The many facets of upper domination

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A B S T R A C T

This paper studies UPPER DOMINATION, *i.e.*, the problem of computing the maximum cardinality of a minimal dominating set in a graph with respect to classical and parameterised complexity as well as approximability.

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1. Introduction

A dominating set of an undirected graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that all vertices outside of S have a neighbour in S. The problem of finding the smallest dominating set of a given graph is one of the most widely studied problems in computational complexity. In this paper, we focus on a related problem that "flips" the optimisation objective. In UPPER DOMINATION we are given a graph and we are asked to find a *maximum cardinality* dominating set that is still minimal. A dominating set is minimal if any proper subset of it is no longer dominating, that is, if it does not contain obviously redundant vertices.

The study of MaxMin or MinMax versions of a problem by "flipping" the objective is not a new idea; in fact, such questions have been posed before for many classical optimisation problems. Some of the most well-known examples are the MINIMUM MAXIMAL INDEPENDENT SET problem [14,13,30,35] (also known as MINIMUM INDEPENDENT DOMINATING SET), the MAXIMUM MINIMAL VERTEX COVER problem [11,45] and the LAZY BUREAUCRAT problem [4,8], which is a MinMax version of KNAPSACK. The initial motivation for this type of question was rather straightforward: most classical optimisation problems admit an easy, naive heuristic algorithm which starts with a trivial solution and then gradually tries to improve it in an

obvious way until it gets stuck. For example, one can produce a (maximal) independent set of a graph by starting with a single vertex and then adding vertices to the current solution while maintaining an independent set. What can we say about the worst-case performance of such a basic algorithm? Motivated by this initial question the study of MaxMin and MinMax versions of standard optimisation problems has gradually grown into a sub-field with its own interest, often revealing new insights into the structure of the original problems. UPPER DOMINATION is a natural example of this family of problems and is also one of the six problems from the so-called *domination chain* (see [32] and Section 2), on which somewhat fewer results are currently known. The goal of this paper is to increase our understanding of this problem by investigating it from the different perspectives of approximability and classical and parameterised complexity.

1.1. Summary of results

We first link minimal dominating sets to a decomposition of the vertex set which turns out to be a useful tool throughout the whole paper.

From a classical complexity point of view, we show that UPPER DOMINATION is NP-hard on planar cubic graphs. Since the problem is easy on graphs of maximum degree 2, our results completely characterise the complexity of the problem in terms of maximum degree. Given the general behaviour of this type of problem, and the above results on UPPER DOMINATION in particular, the questions remains *why* are such problems typically so much harder than their original versions. Consider in this context the following *extension* problem: Given a graph $G = (V, E)$ and a set $S \subseteq V$, does there exist a minimal dominating set *of any size* that contains S? Even though questions of this type are typically trivial for problems such as INDEPENDENT SET, we show that this kind of extension problem for UPPER DOMINATION is NP-hard even for planar cubic graphs. This helps explain the added difficulty of this problem, and more generally of problems of this type, since any natural algorithm that gradually builds a solution would have to contend with (some version of) this extension problem. On the positive side, we derive an exact $O^*(1.348^n)$-algorithm for subcubic graphs which builds on the decomposition derived in Section 2.

From the approximation perspective, we find that while DOMINATING SET admits a greedy $\ln n$ approximation, UPPER DOMINATION does not admit an $n^{1-\epsilon}$ approximation for any $\epsilon > 0$, unless P = NP. We also show that UPPER DOMINATION remains APX-hard on cubic graphs and complement these negative results by giving some approximation algorithms for the problem in restricted cases. Specifically, we show an $O(\Delta)$-approximation on graphs with maximum degree Δ, as well as an EPTAS on planar graphs.

From a parameterised point of view, we show that UPPER DOMINATION is W[1]-hard with respect to standard parameterisation (*i.e.* parameter $k = \Gamma(G)$, where $\Gamma(G)$ denotes the upper domination number). Conversely, CO-UPPER DOMINATION (*i.e.* UPPER DOMINATION with parameterisation $\ell = n - k$), is shown to be in FPT, which we prove by providing both a kernelisation and a branching algorithm.

2. Preliminaries and combinatorial bounds

We only deal with undirected simple connected graphs $G = (V, E)$. The number of vertices $n = |V|$ is known as the order of G. As usual, $N(v)$ denotes the open neighbourhood of v, and $N[v]$ is the closed neighbourhood of v, *i.e.*, $N[v] = N(v) \cup \{v\}$, which easily extends to vertex sets X, *i.e.*, $N(X) = \bigcup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. The cardinality of $N(v)$ is known as the degree of v, denoted as $deg(v)$. The maximum degree in a graph is written as Δ. A graph of maximum degree 3 is called subcubic, and if all degrees equal 3, it is called cubic. In the area of parameterised and exact exponential algorithms, it has become customary not only to suppress constants (as in the O notation), but also polynomial-factors, leading to the so-called O^*-notation.

Given a graph $G = (V, E)$, a subset S of V is a *dominating set* if every vertex $v \in V \setminus S$ has at least one neighbour in S, *i.e.*, if $N[S] = V$. A dominating set is minimal if no proper subset of it is a dominating set. Likewise, a vertex set I is *independent* if $N(I) \cap I = \emptyset$. An independent set is maximal if no proper superset is independent. In the following we use classical notations: $\gamma(G)$ and $\Gamma(G)$ are the minimum and maximum cardinalities over all minimal dominating sets in G, $\alpha(G)$ and $i(G)$ are the maximum and minimum cardinalities over all maximal independent sets, and $\tau(G)$ is the size of a minimum vertex cover, which equals $|V| - \alpha(G)$ by Gallai's identity. A minimal dominating set D of G with $|D| = \Gamma(G)$ is also known as an *upper dominating set* of G, and $\Gamma(G)$ is also called the *upper domination number*.

