires common vertices to have the same coordinates.

For all these problems it is common to also use the problem name to denote a corresponding embedding, that is, we also say that G_1 and G_2 have an SGE, SEFE or SE if they admit solutions to these problems. Moreover, all these problems readily generalize to $k > 2$ input graphs $G_1, \ldots, G_k$, by requiring that the conditions hold for each pair of graphs. In this case a common restriction is to require that all input graphs share exactly the same graph G, that is, $G = G_i \cap G_j$ for $i \neq j$. We call this behavior *sunflower intersection*.

We note that simultaneous embedding problems are closely related to constrained embedding problems. For example if the planar embedding of one of the two graphs of an instance of SEFE is already fixed, the problem of finding a SEFE is equivalent to finding an embedding of the second graph respecting a prescribed embedding for a subgraph, namely the common graph. This constrained embedding problem is known as PARTIALLY EMBEDDED PLANARITY. Angelini et al. [ADF+10] show that this problem can be solved in linear time and, in the spirit of Kuratowski's theorem, Jelínek et al. [JKR11] characterize the yes-instances by forbidden substructures. A similar tie to constrained embedding problems exists in the case of SE. After fixing the drawing of one of the two input graphs it remains to draw a single graph without crossings at prescribed vertex positions. This problem is known as POINT SET EMBEDDING and Pach and Wenger show that this is always possible [PW98]. There are other, less obvious relations between simultaneous embedding and constrained embedding problems, which will be described later.

11.1.2 Overview and Outline

This chapter starts with the three simultaneous embedding problems SGE, SEFE, and SE, and we discuss each of them in one of the following sections. There are three major classes of results on simultaneous embedding problems. The first class contains algorithms that, for given graphs with certain properties, always produce a simultaneous embedding, perhaps with additional quality guarantees. These results show the existence of simultaneous embeddings for the corresponding graph classes. The second class contains counterexamples that do not admit a simultaneous embedding. The third class contains algorithms and complexity results for the problem of testing whether a given instance admits a simultaneous embedding.

We present a survey of the results on SGE in Section 11.2. Due to the strong requirements of SGE results of the first type, which identify classes of graphs that always admit a simultaneous embedding, exist only for very few and strongly restricted graph classes. For

example, even a path and a tree of depth 4 may not have an SGE [AGKN12]. Moreover, it is NP-hard to decide SGE and there are no further results of the third type, that is, algorithms testing whether an instance has an SGE or not, even for restricted instances.

Section 11.3 presents the SEFE problem, which turns out to be much less restrictive than SGE. For example a tree and a path do always admit a SEFE although they do not have an SGE [Fra07]. On the other hand, examples not having a SEFE are also counterexamples for SGE. Moreover, it is still open whether SEFE can be tested in polynomial time for two graphs, whereas it is NP-complete for three or more graphs [GJP+06]. However, for two graphs, there exist several results of the third type, that is, testing algorithms, for restricted inputs. For example, it is possible to decide in linear time whether a pair of graphs admits a SEFE or not, if the common graph is biconnected [ABF+12, HJL10].

In Section 11.4, we consider the least restrictive simultaneous embedding problem, SE, which only requires common vertices to have the same coordinates in all drawings. As every planar graph can be drawn without crossings even if the position of every vertex is fixed [PW98], there are no counterexamples for SE and it is not necessary to have a testing algorithm. The results on SE focus on creating simultaneous embeddings such that edges have few bends and the resulting drawings use small area.

Sections 11.5–11.8 presents several variants of approaches to simultaneous embedding that do not quite fall into the categories of the three main problems. The problem variants discussed in Section 11.5 relax the requirement of having a fixed mapping between the vertices of G_1 and G_2. They rather ask whether a suitable mapping can be found such that a SEFE exists [BCD+07]. Colored SGEs are somewhere between and allow the mapping to identify only vertices having the same color [BEEB+11]. Section 11.6 deals with matched drawings requiring straight-line drawings of the two input graph such that each common vertex has only the same y-coordinate in both drawings. Other work, discussed in Section 11.7, deals with the problem of simultaneously representing a planar graph and its dual [Tut63] and considers different types of simultaneous representations, such as simultaneous intersection representations, as introduced by Jampani and Lubiw [JL09]. Section 11.8 presents several practical approaches to simultaneous embedding problems.

In Section 11.9 results on morphing between different planar drawings of the same graph are presented. A morph aims to preserve the mental map between different drawings of the same graph, which can be seen as the opposite to drawing different graphs such that the common part is drawn the same. Finally, in Section 11.10, we present a list of open questions. The list contains questions that have been open for several years, as well as questions that are motivated by recent research results.
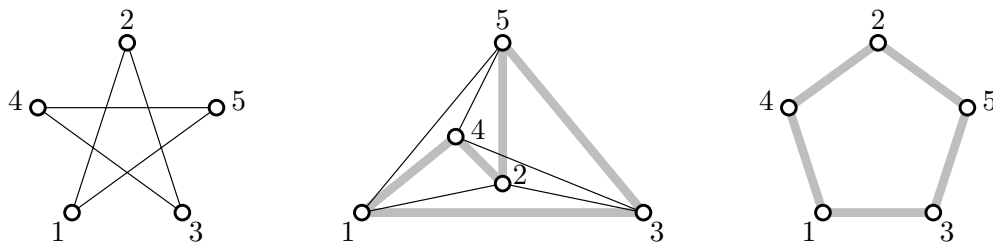
11.2   Simultaneous Geometric Embedding

In this section we consider the most desirable (and most restrictive) kind of simultaneous drawings, the SGEs. Most results on that problem are summarized in Table 11.1. Figure 11.3 illustrates the relation between these results. Before we describe the results in more detail we start with a small example. While it may be tempting to say that if the union of two graphs contains a subdivision of K_5 or $K_{3,3}$ then the two graphs have no simultaneous geometric embedding, this is not the case; see Figure 11.2. In fact, while planarity testing for a single graph can be done in linear time [HT74], Estrella-Balderrama et al. [EBGJ+08] show that the decision problem SGE is NP-hard. Other results concerning the complexity of SGE (for example for restricted graph classes) are not known.

In the following we describe the results illustrated in Figure 11.3. We start with algorithms always creating an SGE when the input is restricted to special graph classes. We then

SGE Instance	Existence	Area	Ref.
G_1 & G_2 paths	✓	$n \times n$	[BCD$^+$07]
G_1 path & G_2 extended star	✓	$O(n^2) \times O(n)$	[BCD$^+$07]
G_1 caterpillar & G_2 path	✓	$n \times 2n$	[BCD$^+$07]
G_1 & G_2 caterpillar	✓	$3n \times 3n$	[BCD$^+$07]
2 stars	✓	$3 \times (n-2)$	[BCD$^+$07]
k stars	✓	$O(n) \times O(n)$	[BCD$^+$07]
G_1 & G_2 cycles	✓	$4n \times 4n$	[BCD$^+$07]
G_1 & G_2 have maximum degree 2	✓	—	[DEK04]
G_1 wheel & G_2 cycle	✓	—	[CvKL$^+$11]
G_1 tree & G_2 matching	✓	—	[CvKL$^+$11]
G_1 outerpath & G_2 matching	✓	—	[CvKL$^+$11]
G_1 tree of depth 2 & G_2 path	✓	—	[AGKN12]
G_1 level-planar w.r.t. path G_2	✓	—	[CEBFK09]
G_1 & G_2 planar	✗	—	[BCD$^+$07]
G_1 path & G_2 planar	✗	—	[BCD$^+$07, EK05a]
G_1 path & G_2 edge disjoint	✗	—	[FKK09]
three paths	✗	—	[BCD$^+$07]
G_1 matching & G_2 planar	✗	—	[CvKL$^+$11]
six matchings	✗	—	[CvKL$^+$11]
G_1 & G_2 outerplanar	✗	—	[BCD$^+$07]
G_1 & G_2 trees	✗	—	[GKV09]
G_1 tree of depth 4 & G_2 edge disjoint path	✗	—	[AGKN12]

Table 11.1 A list of classes of graphs that are either known to always have an SGE or that contain counterexamples. For the positive cases, the area consumption is given, provided that it is known.

continue with graph classes containing counterexamples. Finally, we consider the results not fitting in one of these two cases.

Figure 11.2 The union of the graph on the left and the graph on the right is a K_5, but the middle drawing shows a simultaneous geometric embedding of the two graphs.

