

Solving Problems with Unknown Solution Length at (Almost) No Extra Cost

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ABSTRACT

Most research in the theory of evolutionary computation assumes that the problem at hand has a fixed problem size. This assumption does not always apply to real-world optimization challenges, where the length of an optimal solution may be unknown a priori.

Following up on previous work of Cathabard, Lehre, and Yao [FOGA 2011] we analyze variants of the (1+1) evolutionary algorithm for problems with unknown solution length. For their setting, in which the solution length is sampled from a geometric distribution, we provide mutation rates that yield an expected optimization time that is of the same order as that of the (1+1) EA knowing the solution length.

We then show that almost the same run times can be achieved even if *no* a priori information on the solution length is available.

Finally, we provide mutation rates suitable for settings in which neither the solution length nor the positions of the relevant bits are known. Again we obtain almost optimal run times for the ONEMAX and LEADINGONES test functions, thus solving an open problem from Cathabard et al.

Categories and Subject Descriptors

F.2.2 [Theory of Computation]: Analysis of Algorithms and Problem Complexity—*Nonnumerical Algorithms and Problems*

Keywords

Theory; Run Time Analysis; Evolutionary Computation; Unknown Solution Length; Non-Uniform Mutation Probability

1. INTRODUCTION

While the theory for evolutionary algorithms (EAs) in static problem settings is well developed [1, 6, 9], a topic that is not so well studied in the theory of EA literature

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is the performance of EAs in uncertain environments. Uncertainty can have many faces, for example with respect to function evaluations, the variation operators, or the dynamics of the fitness function. Understanding how evolutionary search algorithms can tackle such uncertain environments is an emerging research topic; see [2] for a survey on examples in combinatorial optimization, but also [7] for an excellent survey also discussing different sources of uncertainty.

In this work we study what evolutionary algorithms can achieve in the presence of uncertainty with respect to the solution length. Quite surprisingly, we show that already some variants of the simplest evolutionary algorithm, the (1 + 1) EA, can be very efficient for such problems.

1.1 Previous Work

Our work builds on previous work of Cathabard, Lehre, and Yao [4], who were the first to consider, from a theoretical point of view, evolutionary algorithms in environments with unknown solution lengths. Cathabard et al. assume that the solution length is sampled from a fixed and known distribution D with finite support. More precisely, they assume that the solution length n is sampled from a truncated version of the geometric distribution, in which the probability mass for values greater than some threshold N is shifted to the event that $n = N$. In this situation, the algorithm designer has access to both the upper bound N for the solution length and the success probability q of the distribution.

Cathabard et al. analyze a variant of the (1 + 1) EA in which each bit is flipped with probability $1/N$ and they also study a variant with non-uniform bit-flip probabilities. In the latter, the i -th bit is flipped independently of all other bits with probability $1/(i + 1)$. They show that these variants have polynomial expected run times on ONEMAX and LEADINGONES function, where the expectation is taken with respect to the solution length and the random decisions of the algorithm. An overview of the precise bounds obtained in [4] is given in Table 2.

1.2 Our Results

We extend the work of Cathabard et al. in several ways. In a first step (Section 3) we show that the regarded mutation probabilities are sub-optimal. Making use of the concentration of the (truncated) geometric distribution, we design bit flip probabilities that yield significantly smaller expected run times (for both the ONEMAX and the LEADINGONES function). We complement this finding by a lower bound that shows the optimality of our result. This proves that no mu-

tation probabilities can yield a performance that is better by more than a constant factor than our suggested ones.

While in the setting of Cathabard et al. we are in the convenient situation that we have full knowledge of the distribution D from which the solution length is sampled, one is sometimes faced with problems for which this knowledge is not readily available. We therefore study in Section 4 what can be done without any a priori knowledge about the solution length. In this situation we require that the algorithm designer chooses bit flip probabilities $(p_i)_{i \in \mathbb{N}}$ such that, regardless of the solution length n , the expected performance of the $(1 + 1)$ EA with bit flip probabilities (p_1, \dots, p_n) is as small as possible. It is not obvious that this can be done in polynomial time. In fact, for both algorithms studied by Cathabard et al. as well as for any uniform choice of the bit flip probabilities, the expected run time on this problem is exponential in n (cf. Theorems 12 and 13).

We show (Theorems 14 and 15) that not only can we tackle this problem with non-uniform bit flip probabilities, but, quite surprisingly, this can be even done in a way that yields almost optimal run times. Indeed, our results are only a $\log^{1+\epsilon} n$ factor worse than the best possible choice of mutation probabilities leading to $\Theta(n \log n)$ and $\Theta(n^2)$ run time bounds for ONEMAX and LEADINGONES, respectively. This factor can be made even smaller as we shall comment at the end of Section 4.2.

Finally, we provide in Section 4.3 a second way to deal with unknown solution lengths. We provide an alternative variant of the $(1+1)$ EA in which the bit flip probabilities are chosen according to some (fixed) distribution at the beginning of each iteration. For suitably chosen distributions Q , the expected run times of the respective $(1+1)$ EA $_Q$ on ONEMAX and LEADINGONES are of the same asymptotic order as those of the previously suggested solution with non-uniform bit flip probabilities. In particular, they are, simultaneously for all possible solution lengths n , almost of the same order as the expected run time of a best possible choice of bit flip probabilities knowing the solution length.

This second approach has an advantage over the non-uniform bit flip probabilities in that it effectively ignores bits that do not contribute anything to the fitness function (*irrelevant bits*). Thus, even if only n bits at unknown positions have an influence on the fitness function, the same run time bounds apply. In contrast, all previously suggested solutions require that the n relevant bits are the leftmost ones. This also answers a question posed by Cathabard et al. [4, Section 6].

Our run time results are summarized in Tables 1 and 2.

2. ALGORITHMS AND PROBLEMS

In this section we define the algorithms and problems considered in this paper. For any problem size n , fitness function $f : \{0, 1\}^n \rightarrow \mathbb{R}$, and vector $\vec{p} = (p_1, \dots, p_n)$ of bit flip probabilities $0 \leq p_i \leq 1$, we consider the $(1 + 1)$ EA $_{\vec{p}}$, as given by Algorithm 1.