11.2.1 Graph Classes with SGE

Brass et al. [BCD$^+$07] give several algorithms for different restricted graph classes always creating an SGE. In the simplest case G_1 and G_2 are both required to be paths. This result is easy to prove and also provides good intuition for most of the positive results:
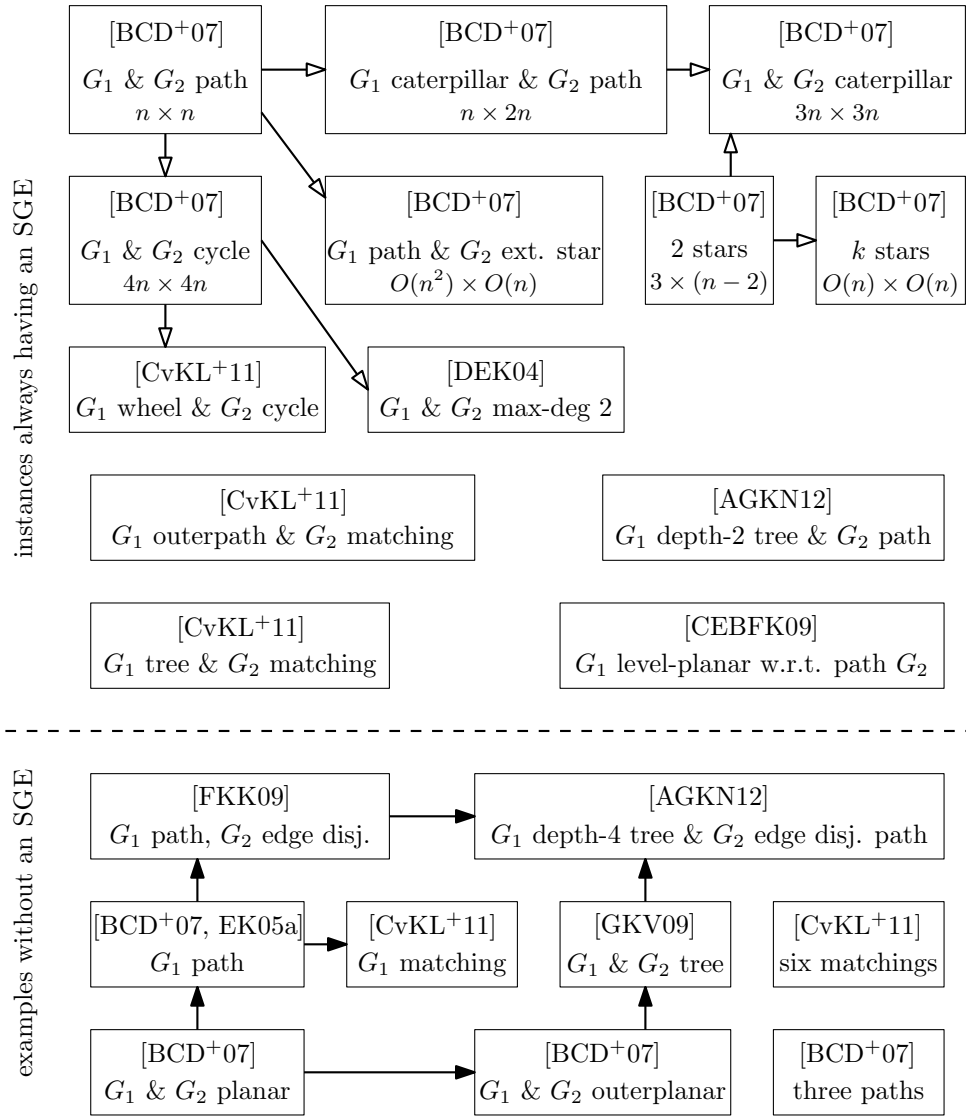
**Figure 11.3** Overview over the so far known results on SGE. Each box represents one result and an arrow highlights that the source-result is extended by the target-result. The arrowheads are empty for the cases in which this is only true if the grid size is neglected. Note that transitive arrows are omitted.

Theorem 11.1 *For two paths P_1 and P_2 on the same vertex set V of size n an SGE on a grid of size $n \times n$ can be found in linear time.*

Proof: For each vertex $u \in V$, we embed u at the integer grid point (p_1, p_2), where $p_i \in \{1, 2, \ldots, n\}$ is the vertex's position in the path P_i, $i \in \{1, 2\}$. Then, P_1 is embedded as an x-monotone polygonal chain, and P_2 is embedded as a y-monotone chain. Thus, neither path is self-intersecting; see Figure 11.4 for an example. $\qquad\square$
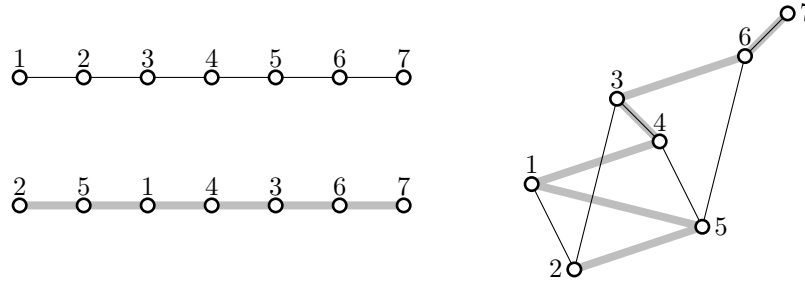
Figure 11.4 Two paths simultaneously embedded such that one path is x-monotone and the other is y-monotone.

Brass et al. [BCD$^+$07] also consider more general graph classes, such as *caterpillars* (trees being paths after the removal of all leaves), *stars* (trees with at most one inner vertex called *center*), and *extended stars* (collection of stars with an additional special root and paths from the special root to the centers of all stars). They show that a caterpillar and a path admit an SGE on a grid of size $n \times 2n$, which can be extended to two caterpillars on a grid of size $3n \times 3n$. Moreover, they can simultaneously embed two stars on a $3 \times (n - 2)$ grid and extend it to the case of k stars on an $O(n) \times O(n)$-grid. Finally, the pairs path plus extended star and cycle plus cycle can be embedded on $O(n^2) \times O(n)$ and $4n \times 4n$ grids, respectively. The latter two results both extend the case of two paths (when neglecting the grid size).

The result for two cycles was further extended by Duncan et al. [DEK04] and Cabello et al. [CvKL$^+$11]. Duncan et al. [DEK04] show that a graph with maximum degree 4 has geometric thickness 2. To this end, they show that two graphs with maximum degree 2 always admit a simultaneous geometric embedding. However, their algorithm computes drawings with potentially large area.

Cabello et al. [CvKL$^+$11] show the existence of an SGE for a *wheel* (union of a star and a cycle on its leaves) and a cycle. They moreover give algorithms for the pairs tree plus *matching* (graph with maximum degree 1) and *outerpath* (outerplanar graph whose weak dual is a path) plus matching. The former algorithm uses only two slope for the matching edges, for the latter one slope suffices.

Given a planar graph and a path on the same vertices, the order of the vertices in the path induces a layering on the vertices. Cappos et al. [CEBFK09] give a linear-time algorithm that computes an SGE of a planar graph and a path if the planar graph is level-planar with respect to the layering induced by the path. Angelini et al. [AGKN12] show that every tree of depth 2 has an SGE with every path.

11.2.2 Examples without SGE

In contrast to the positive results, Brass et al. [BCD$^+$07] give several examples not admitting an SGE. They show the existence of two planar graphs without a simultaneous embedding and extended this result to two outerplanar graphs. Two results we present in more detail are the counterexample for a planar graph and a path by Brass et al. [BCD$^+$07] and Erten and Kobourov [EK05a] and the counterexample of three paths by Brass et al. [BCD$^+$07].

Theorem 11.2 *There exists a planar graph G and a path P not admitting an SGE.*

Proof Sketch: Consider the graph G and the path P as shown in Figure 11.5. Let G' be the subgraph of G induced on the vertices $\{1, 2, 3, 4, 5\}$, and let G'' be the subgraph of
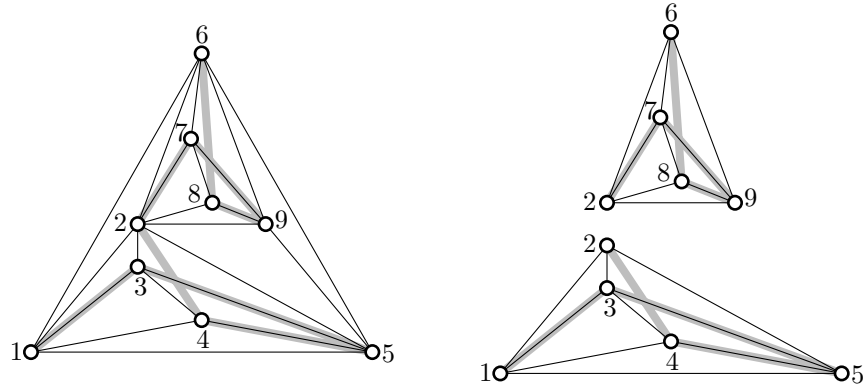
Figure 11.5 A planar graph G and a path P that do not allow an SGE.

G induced on the vertices $\{2, 6, 7, 8, 9\}$. Since G is triconnected fixing the outer face fixes an embedding for G. With the given outer face of G, the path P contains two crossings: one involving $(2, 4)$, and the other one involving $(6, 8)$.

Graph G' has six faces and unless we change the outer face of G' such that it contains the edge $(1, 3)$ or $(3, 5)$, the edge $(2, 4)$ is involved in a crossing in the path. Similarly for G'', unless we change its outer face such that it contains $(2, 7)$ or $(7, 9)$, the edge $(6, 8)$ is involved in a crossing in the path. However G' and G'' do not share any faces and removing both crossings depends on taking two different outer faces, which is impossible. Thus, regardless of the choice for the outer face of G, path P contains a crossing.   □

Theorem 11.3   *There exist three paths P_1, P_2 and P_3 not admitting an SGE.*

Proof: A path of n vertices is simply an ordered sequence of n numbers. The three paths we consider are: 714269358, 824357169 and 758261439. For example, the sequence 714269358 represents the path $(v_7, v_1, v_4, v_2, v_6, v_9, v_3, v_5, v_8)$. We will write ij for the edge connecting v_i to v_j. The union of these paths contain the following twelve edges.

$$E = \{14, 16, 17, 24, 26, 28, 34, 35, 39, 57, 58, 69\}$$

It is easy to see that the graph G consisting of these edges is a subdivision of $K_{3,3}$ and therefore non-planar: collapsing 1 and 7, 2 and 8, 3 and 9 yields the classes $\{1, 2, 3\}$ and $\{4, 5, 6\}$.

It follows that there are two nonadjacent edges of G that cross each other. It is easy to check that every pair of nonadjacent edges from E appears in at least one of the paths given above. Therefore, at least one path will cross itself which completes the proof.   □

Cabello et al. [CvKL+11] extend the counterexample for the case that G_1 is a path to the case where G_1 is a matching. Moreover, they give an example of six matchings not admitting an SGE. Note that this does not directly follow by dividing three paths without an SGE into six matchings, as the resulting matchings allow crossings that were not allowed before. Another extension of the case where G_1 is a path was given by Frati et al. [FKK09] who give a counterexample where G_1 is a path and G is a set of isolated vertices, that is, G_1 and G_2 are edge disjoint.

The question of whether two trees always admit an SGE was open for several years, before it was answered in the negative by Geyer et al. [GKV09] with a construction involving two very large trees. This of course extends the result of two outerplanar graphs not having an

SGE by Brass et al. [BCD+07]. Angelini et al. [AGKN12] further extended it to the case of a tree and a path without an SGE. More precisely, they give an example of a tree of depth 4 and an edge disjoint path not having an SGE. Recall that a tree of depth 2 does always admit a simultaneous embedding with a path, thus in this case the gap between positive and negative results is quite small.
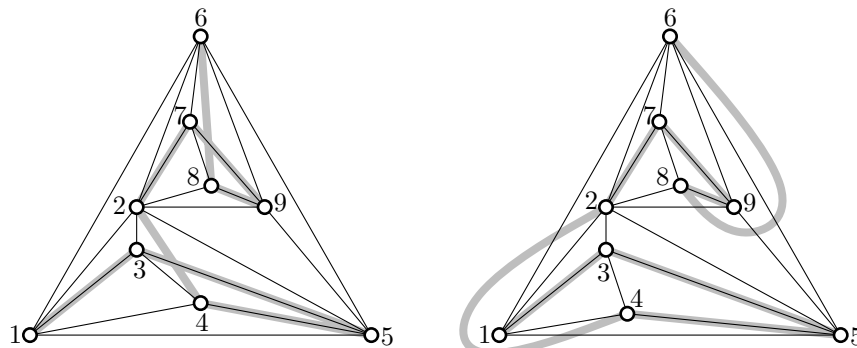
11.2.3 Related Work

Frati et al. [FKK09] consider the restricted case where each input graph has a prescribed combinatorial embedding. They show that the pair path plus star admits an SGE even if the embedding of the star is fixed. They can extend this result to a *double-star* (tree with up to two inner vertices) if it is edge disjoint to the path. On the other hand they show that fixing the embedding of two caterpillars may lead to an counterexample, whereas they admit an SGE if the embedding is not fixed. Another counterexample is the pair outerplanar graph with fixed embedding plus edge-disjoint path.

An interesting additional restriction to SGEs was considered by Argyriou et al. [ABKS12], combining SGE with the RAC drawing convention (RAC – Right-Angular Crossing). They try to find an SGE such that crossings between exclusive edges of different graphs are restricted to right-angular crossings. Argyriou et al. consider only the case where the edge sets of both graphs are disjoint. They present one negative and one positive result for this problem. The negative result consists of a wheel and a cycle not admitting an SGE with right-angular crossings. On the other hand they show the existence of such a drawing on a small integer grid for the case that one of the graphs is a path or a cycle and the other is a matching. Moreover, they give a linear-time algorithm to compute such a drawing.

11.3 Simultaneous Embedding with Fixed Edges

In this section we drop the requirement that edges have to be straight line segments and consider the SEFE problem. Figure 11.6 shows a SEFE of the graph and the path from Figure 11.5 not admitting an SGE. Figure 11.7 and Table 11.2 illustrate the results on the problem SEFE classified in the three categories described before.

Figure 11.6 A graph and a path not admitting an SGE but a SEFE.

instances always having a SEFE

[DL07]

G_1 outerplanar & G_2 path
1 & 0 bends; $O(n) \times O(n^2)$

[DL07]

G_1 outerplanar & G_2 cycle
1 bend; $O(n^2) \times O(n^2)$

[EK05a]

G_1 tree & G_2 path
1 & 0 bends; $O(n) \times O(n^2)$

[DL07]

G_1, G_2 outerpl. & G paths
1 bend; $O(n^2) \times O(n^2)$

[Fra07]
G_1 tree

[Fra07]
G_1 pseudoforest & G forest

[FGJ$^+$09]
G_1 has disjoint cycles & G forest

examples without a SEFE

[FJKS11]
characterization of G_1

[FJKS11]
characterization of G_1
for G_1, G_2 outerplanar

[JS09]
characterization of G

[BCD$^+$07]
G_1 outerplanar

[Fra07]
G_1 & G_2 outerplanar

[BCD$^+$07]
k outerplanar graphs

[BCD$^+$07]
three paths

testing SEFE

[BR13a]
G cycles $(O(n))$

[BR13a]
G fixed edge-orders $(O(n^2))$

[GJP$^+$06]
three graphs NP-complete

[Sch13]
G max-deg 3

[Sch13]
G biconn. components

[Sch13]
G_1 subdiv. of triconn. components

[ABF$^+$12]
G star $(O(n))$

[HJL10, ABF$^+$12]
G biconn. $(O(n))$

[BR13b]
G_1 & G_2 biconn. & G conn. $(O(n^2))$

[FGJ$^+$09]
G_1 pseudoforest $(O(n))$

[FGJ$^+$09]
G_1 has ≤ 2 cycles & G pseudoforest $(O(n))$

Figure 11.7 Overview over the so far known results on SEFE. Each box represents one result and an arrow highlights that the source-result is extended by the target-result. The arrowheads are empty for the cases in which this is only true, if the number of bends per edge, the consumed grid size or the necessary running time is neglected. Note that transitive arrows are omitted.

SEFE Instance	Exist.	Area	Bends	Ref.
G_1 tree & G_2 path	✓	$O(n) \times O(n^2)$	1 & 0	[EK05a]
G_1 outerplanar & G_2 path	✓	$O(n) \times O(n^2)$	1 & 0	[DL07]
G_1 outerplanar & G_2 cycle	✓	$O(n^2) \times O(n^2)$	1	[DL07]
G_1, G_2 outerplanar & G collection of paths	✓	$O(n^2) \times O(n^2)$	1	[DL07]
G_1 tree & G_2 planar	✓	—		[Fra07]
G_1 pseudoforest, G_2 planar & G forest	✓	—		[Fra07]
G_1 has disjoint cycles, G_2 planar & G forest	✓	—		[FGJ$^+$09]
characterization of G	✓/ ✗	—		[JS09]
characterization of G_1	✓/ ✗	—		[FJKS11]
characterization of G_1 (G_1, G_2 outerplanar)	✓/ ✗	—		[FJKS11]
G_1 outerplanar & G_2 planar	✗	—		[BCD$^+$07]
k outerplanar graphs	✗	—		[BCD$^+$07]
three paths	✗	—		[BCD$^+$07]
G_1 & G_2 outerplanar	✗	—		[Fra07]

SEFE Instance	Complexity	Ref.
three planar graphs	NP-complete	[GJP$^+$06]
G_1 pseudoforest & G_2 planar	$O(n)$	[FGJ$^+$09]
G_1 has ≤ 2 cycles, G_2 planar & G pseudoforest	$O(n)$	[FGJ$^+$09]
G star	$O(n)$	[ABF$^+$12]
G consists of disjoint cycles	$O(n)$	[BR13a]
G consists of components with fixed embeddings	$O(n^2)$	[BR13a]
G has maximum degree 3	polynomial	[Sch13]
G_1 subdivision of triconnected components & G_2 planar	polynomial	[Sch13]
G biconnected	$O(n)$	[HJL10]
G biconnected	$O(n)$	[ABF$^+$12]
G consists of biconnected components	polynomial	[Sch13]
G_1, G_2 biconnected & G connected	$O(n^2)$	[BR13b]

Table 11.2 A list of graph classes that are either known to always have a SEFE or that contain counterexamples (table at the top). For the positive examples bounds on the required area and number of bends per edge are given, provided that they are known. The symbol ✓/ ✗ denotes that a complete characterization of positive and negative instances is given. The table at the bottom shows results concerning the computational complexity of SEFE.