The $(1 + 1)$ EA $_{\vec{p}}$ samples an initial search point from $\{0, 1\}^n$ uniformly at random. It then proceeds in rounds, each of which consists of a mutation and a selection step. Throughout the whole optimization process the $(1 + 1)$ EA $_{\vec{p}}$ maintains a population size of one, and the individual in this population is always a best-so-far solution. In the *mutation step* of the $(1 + 1)$ EA $_{\vec{p}}$ the current-best solution x is *mutated* by flipping the bit in position i with probability

Algorithm 1: The $(1 + 1)$ EA $_{\vec{p}}$ for $\vec{p} = (p_1, \dots, p_n)$ optimizing a pseudo-Boolean function $f : \{0, 1\}^n \rightarrow \mathbb{R}$.

```

1 Initialization: Sample  $x \in \{0, 1\}^n$  uniformly at
  random and query  $f(x)$ ;
2 Optimization: for  $t = 1, 2, 3, \dots$  do
3   for  $i = 1, \dots, n$  do
4     With probability  $p_i$  set  $y_i \leftarrow 1 - x_i$  and set
      $y_i \leftarrow x_i$  otherwise;
5   Query  $f(y)$ ;
6   if  $f(y) \geq f(x)$  then  $x \leftarrow y$ ;
```

p_i , $1 \leq i \leq n$. The fitness of the resulting search point y is evaluated and in the *selection step* the parent x is replaced by its offspring y if and only if the fitness of y is at least as good as the one of x . Since we consider maximization problems here, this is the case if $f(y) \geq f(x)$. We are interested in expected *run times*, i.e., the expected number of rounds it takes until the $(1 + 1)$ EA $_{\vec{p}}$ evaluates for the first time a solution of maximal fitness and we therefore do not specify a termination criterion. It is not difficult to see that the $(1 + 1)$ EA $_{\vec{p}}$ indeed generalized the standard $(1 + 1)$ EA. In fact, we obtain the $(1 + 1)$ EA from the $(1 + 1)$ EA $_{\vec{p}}$ if we set $p_i = 1/n$ for all $i \in [n] := \{1, \dots, n\}$. We call such mutation vectors with $p_i = p_j$ for all i, j *uniform* mutation rates, while we speak of *non-uniform* mutation rates if $p_i \neq p_j$ for at least one pair (i, j) .

The two test functions we consider in this work are ONEMAX and LEADINGONES. For a given problem size n , they are defined as

$$\begin{aligned} \text{OM}_n &:= \text{ONEMAX}_n(x) = \sum_{i=1}^n x_i, \text{ and} \\ \text{LO}_n &:= \text{LEADINGONES}_n(x) \\ &= \max\{i \in [0..n] \mid \forall j \leq i : x_j = 1\}, \end{aligned}$$

where $[0..n] := \{0\} \cup [n]$. That is, the ONEMAX function counts the number of ones in a bit string, while the LEADINGONES function counts the number of initial ones. While these two functions are certainly easy to optimize without evolutionary algorithms, the $(1 + 1)$ EA $_{\vec{p}}$ performs exactly the same on all generalized ONEMAX and LEADINGONES functions, which are obtained from the functions above through an XOR of an arbitrary and unknown bit string $z \in \{0, 1\}^n$. Understanding how an evolutionary algorithm behaves on these two functions is an important indicator for how it manages to cope with the easier parts of more complex optimization problems. ONEMAX and LEADINGONES functions are for this reason the two best-studied problems in the theory of evolutionary computation literature.

If a distribution D is known from which the solution length is sampled we consider the expected run time of the $(1 + 1)$ EA $_{\vec{p}}$ on ONEMAX $_D$ and LEADINGONES $_D$, respectively, which are the problems OM $_n$ resp. LO $_n$ with random solution length $n \sim D$. Note here that the expectation is thus taken both with respect to the random solution length and with respect to the random samples of the algorithm.

3. RANDOM SOLUTION LENGTH

We first consider the setting that has been introduced by Cathabard, Lehre, and Yao [4]. After a short presentation

setting	bit flip probability	OneMax		LeadingOnes	
Random Length $\sim \text{Geo}(q)$	uniform and fixed	$O(q^{-1} \log q^{-1})$	Theorem 7	$O(q^{-2})$	Theorem 10
Adversarial Length	uniform and fixed	$2^{\Omega(n)}$	Theorem 12	$2^{\Omega(n)}$	Theorem 13
	fixed	$O(n \log^{2+\varepsilon} n)$	Thm. 14, Cor. 16	$O(n^2 \log^{1+\varepsilon} n)$	Thm. 15, Cor. 16
	uniform and random	$O(n \log^{2+\varepsilon} n)$	Thm. 17, Cor. 19	$O(n^2 \log^{1+\varepsilon} n)$	Thm. 18, Cor. 19

Table 1: Overview of Results for $1/N < q < 1/2$ and $\varepsilon > 0$.

of the model in Section 3.1, a general lower bound for this problem (Section 3.2), and the results of [4] in Section 3.3, we show that the bounds in [4] can be improved by using different (uniform) mutation rates (Section 3.4).

Table 2 summarizes the previously known bounds and our contributions for the setting regarded in this section.

3.1 The Model

Cathabard et al. [4] consider the following model. The algorithm designer knows the distribution D from which the unknown solution length is drawn; only distributions with finite support are considered, so the algorithm designer knows an upper bound N on the actual solution length n . He also knows the class of functions from which the optimization problem is taken (for example ONEMAX or LEADINGONES).

Based on this knowledge, the algorithm designer chooses a vector (p_1, \dots, p_N) of bit flip probabilities indicating with which probability a bit is flipped in each round. In this work we also regard a slightly more general model in which the distributions over \mathbb{N} may possibly have infinite support; the algorithm designer then chooses an infinite sequence of bit flip probabilities $(p_1, p_2, \dots) = (p_i)_{i \in \mathbb{N}}$. After this choice of bit flip probabilities, the actual solution length n is sampled from the given distribution D . Then the $(1+1)$ EA $_{\vec{p}}$ (Algorithm 1) is run with mutation probabilities $\vec{p} = (p_1, \dots, p_n)$ on the given problem with the given problem length.

Cathabard et al. [4] consider as distribution D the following *truncated geometric distribution*, based on a geometric distribution where the probability mass for values greater than n are moved to n .

DEFINITION 1 ([4]). *The truncated geometric distribution $\text{TrunkGeo}(N, q)$ with truncation parameter N and success probability $q \geq 1/N$ satisfies, for all $n \in \mathbb{N}$, that the probability of $\text{TrunkGeo}(N, q) = n$ is*

$$\begin{cases} q(1-q)^{n-1} & \text{if } 1 \leq n \leq N-1, \\ (1-q)^{n-1} & \text{if } n = N, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the truncated geometric distribution recovers the geometric distribution $\text{Geo}(q)$ for $N = \infty$.