## 11.3.1 Positive and Negative Examples

We start with instances that always admit a SEFE. Erten and Kobourov [EK05a] show that a tree and a path can always be embedded simultaneously. They additionally give an algorithm finding a simultaneous embedding in $O(n)$ time on a grid of size $O(n) \times O(n^2)$ such that the edges of G_1 and G_2 have at most one and zero bends per edge, respectively. Note that a grid of size $O(n^2) \times O(n^3)$ is necessary if the bends are required to be drawn on grid points. Di Giacomo and Liotta [DL07] extend this result to the case of an outerplanar graph and a path with the same grid and bend requirements. They extend it further to the case where G_1 and G_2 are outerplanar and the common graph G is a collection of paths and to the case where G_1 is outerplanar and G_2 is a cycle. However, in both cases a grid of size $O(n^2) \times O(n^2)$ and up to one bend per edge are required. If the grid and bend requirements

are completely neglected, the results considering the pairs tree plus path and outerplanar graph plus path can be extended to the case where one of the two graphs is a tree.

Frati [Fra07] shows how a tree G_1 can be simultaneously embedded with an arbitrary planar graph G_2. This algorithm still works if G_1 contains one additional edge that is not a common edge, yielding the result that every graph with at most one cycle (a *pseudoforest*) can be embedded simultaneously with every other planar graph if the common graph does not contain this cycle. Fowler et al. [FGJ+09] extend this result further to the case where G_1 contains only disjoint cycles and the common graph G does not contain a cycle.

Aside from instances always having a SEFE, there are also examples that cannot be simultaneously embedded. Brass et al. [BCD+07] give examples for k outerplanar graphs, three paths and an outerplanar graph plus a planar graph not having a SEFE. The results concerning outerplanar graphs can be extended to the case where both graphs are outerplanar [Fra07].

In between the positive and negative results there are some characterizations stating which instances have a SEFE and which do possibly not. Fowler et al. [FJKS11] give a characterization of the graphs G_1 having a SEFE with every other planar graph. This of course extends all results concerning only G_1. In particular, the results that a tree can be simultaneously embedded with every other graph, whereas an outerplanar graph cannot, are extended. This characterization essentially requires that G_1 must not contain a subgraph homeomorphic to K_3 (a triangle) and an edge not attached to this K_3; see Figure 11.8 for an example. The considerations made for this characterization additionally yield a characterization for the biconnected outerplanar graphs G_1 having a simultaneous embedding with every other outerplanar graph G_2. This of course extends the result that two outerplanar graphs possibly do not have a SEFE.
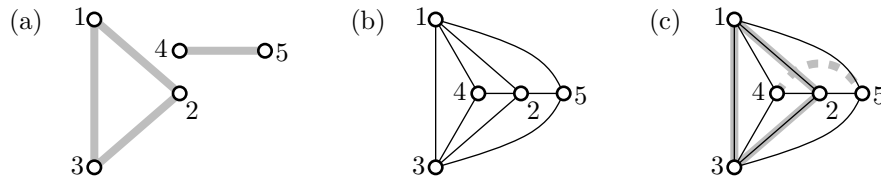
Figure 11.8 G_1 (a) and G_2 (b) do not admit a SEFE (c) as G_2 forces the vertices 4 and 5 to different sides of the triangle $\Delta 123$.

A different characterization, in terms of the common graph, is given by Jünger and Schulz [JS09]. They show that two graphs can be simultaneously embedded if the common graph G has only two embeddings, whereas in all other cases graphs G_1 and G_2 with the common graph G not having a SEFE can be constructed. They additionally show that finding a SEFE is equivalent to finding combinatorial embeddings of G_1 and G_2 inducing the same combinatorial embedding, that is the same orders of edges around vertices and the same relative positions of connected components to one another, on the common graph G [JS09, Theorem 4]. Note that it is not obvious and not even true for more than two graphs [ADF11]. As this result is heavily used in most algorithms solving the decision problem SEFE, we state it as a theorem.

Theorem 11.4   *Two graphs G_1 and G_2 with common subgraph G admit a* SEFE *if and only if they admit combinatorial embeddings inducing the same embedding on G.*

11.3.2 Testing SEFE

Since SEFE has positive and negative instances, it would be nice to have an algorithm deciding for given graphs, whether they can be embedded simultaneously. If more than two graphs are allowed, this problem is known to be NP-complete [GJP$^+$06], whereas the complexity for two graphs is still open. However, there are several results solving SEFE for special cases.

Fowler et al. [FGJ$^+$09] show how to test SEFE, if G_1 is a pseudoforest, that is, a graph with at most one cycle. Note that, as mentioned above, such an instance always has a SEFE if this single cycle is not contained in G. This result can be extended to the case where G_1 contains up to two cycles, if G does not contain the second cycle, that is, G is a pseudoforest. To achieve this result the following auxiliary problem was solved. Given a planar graph G with a designated cycle C and a partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ of the vertices not contained in C, does G admit a planar embedding, such that all vertices in P_i are on the same side of the cycle for every set P_i? Note that this again is a constrained embedding problem, showing that constrained and simultaneous embedding are closely related. Despite early effort [FJKS11], testing SEFE for two outerplanar graphs remains open.

Haeupler et al. [HJL10] give a linear-time algorithm to solve SEFE for the case that the common graph is biconnected. Their solution is an extension of the planarity testing algorithm by Haeupler and Tarjan [HT08]. This planarity testing algorithm starts with a completely unembedded graph and adds vertices iteratively, such that the unembedded part is always connected, ensuring that it can be assumed to lie in the outer face of all embedded components. While inserting vertices, they keep track of the possible embeddings of the embedded parts by representing the possible orders of half-embedded edges around every component with a PQ-tree having these edges as leaves. In a *PQ-tree* every inner node is either a Q-node fixing the order of edges incident to it up to a flip or a P-node allowing arbitrary orders. In this way a PQ-tree represents a set of possible orders of its leaves.

A completely different approach is used by Angelini et al. [ABF$^+$12] to solve SEFE in linear time if the common graph is biconnected. They choose an order for the common graph bottom up in its SPQR-tree such that the private edges can be added.

Another approach by Bläsius and Rutter [BR13b] also uses PQ-trees. They use that the possible orders of edges around every vertex of a biconnected planar graph can be represented by a PQ-tree, yielding a set of PQ-trees, one for each vertex. To obtain a planar embedding, the orders for the PQ-trees have to be chosen consistently. Bläsius and Rutter define the problem SIMULTANEOUS PQ-ORDERING asking for orders in PQ-trees that are chosen consistently, which can, among other applications, be used to represent all planar embeddings of a biconnected graph. This extends to the case of two biconnected planar graphs enforcing shared edges to be ordered the same and thus yields a quadratic time algorithm for SEFE if G_1 and G_2 are biconnected and G is connected. The latter requirement comes from the fact that only orders of edges around vertices are taken into account, relative positions of connected components to one another are neglected. Note that this result extends the case where G is biconnected for the following reason. If G is biconnected, then G is completely contained in a single block (maximal biconnected component) of G_1 and G_2. Thus, even if G_1 or G_2 are not biconnected, they contain only one block that is of interest, all other blocks can simply be attached to this block.

The result by Bläsius and Rutter can be slightly extended to the case where the graphs G_1 and G_2 contain cut-vertices incident to at most two non-trivial blocks (blocks not consisting of a single edge), including the special case where both graphs have maximum degree 5. The SIMULTANEOUS PQ-ORDERING approach again shows the strong relation between simulta-

neous and constrained embedding as in an instance of SEFE the two input graphs constrain the possible orders of some of the edges around vertices of one another with PQ-trees.

Angelini et al. [ABF+12] show the equivalence between SEFE and a constrained version of the Partitioned 2-Page Book Embedding problem. An instance of Partitioned 2-Page Book Embedding is a graph and a partition of its edges into two subsets. It asks whether all vertices can be arranged on a straight line (the *spine*) such that each of the edge partitions can be embedded without crossings in one of the two incident half-planes (*pages* of the book). Partitioned T-Coherent 2-Page Book Embedding additionally has a tree as input with the vertices of the graph as leaves. It is then required that the tree admits an embedding such that the order of its leaves is equal to the order of vertices on the spine. In other words, the allowed orders of vertices on the spine is constrained by a PQ-tree containing no Q-nodes. Angelini et al. [ABF+12] prove the following theorem and we sketch their proof here.

Theorem 11.5   *The problems* SEFE *for two graphs with connected intersection and* Partitioned T-Coherent 2-Page Book Embedding *have the same time complexity.*

Proof Sketch: Angelini et al. [ABF+12] first show that an instance of SEFE where the common graph is connected can be modified (yielding an equivalent instance) such that the common graph is a tree. Moreover, each private edge is incident to leaves of this tree. They then show the equivalence to an instance of Partitioned T-Coherent 2-Page Book Embedding where the common graph is the constraining tree, the leaves of this tree are the vertices that need to be placed on the spine and the private edges of each of the graphs is one of the partitions.

In the following we sketch this construction using the example in Figure 11.9. The instance in (a) having a tree T as common graph such that each private edge is incident to a leaf admits a SEFE. All private edges are embedded outside the dashed cycle around T in (b) containing all its leaves. Choosing another face as outer face and cutting the cycle at an arbitrary position yields a SEFE where all leaves of T are embedded on a straight line (c) with all private edges on the same side. This directly yields the Partitioned T-Coherent 2-Page Book Embedding in (d) of the private edges respecting the tree T. This shows the equivalence of SEFE and Partitioned T-Coherent 2-Page Book Embedding as the constructions works the same in the opposite direction.   □
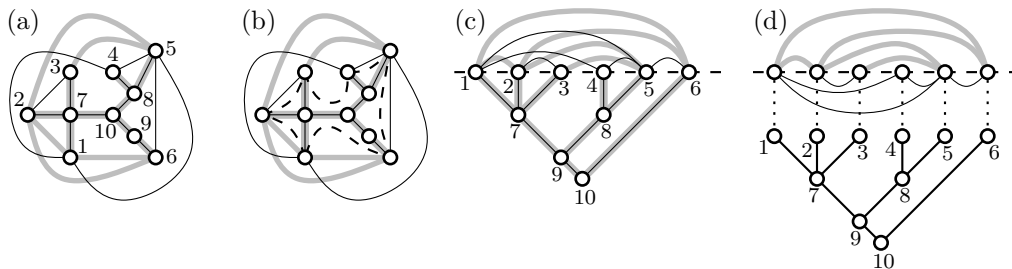
Figure 11.9   Equivalence of an instance of SEFE and the corresponding instance of Partitioned T-Coherent 2-Page Book Embedding.

For the restricted case that T is a star, Partitioned T-Coherent 2-Page Book Embedding reduces to the problem Partitioned 2-Page Book Embedding that can

be solved in linear time [HN09]. Thus the above result directly implies that SEFE can be solved in linear time if the common graph is a star.

All results mentioned thus far require G to be connected and most results also require G_1 and G_2 to be connected. Bläsius and Rutter [BR13a] consider the case where this does not hold. They show that it can be assumed without loss of generality that both graphs G_1 and G_2 are connected.

In the case that G is connected, one only has to deal with orders of edges around vertices and can neglect relative positions of connected components to one another. Bläsius and Rutter approach SEFE from the opposite direction, caring only about the relative positions, neglecting the orders of edges around vertices. More precisely, they give a linear-time algorithm solving SEFE if the common graph is a set of disjoint cycles. They can extend this result to a quadratic-time algorithm for the case where G consists of arbitrary connected components, each with a fixed planar embedding. Both results extend to an arbitrary number of graphs with sunflower intersection. Recall that sunflower intersection means that all graphs intersect in the same common subgraph. Moreover, they give a succinct representation of all simultaneous embeddings.

A completely different, algebraic approach is presented by Schaefer [Sch13]. It is based on the Hanani-Tutte theorem [Cho34, Tut70] stating that a graph is planar if and only if its *independent odd crossing number* is 0. The independent odd crossing number of a drawing is the number of pairs of non-adjacent edges whose number of crossings is odd. The independent odd crossing number of a graph is its minimum over all drawings. Thus, by the Hanani-Tutte theorem, testing planarity is equivalent to testing whether this crossing number is 0. The latter condition can be formulated as a system of linear equations over the field of two elements, leading to a simple polynomial-time planarity algorithm. Schaefer extends this result to other notions of planarity. In particular, it is shown that SEFE can be solved in polynomial time for three interesting cases, namely (1) if the common graph G consists of disjoint biconnected components and isolated vertices, (2) if the common graph has maximum degree 3, and (3) if G_1 is the disjoint union of subdivisions of triconnected graphs. When neglecting the slower running time, this extends several of the results known before; see Figure 11.7.

11.3.3 Related Work

A result not really fitting in one of the three above classes by Duncan et al. [DEK04] considers the restricted case of SEFE where each edge has to be a sequence of horizontal and vertical segments with at most one bend per edge. They show that two graphs with maximum degree 2 always admit such a SEFE on a grid of size $O(n) \times O(n)$ by adapting their linear-time algorithm computing an SGE for these types of graphs (on a larger grid).

Angelini et al. [ADF11] consider the case where the embedding of each of the input graphs is already fixed. With this restriction SEFE becomes trivial for two graphs since it remains to test whether the two graphs induce the same embedding on the common graph. They show that it can also be decided efficiently for three graphs. However, it becomes NP-hard for at least fourteen graphs. They also consider the problem SGE for the case that the embedding of each graph is fixed and show that it is NP-hard for at least thirteen graphs.

Schaefer [Sch13] shows that several other notions of planarity are related to SEFE. In particular, the well-studied cluster planarity problem reduces to SEFE, providing further incentive to study its complexity.

11.4   Simultaneous Embedding

In the most restricted version of the problem, SGE, we insist that vertices are placed in the same position, and edges must be straight-line segments. The SEFE setting relaxes the straight-line condition but maintains that edges common to multiple graphs are realized the same way in each. In the least restrictive setting, SE, we allow the same edge to be realized differently in different graphs.

It has already been mentioned that simultaneous embedding of multiple graphs can be thought of as a generalization of the notion of planarity. A classical result about planar graphs connects the notion of a planar graph with that of a straight-line, crossing-free drawing thereof. Specifically, Wagner in 1936 [Wag36], Fáry in 1948 [Fár48], and Stein in 1951 [Ste51] independently show that if a graph has a drawing without crossings, using arbitrary curves as edges, then there exists a drawing of the graph also without crossings, but with edges drawn as straight-line segments. For multiple graphs, however, this result does not hold. That is, given several graphs on the same n vertices, we can surely realize each graph without crossings, using arbitrary curves as edges and the same vertex positions for each graph. But (except in very special circumstances such as the positive examples in the Section 11.2) we cannot guarantee that there exist vertex positions that allow the realization of each graph with straight-line segments and without crossings. If this were true, then the vertex positions would be a *universal pointset* for graphs on n vertices, and it is known that universal pointsets of linear size do not exist [dFPP90].

Pach and Wenger [PW98] show that every planar graph can be drawn without crossings with a prespecified position for every vertex. Thus, for every pair of planar graphs an SE can be created by drawing the first graph arbitrarily and the second graph to the vertex positions specified by the first drawing. Thus, there are neither negative examples nor is it necessary to have testing algorithms. However, the drawing of the second graph may have linearly many bends per edge, thus it is of interest to find an SE with fewer bends.

Erten and Kobourov [EK05a] show that every two graphs can be drawn simultaneously in $O(n)$ time with at most three bends per edge on an $O(n^2) \times O(n^2)$ grid ($O(n^3) \times O(n^3)$ if bends need to be placed on grid points), where n is the number of vertices. To achieve this result, they combine the construction of Brass et al. [BCD$^+$07] to create an SGE of two paths (see Theorem 11.1 in Section 11.2) with a technique by Kaufmann and Wiese [KW02], who show that every planar graph can be drawn with at most two bends per edge if the allowed vertex positions are restricted to a set of points. We include the main result from this paper along with a proof sketch.
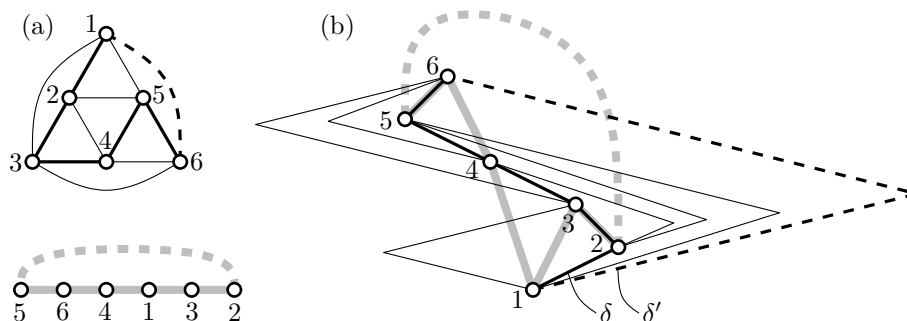
Figure 11.10   (a) The cycle H_2 (gray) with the path P_2 (not dashed) and the graph G_1 containing the Hamiltonian cycle H_1 (bold) and the Hamiltonian path P_1 (bold, not dashed). (b) The drawing of G_1 and P_2 according to the construction of Theorem 11.6.