It is well known, respectively can be found in [4, Proposition 1], that for $X = \text{Geo}(q)$ and $Y = \text{TrunkGeo}(N, q)$ with $q \geq 1/N$

$$\mathbb{E}[X] = q^{-1} \text{ and } \mathbb{E}[Y] = \Theta(q^{-1}). \quad (1)$$

Note that we trivially have $\mathbb{E}[Y] \leq \mathbb{E}[X]$.

3.2 A General Lower Bound

What is a good lower bound for the expected run time of any $(1+1)$ EA $_{\vec{p}}$ on ONEMAX or LEADINGONES when

the length is sampled from some given distribution D on \mathbb{N} ? If the algorithm designer would know the true length n before he has to decide upon the mutation probabilities (p_1, \dots, p_n) , then the optimal bit flip probability for this solution length could be chosen. For ONEMAX, the best choice is to set $\vec{p} = (1/n, \dots, 1/n)$ as has been proven in [10, 11] (note here that for fixed problem sizes, due to the symmetry of ONEMAX, non-uniform mutation rates cannot be advantageous over uniform ones). This results in an expected run time of $\Theta(n \log n)$.

For LEADINGONES, if the true length n is known, any setting of the bit-flip probabilities leads to an expected run time of $\Omega(n^2)$ regardless of the choice of \vec{p} , as the next lemma shows.

LEMMA 2. *For any fixed solution length n and any vector $\vec{p} = (p_1, \dots, p_n)$ of mutation probabilities, the expected run time of the $(1+1)$ EA $_{\vec{p}}$ on LEADINGONES $_n$ is $\Omega(n^2)$.*

Using these lower bounds for fixed solution lengths, Jensen's Inequality and the convexity of $n \mapsto n \log n$ and $n \mapsto n^2$, respectively, we get the following general lower bound.

THEOREM 3. *Let D be any distribution on \mathbb{N} with a finite expectation of m . Then the expected run time of any $(1+1)$ EA $_{\vec{p}}$ on ONEMAX $_D$ is $\Omega(m \log m)$ and the expected run time of any $(1+1)$ EA $_{\vec{p}}$ on LEADINGONES $_D$ is $\Omega(m^2)$. Both bounds apply also to the setting in which the algorithm designer can choose the mutation probabilities $\vec{p} = (p_1, \dots, p_n)$ after the solution length $n \sim D$ has been drawn.*

Using Equation (1), we get the following corollary.

COROLLARY 4. *Let $N \in \mathbb{N}$ and $q \geq 1/N$. Let $D = \text{TrunkGeo}(N, q)$ or $D = \text{Geo}(q)$. The expected run time of any $(1+1)$ EA $_{\vec{p}}$ on ONEMAX $_D$ is $\Omega(q^{-1} \log q^{-1})$ and the expected run time of any $(1+1)$ EA $_{\vec{p}}$ on LEADINGONES $_D$ is $\Omega(q^{-2})$. Both bounds apply also to the setting in which the algorithm designer can choose the mutation probabilities $\vec{p} = (p_1, \dots, p_n)$ after the solution length $n \sim D$ has been drawn.*

3.3 Known Upper Bounds

Cathabard et al. [4] analyze the run time of the $(1+1)$ EA $_{\vec{p}}$ with uniform mutation probabilities $p_1 = \dots = p_N = 1/N$ and of the $(1+1)$ EA $_i$ with $p_i = 1/(i+1)$, $1 \leq i \leq N$.

For ONEMAX they obtain the following results.

THEOREM 5 (RESULTS FOR ONEMAX FROM [4]). *Let $N \in \mathbb{N}$, $\varepsilon \in (0, 1)$, and $q = N^{-\varepsilon}$. For $D = \text{TrunkGeo}(N, q)$ the expected run time of the $(1+1)$ EA $_{\vec{p}}$ with*

problem	lower bound	Results from [4]		Theorems 7 (OM) resp. 10 (LO)	
	Corollary 4	$p_i = 1/N$	$p_i = 1/(i+1)$	$p_i = q/2$	$p_i = q$
ONEMAX _D	$\Omega(q^{-1} \log q^{-1})$	$\Theta(N \log q^{-1})$	$O(q^{-2} \log N)$	$\Theta(q^{-1} \log q^{-1})$	$\Theta(N \log N)$
LEADINGONES _D	$\Omega(q^{-2})$	$\Theta(Nq^{-1})$	$\Theta(q^{-3})$	$\Theta(q^{-2})$	$\Theta(Nq^{-1})$

Table 2: Expected run times of the (1 + 1) EA \vec{p} with $\vec{p} = (p_i)_{i=1}^N$ for $D = \text{TrunkGeo}(N, q)$ and $1/N \leq q \leq 1/2$

$\vec{p} = (1/N, \dots, 1/N)$ on ONEMAX_D is $\Theta(N \log q^{-1})$, while the expected run time of the (1 + 1) EA_i on ONEMAX_D is $O(q^{-2} \log N)$.

This result shows that the (1 + 1) EA \vec{p} with $\vec{p} = (1/N, \dots, 1/N)$ outperforms the (1 + 1) EA_i for $q < 1/\sqrt{N}$, while the latter algorithm is preferable for larger q . As we shall see in the following section one should not conclude from this result that non-uniform bit flip probabilities are the better choice for this problem.

Remark: By using a slightly more careful analysis than presented in [4], the bound for the (1 + 1) EA_i on ONEMAX_D can be improved to $O(q^{-2} \log q^{-1})$. In fact, an analysis similar to the one in Section 3.4, that is disregarding outcomes that are much larger than the expectation, will give that result. It can also be shown that the requirement $q = N^{-\varepsilon}$ is not needed as the $O(q^{-2} \log q^{-1})$ holds for all $q > 1/N$. It also holds for the (non-truncated) geometric distribution $D = \text{Geo}(q)$.

For LEADINGONES, Cathabard et al. show the following results.

THEOREM 6 (RESULTS FOR LEADINGONES FROM [4]). For N, ε, q , and D as in Theorem 5, the expected run time of the (1 + 1) EA \vec{p} with $\vec{p} = (1/N, \dots, 1/N)$ on LEADINGONES_D is $\Theta(Nq^{-1})$, while the expected run time of the (1 + 1) EA_i on LEADINGONES_D is $\Theta(q^{-3})$.

Thus also for LEADINGONES the (1 + 1) EA_i performs better than the (1 + 1) EA \vec{p} with $\vec{p} = (1/N, \dots, 1/N)$ when $q > 1/\sqrt{N}$ while the uniform (1 + 1) EA \vec{p} should be preferred for smaller q .

Remark: As in the ONEMAX case the $\Theta(q^{-3})$ bound for the (1 + 1) EA_i holds more generally for all geometric distributions $\text{Geo}(q)$ with parameter $q > 1/N$.

From Theorems 5 and 6 we can see that for both ONEMAX_D and LEADINGONES_D the (1 + 1) EA_i loses a factor of $1/q$ with respect to the lower bound given by Corollary 4. This will be improved in the following section.

3.4 Optimal Upper Bounds With Uniform Mutation Probabilities

We show that for D being the (truncated or non-truncated) geometric distribution there exist bit flip probabilities $\vec{p} = (p_1, \dots, p_N)$ and $\vec{p} = (p_i)_{i \in \mathbb{N}}$, respectively, such that the expected run time of the (1 + 1) EA \vec{p} on ONEMAX_D and LEADINGONES_D is significantly lower than those of the two algorithms studied by Cathabard et al. The expected run times of our algorithm match the lower bounds given in Corollary 4 and are thus optimal in asymptotic terms.

In both cases, i.e., both for ONEMAX_D and for LEADINGONES_D, the mutation rates yielding the improvement over the results in [4] are uniform. Our results therefore imply that for these two problems, unlike conjectured

in [4], one cannot gain more than constant factors from using non-uniform mutation probabilities.

The key observation determining our choice of the mutation probability is the fact that the (truncated) geometric distribution is highly concentrated. Hence, if we know the parameters of the distribution, we can choose the mutation probability such that it is (almost) reciprocal in each position to the expected length of the solution. Thus, in the setting of [4], i.e., for the truncated geometric distribution with parameters N and q , we set $p_i := q/2$ for all $i \in [N]$ (recall equation (1)). Our approach naturally also works for the (non-truncated) geometric distribution $\text{Geo}(q)$, which is also highly concentrated around its mean $1/q$.

We remark without proof that similar results hold for other distributions that are highly concentrated around the mean, e.g., binomial distributions, and also highly concentrated unbounded distributions, such as Poisson distributions.

THEOREM 7. For $N \in \mathbb{N}$ let $1/N \leq q < 1/2$, q possibly depending on N . For $D = \text{Geo}(q)$ and $D = \text{TrunkGeo}(N, q)$ the expected run time of the (1 + 1) EA \vec{p} with $\vec{p} = (q/2, \dots, q/2)$ on ONEMAX_D is $\Theta(q^{-1} \log q^{-1})$.

For the proof we will use the following upper bound for the expected run time of the (1 + 1) EA on ONEMAX. A similar upper bound can be found in [11, Theorem 4.1].

LEMMA 8 ([10, THEOREM 8]). For a fixed length n and a uniform mutation vector $\vec{p} = (p, \dots, p)$ with $0 < p < 1$, the expected run time of the (1 + 1) EA \vec{p} on ONEMAX_n is at most $(\ln(n) + 1)/(p(1 - p)^n)$.

PROOF OF THEOREM 7. We first consider $D = \text{TrunkGeo}(N, q)$. We do not worry about constant factors in this analysis and thus bound some expressions generously.

Using Lemma 8 we can bound the expected run time of the (1 + 1) EA \vec{p} on ONEMAX_D from above by

$$\sum_{n=1}^{N-1} \frac{q(1-q)^{n-1}(\ln(n) + 1)}{q/2(1-q/2)^n} + \frac{(1-q)^{N-1}(\ln(N) + 1)}{q/2(1-q/2)^N}. \quad (2)$$

To bound the last summand in this expression, we first observe that, for all positive n ,

$$(1 - \frac{q}{2})^n = (1 - q + \frac{q^2}{4})^{n/2} > (1 - q)^{n/2}. \quad (3)$$

This shows that the last summand in (2) is at most

$$2(1 - q)^{N/2-1}(\ln(N) + 1)/q,$$

which is $O(q^{-1} \log q^{-1})$. This can be seen as follows. For $q \geq 2 \ln(N)/N$ it holds (using the inequality $1 - q \leq \exp(-q)$) that $(1 - q)^{N/2-1} \leq \exp(-qN/2) \leq 1/\ln(N)$ and

thus $2(1-q)^{N/2-1}(\ln(N)+1)/q = O(1/q)$, while for $1/N \leq q \leq 2 \ln(N)/N$ we have (for some suitably chosen constant C) $(1-q)^{N/2} \ln(N) \leq (1-1/N)^{N/2} \ln(N) \leq C(\ln(N) - \ln(2 \ln \ln N)) = C \ln(N/(2 \ln \ln N)) \leq C \ln(1/q)$.

Using again (3) we bound the first part of the sum (2) by

$$\begin{aligned} & \frac{2}{1-q} \sum_{n=1}^{N-1} \frac{(1-q)^n (\ln(n)+1)}{(1-q/2)^n} \\ & \leq \frac{2}{1-q} \sum_{n=1}^{N-1} (\ln(n)+1)(1-q)^{n/2} \\ & = 2 \sum_{n=1}^{N-1} (\ln(n)+1)(1-q)^{n/2-1}. \end{aligned}$$

To show that this expression is $O(q^{-1} \log q^{-1})$ we split the sum into blocks of length $k := \lceil 1/q \rceil$ and use again the inequality $1-q \leq \exp(-q)$. This shows that the last expression is at most

$$\begin{aligned} & 2 \sum_{j=0}^{\lceil N/k \rceil - 1} \sum_{\ell=1}^k \exp(-q(\frac{jk+\ell}{2} - 1)) (\ln(jk+\ell)+1) \\ & \leq 2k \sum_{j=0}^{\lceil N/k \rceil - 1} \exp(-\frac{1}{k}(\frac{jk}{2} - 1)) (\ln(j+1) + \ln(k)+1) \\ & = O(k \ln k), \end{aligned}$$

where the last equality can be best seen by first considering that $\sum_{j=0}^{\lceil N/k \rceil - 1} \exp(-\frac{1}{k}(\frac{jk}{2} - 1)) (\ln(k)+1) = \Theta(\log k)$, while $\sum_{j=0}^{\lceil N/k \rceil - 1} \exp(-\frac{1}{k}(\frac{jk}{2} - 1)) (\ln(j+1)) = O(1)$. Summarizing the computations above we see that (2) is of order at most $q^{-1} \log q^{-1}$.