Theorem 11.6 *For two planar graphs G_1 and G_2 an* SE *with at most three bends per edge on an $O(n^2) \times O(n^2)$ grid can be found in linear time.*

Proof Sketch: Initially, assume that G_1 and G_2 are 4-connected. This assumption is removed later using the technique of Kaufmann and Wiese.

We can compute Hamiltonian cycles H_1 and H_2 of G_1 and G_2, respectively, using the algorithm of Chiba and Nishizeki [CN89]. Let P_1 and P_2 be Hamiltonian paths contained in H_1 and H_2, respectively; see Figure 11.10(a) for an example. As in the proof of Theorem 11.1, we can construct an SGE of P_1 and P_2 such that P_1 is y-monotone, while P_2 is x-monotone. We show how to add the remaining edges of G_1 and the construction is similar for G_2.

We consider the absolute values of the slopes the edges in P_1 have and define δ to be their minimum. Let further δ' be slightly smaller. We first close the cycle H_1 by adding the missing edge using two straight-line segments with slopes δ' and $-\delta'$; see Figure 11.10(b). Similarly, all remaining edges of G_1 are drawn with two straight-line segments with slopes appropriately chosen between δ' and δ and between $-\delta$ and $-\delta'$. Dealing similarly with the remaining edges of G_2 yields an SE with at most one bend per edge on a grid of size $O(n^2) \times O(n^2)$.

For the case that G_1 and G_2 are not 4-connected, Kaufmann and Wiese [KW02] showed how they can be augmented to 4-connected planar graphs by adding new edges and subdividing every edge at most once. Drawing these augmented graphs as described above, removing the additional edges and replacing each subdivision vertex with a bend yields an SE of G_1 and G_2 with at most three bends per edge on an $O(n^2) \times O(n^2)$ grid. $\square$

The result of Erten and Kobourov was improved by Di Giacomo and Liotta [DL05, DL07] to at most two bends per edge in general and one bend per edge, if G_1 and G_2 are both sub-Hamiltonian. That is, they can be augmented to become Hamiltonian maintaining planarity, and an augmentation together with a Hamiltonian cycle is given with the input. Similar results were obtained by Kammer [Kam06]. As series-parallel graphs [DDLW06], trees and outerplanar graphs [CLR87, BK79] are always sub-Hamiltonian and an augmentation together with a Hamiltonian cycle can be computed in linear-time this result yields a linear time algorithm to compute an SE of G_1 and G_2 with one bend per edge on a grid of size $O(n^2) \times O(n^2)$ if each of the graphs G_1 and G_2 is series-parallel, a tree or outerplanar.

Cappos et al. [CEBFK09] show that a path and an outerplanar graph can be simultaneously embedded in linear time such that edges in the outerplanar graph are straight-line segments and each edge in the path consists of a single circular arc. Alternatively, the path edges may be piecewise linear with at most two bends per edge.

11.5 Colored Simultaneous Embedding

Since SGE can be too restrictive, various relaxations have been considered. The two relaxed versions already mentioned, SEFE and SE relax the requirement of straight-line edges, and even the requirement that common edges are drawn the same way in both drawings. Another way to relax the constraints of the original SGE problem is to allow changes in vertex positions in different graphs.

Until this point we had assumed that multiple input graphs have labeled vertices and thus the mapping between the vertices of the graphs is part of the input. In *simultaneous embedding without mapping* we are interested in computing plane drawings for each of the given graphs on the same set of points, where any vertex can be placed at any of the points
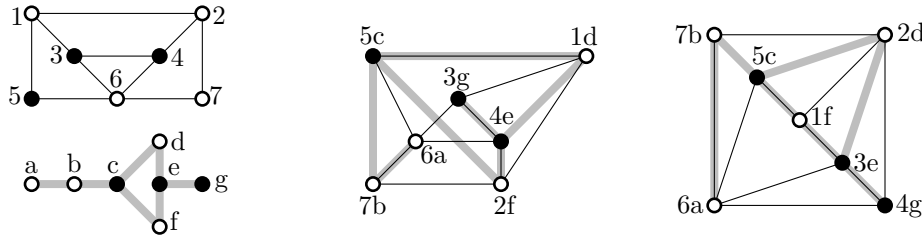
Figure 11.11   Two 2-colored graphs with two CSEs corresponding to different mappings.

in the point set. This setting of the problem was investigated in the very first paper on SGE [BCD$^+$07] and is the source of one of the longest-standing open problems in the area.

A common generalization of the problems above is COLORED SIMULTANEOUS EMBEDDINGS (CSE), which was introduced by Brandes et al. [BEEB$^+$11], and contains both, the version with and without mapping. Formally, the problem of CSE is defined as follows. The input is a set of planar graphs $G_1 = (V, E_1), G_2 = (V, E_2), \ldots, G_k = (V, E_k)$ on the same vertex set V and a partition of V into c classes, which we refer to as colors. The goal is to find plane straight-line drawings D_i of G_i using the same $|V|$ points in the plane for all $i = 1, \ldots, k$, where vertices mapped to the same point are required to be of the same color. We call such graphs c-colored graphs; see Figure 11.11 for an example. Given the above definition, simultaneous embeddings with and without mapping correspond to colored simultaneous embeddings with $c = |V|$ and $c = 1$, respectively. Thus, when a set of input graphs allows for a simultaneous embedding without mapping but does not allow for a simultaneous embedding with mapping, there must be a threshold for the number of colors beyond which the graphs can no longer be embedded simultaneously.

Colored simultaneous embeddings provide a way to obtain near-simultaneous embeddings, where we place corresponding vertices nearly, but not necessarily exactly, at the same locations. Relaxing the constraint on the size of the pointset allows for a way to more easily obtain near-simultaneous embeddings, where we attempt to place corresponding vertices relatively close to one another in each drawing. For example, if each cluster of points in the plane has a distinct color, then even if a red vertex v placed at a red point $p \in G_1$ has moved to another red point $q \in G_2$, the movement is limited to the area covered by the red points.

Brandes et al. [BEEB$^+$11] show several positive and negative results about CSE. In particular they show that there exist universal pointsets of size n for 2-colored paths and spiders as well as 3-colored paths and caterpillars. It is also shown that a 2-colored tree (or even a 2-colored outerplanar graph) and any number of 2-colored paths can be simultaneously embedded. In the negative direction, there exist a 2-colored planar graph and pseudo-forest, three 3-colored outerplanar graphs, four 4-colored pseudo-forests, three 5-colored pseudo-forests, five 5-colored paths, two 6-colored biconnected outerplanar graphs, three 6-colored cycles, four 6-colored paths, and three 9-colored paths that cannot be simultaneously embedded.

Frati et al. [FKK09] continue the investigation of near-SGE's, that is, they try to find straight-line drawings of the input graphs with a small distance between every pair of common vertices in different drawings. As a negative result, they present a pair of graphs such that in every pair of drawings there exists a common vertex with distance linear in the size of the input. On the other hand, they present positive results for a sequence of paths and a sequence of trees for the case that every two consecutive graphs in the sequence are similar with respect to a parameter measuring their similarity. It can then be shown
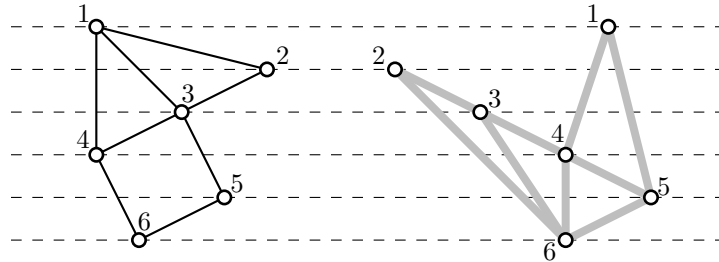
Figure 11.12 A matched drawing: corresponding vertices have the same y-coordinate.

that the distance of a common vertex in two consecutive drawings depends linearly on this parameter.

11.6 Matched Drawings

Another approach to relax requirements of SGE are the so-called *matched drawings* introduced by Di Giacomo et al. [DDvK+09]. A matched drawing of a pair of graphs is a planar straight-line drawing of each of the graphs such that each common vertex has the same y-coordinate in both drawings (instead of the same y- and x-coordinate as required for SGE); see Figure 11.12 for an example.

Di Giacomo et al. [DDvK+09] give a small counterexample consisting of two small triconnected planar graphs not admitting a matched drawing. Moreover, they give a larger example (620 vertices) of a biconnected graph and a tree not having a matched drawing.