For $D = \text{Geo}(q)$ the computations are almost identical. By Lemma 8 and (3) the expected run time of the (1+1) $EA_{\vec{p}}$ on ONEMAX_D is at most

$$\begin{aligned} & \frac{2}{1-q} \sum_{n=1}^{\infty} \frac{(1-q)^n (\ln(n)+1)}{(1-q/2)^n} \\ & \leq 2 \sum_{n=1}^{\infty} (1-q)^{n/2-1} (\ln(n)+1) = O(q^{-1} \log q^{-1}), \end{aligned}$$

which can be seen in a similar way as above by splitting the sum into blocks of size $k := \lceil 1/q \rceil$ and using $1-q \leq \exp(-q)$. \square

It is interesting to note that the expected run time increases to between $\Omega(N)$ and $O(N \log N)$ when the mutation probability is chosen to be $\vec{p} = (q, \dots, q)$. This can easily be seen as follows. For the upper bound we use Lemma 8 (ignoring the “+1” terms which are easily seen to play an insignificant role) to obtain that the expected run time of the (1+1) $EA_{\vec{p}}$ with $\vec{p} = (q, \dots, q)$ on $\text{ONEMAX}_{\text{TrunkGeo}(N,q)}$ is at most $\sum_{n=1}^{N-1} \frac{q(1-q)^{n-1} \ln n}{q(1-q)^n} + \frac{(1-q)^{N-1} \ln N}{q(1-q)^N} = \sum_{n=1}^{N-1} \frac{\ln n}{1-q} + O(\log(N)/q) = \frac{\ln((N-1)!)}{1-q} + O(N \log N) = O(N \log N)$.

We can derive a strong lower bound of $\Omega(N \log N)$ in the case of $2^{-N/3} \leq q \leq 1/N$ from the following one for static solution lengths.

LEMMA 9 ([10, THEOREM 9], [11, THEOREM 4.1]).

For a fixed length n and a uniform mutation vector $\vec{p} = (p, \dots, p)$, the expected run time of the (1+1) $EA_{\vec{p}}$

on ONEMAX_n is at least $(\ln(n) - \ln \ln n - 3)/(p(1-p)^n)$ for $2^{-n/3} \leq p \leq 1/n$ and at least $(\ln(1/(p^2 n)) - \ln \ln n - 3)/(p(1-p)^n)$ for $1/n \leq p \leq 1/(\sqrt{n} \log n)$.

Thus, the expected run time of the (1+1) $EA_{\vec{p}}$ with $\vec{p} = (q, \dots, q)$ and $2^{-N/3} \leq q \leq 1/N$ on $\text{ONEMAX}_{\text{TrunkGeo}(N,q)}$ is at least $\sum_{n=1}^{N-1} q(1-q)^{n-1} \frac{(\ln(n) - \ln \ln n - 3)}{q(1-q)^n} \geq \sum_{n=1}^{N-1} \frac{1}{2} \frac{\ln n}{1-q} = \frac{1}{2} \frac{\ln((N-1)!)}{1-q} = \Omega(N \log N)$. Similarly we can get a lower bound of $\Omega(N)$ in case of $1/N \leq q \leq 1/(\sqrt{N} \log N)$ by using the lower bound of $1/(q(1-q)^n)$ for any fixed solution length n .

We now turn our attention to the LEADINGONES problems, where a similar approach as above yields the following result.

THEOREM 10. Let $N \in \mathbb{N}$ and $1/N \leq q \leq 1/2$. For $D = \text{TrunkGeo}(N, q)$ and $D = \text{Geo}(q)$ the expected run time of the (1+1) $EA_{\vec{p}}$ with $\vec{p} = (q/2, \dots, q/2)$ on LEADINGONES_D is $\Theta(q^{-2})$.

We will derive this result from the following lemma, which was independently proven in [3, Theorem 3], [10, Corollary 2], and in a slightly weaker form in [8, Theorem 1.2].

LEMMA 11 ([3], [10], AND [8]). For a fixed length n and a mutation vector $\vec{p} = (p, \dots, p)$ with $0 < p < 1/2$, the expected run time of the (1+1) $EA_{\vec{p}}$ on LEADINGONES_n is exactly $1/(2p^2) ((1-p)^{-n+1} - (1-p))$.

PROOF OF THEOREM 10. We first consider the case that the solution length is sampled from the truncated geometric distribution $\text{TrunkGeo}(N, q)$. Using Lemma 11 and (3) (in the third and in the last step) the expected run time of the (1+1) $EA_{\vec{p}}$ on LEADINGONES_D is

$$\begin{aligned} & \sum_{n=1}^{N-1} q(1-q)^{n-1} \frac{2}{q^2} ((1-q/2)^{-n+1} - (1-q/2)) + A \\ & \leq \frac{2}{q} \sum_{n=1}^{N-1} \left(\frac{(1-q)^{n-1}}{(1-q/2)^{n-1}} \right) + A \\ & \leq \frac{2}{q} \sum_{n=0}^{\infty} (1-q)^{n/2} + A \\ & = \frac{2}{q} \frac{1}{1-(1-q)^{1/2}} + A = O(q^{-2}) + A, \end{aligned}$$

where A is the summand that accounts for the event that the solution length is N , i.e.,

$$A = (1-q)^{N-1} \frac{1}{2q^2} ((1-q)^{-N+1} - (1-q)) = O(q^{-2}).$$

Similarly for $D = \text{Geo}(q)$ the expected run time of the (1+1) $EA_{\vec{p}}$ on LEADINGONES_D is bounded from above by

$$\begin{aligned} & \frac{2}{q} \sum_{n=1}^{\infty} \frac{(1-q)^{n-1}}{(1-q/2)^{n-1}} \leq \frac{2}{q} \sum_{n=0}^{\infty} (1-q)^{n/2} \\ & = \frac{2}{q} \frac{1}{1-(1-q)^{1/2}} \leq \frac{4}{q^2}, \end{aligned}$$

where we recall that the last step follows from (3) for $n=1$, which provides $(1-q)^{1/2} \leq 1-q/2$. \square

Just as for ONEMAX_D (with $D = \text{TrunkGeo}(N, q)$) we see that also on LEADINGONES_D the expected run time increases

(in this case to $\Theta(N/q)$) when the mutation probability is chosen to be $\vec{p} = (q, \dots, q)$. By Lemma 11 this run time equals $\sum_{n=1}^{N-1} q(1-q)^{n-1} \frac{1}{2q^2} ((1-q)^{-n+1} - (1-q)) + A = \frac{1}{2q} \sum_{n=1}^{N-1} (1 - (1-q)^n) + A = \frac{1}{2q} \left(N - 1 - \frac{1 - (1-q)^N}{q} + 1 \right) + A = \Theta(N/q) + A$, where A is the summand that accounts for the event that the solution length is N , i.e.,

$$A = (1-q)^{N-1} \frac{1}{2q^2} \left((1-q)^{-N+1} - (1-q) \right) = \Theta(q^{-2}).$$

4. ARBITRARY SOLUTION LENGTHS

In the setting described in Section 3 it is assumed that the algorithm designer has quite a good knowledge about the solution length. Not only does he know an upper bound N on the solution length, but he may also crucially exploit its distribution. Indeed, we make quite heavy use in Theorems 7 and 10 of the fact that the (truncated) geometric distribution is highly concentrated around its expected value. That so much information is available to the algorithm designer can be a questionable assumption in certain applications. We therefore regard in this section a more general setting in which *no* a priori information is given about the possible solution length n . That is, we regard a setting in which the solution length can be an arbitrary positive integer. In this setting neither do we have any upper bounds on n nor any information about its distribution.

As before, our task is to decide upon on a sequence $(p_i)_{i \in \mathbb{N}}$ of mutation probabilities $0 \leq p_i \leq 1$. An adversary may then choose the solution length n and we run the $(1+1)$ EA $_{\vec{p}}$ with $\vec{p} = (p_1, \dots, p_n)$. In practical applications, this can be implemented with a (possibly generous) upper bound on the problem size.

We first show that uniform fixed bit flip probabilities necessarily lead to exponential run times (see Section 4.1). We then show two ways out of this problem. In Section 4.2 we consider non-uniform bit flip probabilities and in Section 4.3 we show that we can have an efficient algorithm with uniform bit flip probabilities if we choose the bit flip probability randomly in each iteration.

4.1 Uniform Bit Flip Probabilities

It seems quite intuitive that if nothing is known about the solution length there is not much we can achieve with uniform bit flip probabilities. In fact, for any fixed mutation probability $p \in [0, 1]$, we just need to choose a large enough solution length n to see that the $(1+1)$ EA $_{\vec{p}}$ with uniform mutation probability p is very inefficient. More precisely, using [11, Theorem 6.5] we get the following theorem regarding optimizing ONEMAX with uniform bit flip probabilities.

THEOREM 12. *Let $p \in [0, 1]$ be a constant. Then there exists a positive integer $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the expected run time of the $(1+1)$ EA $_{\vec{p}}$ with $\vec{p} = (p, \dots, p)$ on ONEMAX $_n$ is $2^{\Omega(n)}$.*

It is quite intuitive that for large p the expected optimization time of the $(1+1)$ EA $_{\vec{p}}$ with $\vec{p} = (p, \dots, p)$ is very large also for small problem sizes, as in this case typically too many bits are flipped in each iteration. This has been made precise by Witt, who showed that for p, n with $p = \Omega(n^{\varepsilon-1})$, the expected run time of the $(1+1)$ EA $_{\vec{p}}$ is $2^{\Omega(n^\varepsilon)}$ with probability at least $1 - 2^{-\Omega(n^\varepsilon)}$ [11, Theorem 6.3].

For LEADINGONES we get a similar lower bound from Lemma 11.

THEOREM 13. *Let $p \in (0, 1/2)$. Then the expected run time of the $(1+1)$ EA $_{\vec{p}}$ with $\vec{p} = (p, \dots, p)$ on LEADINGONES $_n$ is $2^{\Omega(n)}$.*

PROOF. From Lemma 11 we have that the expected run time of the $(1+1)$ EA $_{\vec{p}}$ is, for n large enough,

$$\frac{1}{2p^2} \left((1-p)^{-n+1} - (1-p) \right) \geq \frac{1}{2} \left(e^{pn-p} - 1 \right) = 2^{\Omega(n)}.$$

□

4.2 Non-Uniform Bit Flip Probabilities

One way to achieve efficient optimization with unknown solution length is by using non-uniform mutation rates, that is, different bit positions have different probabilities associated for being flipped during a mutation operation.

To state our results we need the concept of *summable sequences*. Such sequences will be the basis for the sequence of bit flip probabilities. In short, a sequence $(p_i)_{i \in \mathbb{N}}$ is *summable* if its series $(\sum_{k=1}^n p_k)_{n \in \mathbb{N}}$ converges (that is, if it is bounded). The advantage of using summable sequences is that the probability of flipping only one single bit is *always* constant, regardless of the total number of bits considered, i.e., regardless of the problem length n . This is in contrast to the sequence $(1/(i+1))_{i \in \mathbb{N}}$ considered in [4], which is not summable, and which has a chance of $(1/2) \prod_{i=2}^n (1 - 1/(i+1)) = 1/n$ of flipping only the first bit and a chance of $(1/n) \prod_{i=1}^{n-1} (1 - 1/(i+1)) = 1/n^2$ of flipping only the n th bit. For this reason the $(1+1)$ EA $_i$ is very inefficient for the setting in which the solution length can be arbitrary.

Theorems 14 and 15 show that not knowing the solution length n does not harm the run time more than by a factor of order $\log^{1+\varepsilon} n$ with respect to the optimal bound when the problem length is known a priori, cf. also Corollary 16 for an explicit sequence yielding this bound. In fact, we prove that the additional cost caused by not knowing the solution length in advance is even a bit smaller, cf. the comments after Corollary 16.

We start with the theorem regarding ONEMAX.