Apart from that they also have some results on the positive side. They show that two trees are always matched drawable. Moreover, they observe that any planar graph has a matched drawing with a so-called *unlabeled level planar (ULP)* graph, that is, a graph that admits a planar straight-line drawing even if the y-coordinate of each vertex is prespecified such that no two vertices have the same y-coordinate. A characterization of ULP graphs is given by Fowler and Kobourov [FK08]. Di Giacomo et al. [DDvK+09] moreover show for a graph class containing non-ULP graphs (the *carousel graphs*) that they admit matched drawings with arbitrary planar graphs. A special case of a carousel graph is a graph consisting of a single vertex v_0 and a set of disjoint subgraphs $S_1, \ldots, S_k$, each S_i connected to v_0 over a single edge $\{v_0, v_i\}$ such that S_i is either a caterpillar with v_i on its spine, a radius-2 star with v_i as center or a cycle.

Grilli et al. [GHL+09] present further positive results on matched drawings. They show how to draw the pairs outerplane plus wheel, wheel plus wheel, outerplane plus *maximal outerpillar* (outerplane graph with triangulated inner faces and caterpillar as weak dual), and outerplane plus *generalized outerpath* (outerpath where some edges on the outer face may be replaced by some small subgraphs). Moreover, they consider matched drawings for graph triples and give algorithms creating matched drawings of three cycles, and a caterpillar and two ULP graphs.

11.7 Other Simultaneous Representations

Apart from simultaneously drawing two graphs sharing some common parts there are other ways to represent graphs simultaneously. In this section we describe how a plane graph and its dual can be represented simultaneously, and what is known about simultaneous intersection representations of (not necessarily planar) graphs.

11.7.1 A Plane Graph and Its Dual

In a simultaneous drawing of a planar graph and its dual each vertex in the dual graph
is required to be placed inside the corresponding face of the primal graph. Moreover, no
crossings are allowed except for crossings between a dual and its corresponding primal edge.
Tutte [Tut63] first considered this problem and showed that every triconnected planar graph
admits a simultaneous straight-line drawing with its dual. However, the resulting drawings
may have exponentially large area. Erten and Kobourov [EK05b] provide a linear-time
algorithm simultaneously embedding a triconnected planar graph and its dual on a grid of
size $(2n - 2) \times (2n - 2)$ such that all edges are drawn as straight-line segments. Zhang and
He [ZH06] improved this result to a grid of size $(n - 1) \times n$.

Brightwell and Scheinerman [BS93] show the existence of a simultaneous straight-line
drawing of a triconnected planar graph and its weak dual such that the crossings between
dual and the corresponding primal edges are right-angular crossings. A *circle packing*
of a planar graph represents the vertices as non-crossing circles such that two vertices are
adjacent if and only if their corresponding circles touch. Given a circle packing of a planar
graph, one obtains a planar straight-line drawing by placing each vertex at the center of
its corresponding circle. Mohar [Moh97] shows that every triconnected planar graph has
a simultaneous circle packing with its dual such that in the corresponding straight-line
drawings primal and dual edges have right-angular crossings. Argyriou et al. [ABKS12]
give a simple example of a graph that is not triconnected not admitting such a drawing.
On the positive side they give an algorithm that creates such drawings for the case that the
primal graph is outerplanar.

Another way of simultaneously representing a planar graph and its dual is the *tessellation
representation* introduced by Tamassia and Tollis [TT89]. In a tessellation representation,
every edge, every vertex and every face is represented by a (possibly degenerated) rectangle,
a so-called *tile*, such that the interiors of these tiles are pairwise disjoint, that their union
forms a rectangle, and that the incidences in the graph are represented by side contacts of
the tiles in the following way: (i) Two tiles share a horizontal line segment if and only if
they represent an edge and an incident face; and (ii) two tiles share a vertical line segment
if and only if they represent an edge and an incident vertex. Tamassia and Tollis [TT89], in
particular, showed that every biconnected planar graph admits a tessellation representation
where the tiles representing vertices and faces are vertical and horizontal line segments,
respectively.

The textbook by Di Battista et al. [DETT99, Sections 4.3 and 4.4] contains a short
description of the algorithm computing tessellation representations and of the relation to
visibility representations. Moreover, tessellation representations were also considered on
other surfaces such as the torus [MR98].

11.7.2 Intersection Representations

Jampani and Lubiw [JL09] introduce the concept of simultaneous graph representations
for other representations than drawings. An intersection representation of a graph assigns
a geometric object to each vertex such that two vertices are adjacent if and only if their
corresponding geometric objects intersect. Two graphs sharing a common subgraph are
simultaneous intersection graphs if each of them has an intersection representation such that
the common vertices are represented by the same objects. Note that every planar drawing
of a graph can be interpreted as intersection representation, each vertex is represented by
the union of its edges. This shows that deciding SEFE as a special case of recognizing
simultaneous intersection graphs.

Other popular intersection representations are the following. In an *interval representation* of a graph each vertex is represented by an interval on the real line. A graph is *chordal* if each induced cycle has length three. Gavril [Gav74] shows that chordal graphs are exactly the intersection graphs of subtrees in a tree. This shows that the class of interval graphs is contained in the class of chordal graphs. *Permutation graphs* are the intersection graphs that can be represented by a set of line segments connecting two parallel lines. Jampani and Lubiw [JL09] give $O(n^3)$-time algorithms recognizing simultaneous permutation graphs and simultaneous chordal graphs. The algorithm for simultaneous permutation graphs can be extended to more than two graphs with sunflower intersection. On the other hand, it is NP-hard to recognize simultaneous chordal graphs of this kind (for a constant number k of graphs, the complexity is still open).

In a follow-up paper Jampani and Lubiw [JL10] give an algorithm recognizing simultaneous interval graphs in $O(n^2 \log n)$ time. As interval graphs can be characterized in terms of PQ-trees, recognizing simultaneous interval graphs leads to a problem of finding orders in several PQ-trees simultaneously. Bläsius and Rutter [BR13b] consider this kind of problem in a more general leading to a $O(n)$-time algorithm recognizing simultaneous interval graphs.

Related to simultaneous intersection graphs are simultaneous comparability graphs also introduced by Jampani and Lubiw [JL09]. A *comparability graph* is a graph that can be oriented transitively where transitively means that a directed path implies the existence of a directed edge. Two graphs are *simultaneous comparability graphs* if each of them can be oriented transitively such that common edges are oriented the same in both. Jampani and Lubiw give an $O(nm)$-time algorithm recognizing simultaneous comparability graphs. It can also be used to recognize an arbitrary number of comparability graphs with sunflower intersection. Comparability graphs are related to intersection graphs as comparability graphs are exactly the graphs whose complement is a *function graph*, that is the intersection graph with respect to continuous functions on an interval [GRU83].

As for the problem SEFE, finding simultaneous representations is related to extending a representation of a subgraph to one of the whole graph. For interval graphs Klavík et al. [KKV11] give a $O(nm)$-time algorithm testing whether a partial interval representation can be extended. Bläsius and Rutter [BR13b] were able to improve the running time to $O(m)$ by constructing a second graph such that both graphs are simultaneous interval graphs if and only if the partial interval representation can be extended.

11.8 Practical Approaches to Dynamic Graph Drawing

The majority of the results reviewed above focused on the theoretical aspects of dynamic graph drawing. In this section we review practical approaches to this problem. As we have seen in the previous sections, numerous negative results show that in many of the interesting settings we cannot guarantee simultaneous embeddings. On the other hand, several efficient algorithms for different variants of the problem do exist, but they usually place additional restrictions on the number of input graphs, or limit the graphs to special sub-classes of planar graphs.

As discussed in the introduction, the problem is well motivated in practice. Of particular interest are applications to visualization of dynamic graphs and the related issues of mental map preservation and good graph readability. With this in mind we mention several more practical results here. First we focus on drawing algorithms that aim to produce simultaneous embeddings or layouts that are in some sense close to being a simultaneous embedding. Afterwards, we briefly discuss other approaches to dynamic graph drawing.

Erten et al. [EKLN05] adapt force-directed algorithms to create drawings of a series of graphs sharing subgraphs finding a tradeoff between nice drawings and similarities of common parts. Kobourov and Pitta [KP05] describe an interactive system that allows multiple users to interactively modify a pair of graphs simultaneously using a multi-user, touch-sensitive input device. While those two approaches focus on straight-line drawings (corresponding to SGE), the GraphSET system by Estrella-Balderrama et al. [EBFK10] also allows edges to have bends. GraphSET is a tool helping the user to investigate the theoretical problems SGE and SEFE and it contains implementations of several testing and drawing algorithms. Chimani et al. [CJS08] create simultaneous drawings of graphs by drawing the union of the graphs. Their objective is to minimize the number of crossings in the drawing, where crossings between edges of different graphs do not count, yielding a simultaneous embedding if and only if the number of crossings is zero.