THEOREM 14. *Let $(p_i)_{i \in \mathbb{N}}$ be a monotonically decreasing summable sequence with $\Sigma := \sum_{i=1}^{\infty} p_i < 1$. Then, for any $n \in \mathbb{N}$, the expected run time of the $(1+1)$ EA $_{\vec{p}}$ with $\vec{p} = (p_1, \dots, p_n)$ on ONEMAX $_n$ is at most $\log n / (p_n(1 - \Sigma)) = O(\log n / p_n)$.*

PROOF. We make use of the multiplicative drift theorem [5, Theorem 3] and show that for every n and every search point x with $n - k$ ones, the probability to create in one iteration of the $(1+1)$ EA $_{\vec{p}}$ with $\vec{p} = (p_1, \dots, p_n)$ a search point y with $\text{ONEMAX}_n(y) > \text{ONEMAX}_n(x)$ is at least of order k/p_n . This can in fact be seen quite easily by observing that the probability to increase the ONEMAX-value of x by exactly one is at least

$$\begin{aligned} kp_n \prod_{j=1}^n (1 - p_j) &\geq kp_n \left(1 - \sum_{j=1}^n p_j \right) \geq kp_n \left(1 - \sum_{j=1}^{\infty} p_j \right) \\ &= kp_n (1 - \Sigma). \end{aligned}$$

From this an upper bound of $\log n / (p_n(1 - \Sigma))$ for the run time of the $(1+1)$ EA $_{\vec{p}}$ follows immediately from the multiplicative drift theorem. □

Next we consider LEADINGONES. The proof follows along similar lines as the one for ONEMAX and uses a fitness level argument instead of multiplicative drift (using additive drift would also be possible).

THEOREM 15. *Let $(p_i)_{i \in \mathbb{N}}$ be a monotonically decreasing summable sequence with $\Sigma := \sum_{i=1}^{\infty} p_i < 1$. Then, for any $n \in \mathbb{N}$, the expected run time of the $(1+1)$ EA $_{\vec{p}}$ with $\vec{p} = (p_1, \dots, p_n)$ on LEADINGONES $_n$ is at most $n/(p_n(1-\Sigma)) = O(n/p_n)$.*

PROOF. Let $n, k \in \mathbb{N}$ with $k < n$ and let $x \in \{0, 1\}^n$ with $\text{LO}(x) = k - 1$. The probability to get in one iteration of the $(1+1)$ EA $_{\vec{p}}$ with $\vec{p} = (p_1, \dots, p_n)$ a search point y with $\text{LO}(y) > \text{LO}(x)$ is at least

$$p_k \prod_{j=1}^{k-1} (1 - p_j) \geq p_k (1 - \sum_{j=1}^{k-1} p_j) \geq p_k (1 - \Sigma) \geq p_n (1 - \Sigma).$$

By a simple fitness level argument (see, e.g., the work by Sudholt [10] for background and examples of this method), the expected run time of the $(1+1)$ EA $_{\vec{p}}$ on LEADINGONES $_n$ is thus at most $n/(p_n(1-\Sigma))$. \square

It is well known that for every constant $\varepsilon > 0$ the sequence $(1/(i \log^{1+\varepsilon} i))_{i \in \mathbb{N}}$ is summable (this can be proven via Cauchy's condensation test). It is obviously also monotonically decreasing in i . Theorems 14 and 15, together with the sequence $(p_i)_{i \in \mathbb{N}} := (1/(2Si \log^{1+\varepsilon} i))_{i \in \mathbb{N}}$ for $S := \sum_{i=1}^{\infty} 1/(i \log^{1+\varepsilon} i)$, therefore imply the following corollary.

COROLLARY 16. *For every positive constant ε there exists a sequence of mutation probabilities $(p_i)_{i \in \mathbb{N}}$ such that for any n the expected run time of the $(1+1)$ EA $_{\vec{p}}$ with $\vec{p} = (p_1, \dots, p_n)$ on ONEMAX $_n$ is $O(n \log^{2+\varepsilon} n)$ and is of order $n^2 \log^{1+\varepsilon} n$ for LEADINGONES $_n$.*

The bound from Corollary 16 can be improved by regarding the following summable sequences.

For any $r \in \mathbb{R}$ and any $i \in \mathbb{N}_{\geq 2}$ let

$$\log^{(i)} r := \begin{cases} \log_2(\log^{(i-1)} r), & \text{if } \log^{(i-1)}(r) \geq 2; \\ 1, & \text{otherwise;} \end{cases}$$

where $\log^{(1)} r := \log_2 r$ if $r \geq 2$ and $\log^{(1)} r := 1$ otherwise. For every constant $\varepsilon > 0$ and all positive integers s, i let

$$p_i^{s, \varepsilon} := 1 / \left(i (\log^{(s)}(i))^{1+\varepsilon} \prod_{j=1}^{s-1} \log^{(j)}(i) \right). \quad (4)$$

For every $\varepsilon > 0$ and every $s \geq 1$ the sequence $(p_i^{s, \varepsilon})_{i \in \mathbb{N}}$ is summable. Furthermore, this sequence clearly is monotonically decreasing. Choosing larger and larger s therefore gives better and better asymptotic run time bounds in Theorems 14 and 15.

4.3 Randomized Bit Flip Probability

In the conclusions of [4] the authors ask the following: how can we optimize efficiently when an upper bound N on the problem length is known, but only n bits at *unknown positions* are relevant for the fitness? It is not difficult to see that our previous solutions with non-uniform bit flip probabilities will not be able to assign appropriate bit flip probabilities to the *relevant* bit positions. However, any uniform

choice of bit flip probabilities will effectively ignore irrelevant bit positions. In this section we consider a variation of the $(1+1)$ EA where the bit flip probability p is chosen randomly from a distribution Q on $(0, 1)$ in each iteration (the distribution Q does not change over time). This mutation probability is then applied independently to each bit, i.e., each bit of the current best solution is independently flipped with probability p . See Algorithm 2 for the detailed description of the $(1+1)$ EA $_Q$.

Algorithm 2: The $(1+1)$ EA $_Q$ for a distribution Q on $(0, 1)$ optimizing a pseudo-Boolean function $f : \{0, 1\}^n \rightarrow \mathbb{R}$.

```

1 Initialization: Sample  $x \in \{0, 1\}^n$  uniformly at
   random and query  $f(x)$ ;
2 Optimization: for  $t = 1, 2, 3, \dots$  do
3   Sample bit flip probability  $p_t$  from  $Q$ ;
4   for  $i = 1, \dots, n$  do
5     With probability  $p_t$  set  $y_i \leftarrow 1 - x_i$  and set
      $y_i \leftarrow x_i$  otherwise;
6   Query  $f(y)$ ;
7   if  $f(y) \geq f(x)$  then  $x \leftarrow y$ ;
```

To make the problem more explicit, we are asked to find a distribution Q on $(0, 1)$ such that the $(1+1)$ EA $_Q$ efficiently optimizes for any $n \in \mathbb{N}$ and any pairwise different $b_1, \dots, b_n \in \mathbb{N}$ the functions

$$\text{ONEMAX}_{b_1, \dots, b_n}(x) := \sum_{i=1}^n x_{b_i}, \text{ respectively}$$

$$\text{LEADINGONES}_{b_1, \dots, b_n}(x) := \max\{i \in [0..n] \mid \forall j \leq i : x_{b_j} = 1\}.$$

In Theorems 17 and 18 we show that such a distribution Q exist. That is, there is a distribution Q such that the corresponding $(1+1)$ EA $_Q$ efficiently optimizes any ONEMAX $_{b_1, \dots, b_n}$ and any LEADINGONES $_{b_1, \dots, b_n}$ function, regardless of the number of relevant bits and regardless of their positions.

We start with our main result regarding ONEMAX.

THEOREM 17. *Let $(p_i)_{i \in \mathbb{N}} \in (0, 1)^{\mathbb{N}}$ be a monotonically decreasing summable sequence. Set $\Sigma := \sum_{j=1}^{\infty} p_j$. Let Q be the distribution which assigns the mutation probability $1/i$ a probability of p_i/Σ .*

For any $n \in \mathbb{N}$ and any pairwise different positive integers b_1, \dots, b_n the expected run time of the $(1+1)$ EA $_Q$ on ONEMAX $_{b_1, \dots, b_n}$ is $O(\log(n)/p_{2n})$.

PROOF. The probability to sample a mutation probability between $1/(2n)$ and $1/n$ is

$$\sum_{j=n}^{2n} p_j \geq np_{2n}.$$

We disregard all iterations in which we do not sample a mutation probability between $1/(2n)$ and n (they can only be beneficial). Thus, on average, we consider at least one iteration out of $1/(np_{2n})$.

Assuming that x is a search point with $n - \ell$ ones (in the relevant positions) and that the sampled bit flip probability p satisfies $1/(2n) \leq p \leq 1/n$, the probability to make a

progress of exactly one is at least

$$\ell p(1-p)^{n-1} \geq \ell/(2n)(1-1/n)^{n-1} \geq \ell/(2en).$$

Thus, we have an expected progress in each iteration of at least $\frac{\ell}{2en}np_{2n} = O(\ell p_{2n})$. Therefore, by the multiplicative drift theorem [5, Theorem 3], we need in expectation $O(\log(n)/p_{2n})$ iterations to optimize function $\text{ONEMAX}_{b_1, \dots, b_n}$. \square

For **LEADINGONES** we obtain the following.

THEOREM 18. *Let $(p_i)_{i \in \mathbb{N}}$ and Q as in Theorem 17.*

For any $n \in \mathbb{N}$ and any pairwise different $b_1, \dots, b_n \in \mathbb{N}$ the expected run time of the $(1+1)$ EA_Q on $\text{LEADINGONES}_{b_1, \dots, b_n}$ is $O(n/p_{2n})$.

PROOF. This proof follows along similar lines as the one for **ONEMAX**. We have again that the probability to have a bit flip probability between $1/(2n)$ and $1/n$ in an iteration is at least np_{2n} .

Let x be a search point with $\text{LEADINGONES}_{b_1, \dots, b_n}(x) = \ell$. Given a mutation probability p between $1/(2n)$ and $1/n$, the probability to create in one iteration of the $(1+1)$ EA_Q a search point y of fitness greater than ℓ is at least

$$p(1-p)^{\ell-1} \geq 1/(2n)(1-1/n)^{n-1} \geq 1/(2en).$$

Thus, we have an expected progress in each iteration of at least $\frac{1}{2en}np_{2n} = O(p_{2n})$. Therefore, by the fitness level method (see again [10] for a discussion of this method), we need in expectation $O(n/p_{2n})$ iterations to optimize $\text{LEADINGONES}_{b_1, \dots, b_n}$. \square

By choosing the summable sequence with entries as in (4) and $s = 1$, the two theorems above immediately yield the following result.

COROLLARY 19. *The expected run time of the described $(1+1)$ EA_Q with Q using the summable sequence (4) with $s = 1$ on $\text{ONEMAX}_{b_1, \dots, b_n}$ is $O(n \log^{2+\varepsilon} n)$ and on $\text{LEADINGONES}_{b_1, \dots, b_n}$ it is $O(n^2 \log^{1+\varepsilon} n)$.*

Note that, just as discussed after Corollary 16, choosing larger and larger s gives asymptotically better and better bounds.

5. SUMMARY AND OUTLOOK

We have analyzed the performance of variants of the $(1+1)$ EA in the presence of unknown solution lengths. While for highly concentrated solution length non-uniform mutation probabilities are not advantageous (or at least not to a significant degree), they are crucial in a setting in which we do not have any knowledge about the solution length. Surprisingly, even in the latter situation, a sequence of (non-uniform) mutation probabilities exists such that the corresponding $(1+1)$ EA is almost optimal, simultaneously for all possible solution lengths.

We have also investigated a setting in which the relevant bit positions can be arbitrary in number and position. Possibly even more surprisingly, even this can be handled quite efficiently by a $(1+1)$ EA variant for the two test functions **ONEMAX** and **LEADINGONES**.

We believe the setting of unknown solution length to be relevant for numerous real-world applications. As a next step toward a better understanding of how this uncertainty

can be tackled efficiently with evolutionary algorithms, we suggest to investigate more challenging function classes, e.g., starting with the class of all linear functions. It is not clear a priori if bounds similar to the ones presented in Section 4 can be achieved for such problems.

From a mathematical point of view it would also interesting to investigate the tightness of our bounds in Section 4. We do not know whether some choice of mutation probabilities gives an upper bound of $O(n \log n)$ for **ONEMAX** or $O(n^2)$ for **LEADINGONES**. We recall that the sequences $(1/(n \log(n)))_{n \in \mathbb{N}}$ as well as $(1/p_i^{\infty, \varepsilon})_{i \in \mathbb{N}}$ with $p_i^{\infty, \varepsilon} := \lim_{s \rightarrow \infty} p_i^{s, \varepsilon}$ are not summable. Removing the gap entirely is therefore likely to require a substantially different approach.

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