Misue et al. [MELS95] initiated the study of drawing dynamically changing graphs and first proposed several models to capture the notion of preserving the user's mental map. In particular they suggested preservation of orthogonal orderings, proximity relations, or the topology as a formalization. Bridgeman and Tamassia [BT98] describe and evaluate difference metrics that are specialized to orthogonal graph drawings. Purchase et al. [PHG07] provide empirical evidence that preserving the user's mental map indeed assists in comprehending the evolving graph. Purchase and Samra [PS08] argue that for minimizing the node movement, finding a trade-off is worse than either keeping the exact node positions or just layouting the next graph from scratch for memorizing tasks. In a recent study, Archambault and Purchase [AP13] observed positive effects of mental map preservation for localization tasks, both in terms of speed and accuracy. Sallaberry et al. [SMM13] consider mental map preservation for large graphs and argue that restricting node movements to small distances is not sufficient for this case. They propose to cluster nodes into groups that perform the same movement in order to increase the stability of the drawing.

Bridgeman et al. [BFG$^+$97] present InteractiveGiotto, a bend-minimization algorithm for orthogonal drawings that is designed for dynamic and interactive scenarios. Their algorithm supports arbitrary graph changes and preserves the embedding, all edge crossings, and the bends of edges.

Brandes and Wagner [BW97] suggest a Bayesian framework for dynamic graph drawing that can in principle be applied to all layout styles and allows to choose a trade-off between quality and stability. Diel and Görg [DG02] introduce foresighted layouts, where the basic idea is to layout the union of the graph over all time steps and to combine vertices and edges whose life times are disjoint, in order to reduce the size of the drawing. This automatically guarantees a high stability of the layout, but possibly incurs a negative impact on the quality of individual drawings. Görg et al. [GBPD04] enhance this method by an additional step that improves the quality of the individual layouts while keeping them close to the foresighted layout.

North and Woodhull [NW02] propose a heuristic for online hierarchical graph drawing by dynamizing the classical Sugiyama algorithm [STT81]. Collberg et al. [CKN$^+$03] describe a system for visualizing the evolution of software based on force-directed methods applied to so-called time-sliced graphs. A time-sliced graph consists of disjoint copies of the graph at each point in time together with time-slice edges, which connect corresponding vertices from different points in time. The algorithm attempts to place vertices that are connected by a time-slice edge in roughly the same position. Frishman and Tal [FT08] describe an algorithm for online dynamic graph drawing that can be implemented to run on a GPU.

11.9   Morphing Planar Drawings

The main motivation for simultaneously embedding different (but related) graphs is to preserve the mental map between the unchanged parts by drawing them the same. As opposed to this, *morphing* tries to match different drawings of the same graph. More precisely, let Γ_1 and Γ_2 be two drawings of the same graph G, a morph between them is a motion of the vertices along trajectories starting at the vertex positions in Γ_1 and ending at their positions in Γ_2.

The simplest possible morph between two drawings Γ_1 and Γ_2 is the *linear morph* where each vertex moves at constant speed along a line segment from its origin in Γ_1 to its destination in Γ_2. However, the intermediate drawings of linear morphs may be pretty bad, in fact, it may even happen that the whole graph collapses to a single point. To resolve this problem Cairns [Cai44] introduced the notion of morphing planar graphs, requiring that every intermediate drawing is also planar. He showed that two planar drawings of a triangulated plane graph with an equally drawn outer face can be morphed into each other in a planar way using a sequence of linear morphs. However, this sequence of linear morphs has exponential size.

Thomassen [Tho83] extends this to drawings of general (not necessarily triangulated) planar graphs with an equally drawn outer face and convex faces by augmenting the drawings to *compatible triangulations*, that is, one must be able to add all new vertices and edges to both given drawings without violating the planarity or straight-line requirement. Compatible triangulations were further investigated by Aronov et al. [ASS93] who show that two drawings admit compatible triangulations with only $O(n^2)$ new vertices. They moreover show that $\Theta(n^2)$ new vertices are sometimes necessary. This result has the following general implication. If there exist planar morphs between drawings of triangulated graphs using $O(f(n))$ linear morphing steps, then there are morphs between drawings of arbitrary plane graphs using $O(f(n^2))$ steps.

To be able to morph with a polynomial number ($O(n^6)$) of linear steps Lubiw and Petrick [LP08] relaxed the straight-line requirement and showed how to morph between two planar drawings when edges are allowed to be bent during the morph. However, this result can also be achieved without this relaxation. Alamdari et al. [AAC$^+$13] show that for every pair of planar straight-line drawings of a triangulated graph with an equally drawn outer face there exists a planar morph consisting of a sequence of $O(n^2)$ linear morphs. This is the first result showing that a polynomial number of morphing steps is sufficient. Using the results on compatible triangulations mentioned above [ASS93] this yields a morph with $O(n^4)$ linear steps for general plane graphs.

Floater and Gotsman [FG99] introduced a completely different approach to planar morphing of triangulations. They make use of the fact that in a planar drawing the position of each vertex is a convex combination of the neighboring vertices and that conversely fixing the coefficients of the convex combinations and fixing the outer face yields a planar drawing. This was shown by Floater [Flo97] extending the results by Tutte [Tut60, Tut63]. Floater and Gotsman [FG99] create a morph between two planar drawings by transforming the coefficients of the corresponding convex combinations into one another, yielding a sequence of coefficients and thus a sequence of planar drawings. Surazhsky and Gotsman [SG01, SG03] improve this approach further to obtain aesthetically more appealing morphs.

The approach based on convex combinations has the disadvantage that the trajectories are not explicitly computed and that it is not clear how many linear morphing steps are necessary to obtain a planar and smooth morph. Despite its theoretical shortcomings, in practice this algorithm leads to nice morphs, as shown by Erten et al. [EKP04b, EKP04a], who combine this approach with rigid motion (translation, rotation, scaling and shearing)

and the triangulation algorithm by Aronov et al. [ASS93]. Moreover, they are able to morph edges with bends to straight-line edges and vice versa.

Biedl et al. [BLS06] consider a related problem of morphing so-called *parallel* straight-line drawings, that is, straight-line drawings such that for every edge e, the slope of e is the same in both drawings. Moreover, the edge slopes have to be preserved throughout the whole morph. They show that for orthogonal drawings (without bends) such a morph always exists. On the other hand, testing for the existence of such a morph becomes NP-hard if the edges are allowed to have three or more slopes. Lubiw et al. [LPS06] investigate morphs between general orthogonal drawings of planar graphs where edges may have bends. They show that for every pair of drawings there is a morph preserving planarity and orthogonality consisting of polynomial many steps, where each step is either a movement of vertices or a "twist" around a vertex that introduces new bends at the edges incident to this vertex.

Of course, problems similar to planar morphing can be considered for non-planar graphs. Examples are the results by Friedrich and Eades [FE02] and Friedrich and Houle [FH02].

11.10   Open Problems

There are many interesting problems, some of which have been open for a decade and have resisted efforts to address them. Here we list several of the current open problems.

1. Given two arbitrary planar graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with the same number of vertices, $|V_1| = |V_2|$, does there always exist a mapping from the vertex set of the first graph onto the vertex set of the second graph $V_1 \rightarrow V_2$ such that the two graphs have a SGE? That is, do pairs of planar graphs always have an SGE without mapping?

2. Given two graphs of max-degree 2, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with the same number of vertices, an SGE with mapping does always exist. Unlike most other results where the pair of graphs has an SGE the area of the necessary grid is not bounded. Is it possible to guarantee polynomial integer grid for the simultaneous embedding?

3. What is the complexity of SGE for two graphs with fixed planar embeddings?

4. Is it possible to decide SGE for restricted cases, for example if the common graph is highly connected?

5. What is the complexity of the decision problem SEFE for two graphs?

6. Are there interesting parameters for which SEFE or SGE are FPT? For example, tree-distance of G? What about maximum degree Δ?

7. What is the complexity of SEFE for more than two graphs with sunflower intersection?

8. What is the complexity of SEFE for four graphs, each with a fixed planar embedding?

9. What is the complexity of the optimization version of SEFE where one asks for drawings such that as many common edges as possible are drawn the same?

10. Let G_1 and G_2 be two planar graphs with given combinatorial embeddings inducing the same embedding on their intersection G, that is, a SEFE is given with the input. What is the complexity of minimizing the number of crossings in a corresponding drawing?

11. Let G_1 and G_2 be two planar graphs with given combinatorial embeddings inducing the same embedding on their intersection G, that is, a SEFE is given

with the input. Do G_1 and G_2 admit drawings with few bends on a small grid respecting the given SEFE?

12. There are many open problems in the CSE setting. A particularly interesting one concerns pairs of trees. It is known that two n-vertex trees without mapping (1-colored) have a simultaneous geometric embedding (any set of n points in convex position suffices). It is also known that at the other extreme when the mapping is given (n-colored) such geometric embedding may not exist. However, the problem is open for any number of colors $c \in \{2, \ldots, n-1\}$.

13. Similarly to the previous problem, the status of the tree-path CSE problem is open for any number of colors $c \in \{3, \ldots, n-1\}$.

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