For any subset $S \subseteq V$ and $v \in V$ we define the private neighbourhood of v with respect to S as $pn(v, S) := N[v] \setminus N[S \setminus \{v\}]$. Any $w \in pn(v, S)$ is called a *private neighbour of v with respect to S*. A set S is called *irredundant* if every vertex in it has at least one private neighbour, *i.e.*, if $|pn(v, S)| > 0$ for every $v \in S$. The cardinality of the largest irredundant set in G is denoted by $IR(G)$, while $ir(G)$ denotes the cardinality of the smallest maximal irredundant set in G. We can now observe the validity of the well-known *domination chain*:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G)$$

The domination chain is largely due to the following two combinatorial properties: (1) Every maximal independent set is a minimal dominating set. (2) A dominating set $S \subseteq V$ is minimal if and only if $|pn(v, S)| > 0$ for every $v \in S$. Observe that v can be a private neighbour of itself, *i.e.*, a dominating set is minimal if and only if it is also an irredundant set. Actually, every minimal dominating set is also a maximal irredundant set.
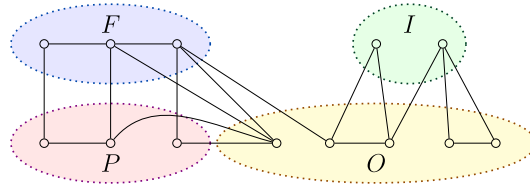
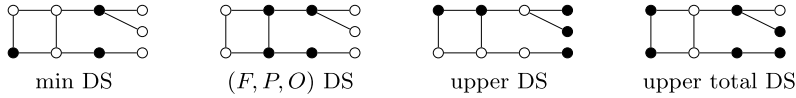**Fig. 1.** Illustration of the FIPO structure imposed by minimal dominating sets.

| min DS | (F, P, O) DS | upper DS | upper total DS |

Fig. 2. Differences between Minimum, (F, I, P, O)-, Upper and Upper Total Domination.

The following exposition is crucial for the development of the algorithms we derive in this paper and also for the general investigation of properties of minimal dominating sets. Any minimal dominating set D for a graph $G = (V, E)$ can be associated with a partition of the set of V into four sets F, I, P, O given by: $I := \{v \in D : v \in pn(v, D)\}$, $F := D \setminus I$, $P \in \{B \subseteq N(F) \setminus D : |pn(v, D) \cap B| = 1 \text{ for all } v \in F\}$ and $O := V \setminus (D \cup P)$, see Fig. 1. This representation is not necessarily unique since there might be different choices for the sets P and O, but for every partition of this kind, the following properties hold:

1. Every vertex $v \in F$ has at least one neighbour in F, called a **f**riend.
2. The set I is an **i**ndependent set in G.
3. The subgraph induced by the vertices $F \cup P$ has an edge cut set separating F and P that is a perfect matching; hence, P can serve as the set of **p**rivate neighbours for F.
4. The neighbourhood of a vertex in I is always a subset of O. As the vertices in O can be deleted from the graph without changing the property of D being a minimal dominating set, they are called **o**utsiders.

This partition is also related to a different characterisation of $\Gamma(G)$ in terms of so-called upper perfect neighbourhoods [32]. Observe two important special cases: If $F = \emptyset$, then I is an independent dominating set. If $I = \emptyset$, then F is a minimal total dominating set, *i.e.*, a set $S \subseteq V$ such that $V = N(S)$ and $N(S') \neq V$ for all $S' \subset S$; both classical variations of domination in graphs (see [32]). Observe that finding a maximum cardinality minimal dominating set for which $I = \emptyset$ holds in an (F, I, P, O) partitioning (called (F, P, O)-Domination set in the following) is not equivalent to the problem Upper Total Domination, which asks for a maximum cardinality minimal total dominating set. Fig. 2 illustrates the differences between optimal solutions (illustrated by the black vertices) for Minimum, (F, P, O)-, Upper and Upper Total Domination.

Lemma 1. *For any graph G and any upper dominating set D for G with an associated partition (F, I, P, O), if $|D| = \Gamma(G) > \alpha(G)$ then $|I| \leq \alpha(G) - 2$.*

Proof. Let D be an upper dominating set for a graph G with an associated partition (F, I, P, O). First observe that if $\Gamma(G) > \alpha(G)$ then $|F| \geq 2$. Indeed, if $|F| = 0$, then the upper dominating set is also an independent set, and thus $\Gamma(G) = \alpha(G)$, and according to our definition of partition (F, I, P, O), we have $|F| \neq 1$ (see Property 1 of this partition). Now, if $|F| \geq 2$ then the subgraph of G induced by $F \cup P$ contains an independent set of size 2 consisting of a vertex in F, say v, and a vertex in P, say u, such that v and u are not adjacent. Since in the original graph G, there are no edges between the vertices in I and the vertices in $F \cup P$ (Property 4), $I \cup \{u, v\}$ forms an independent set of size $|I| + 2$. This sets a lower bound on the independence number and we have $\alpha(G) \geq |I| + 2$, that is, $|I| \leq \alpha(G) - 2$. $\square$

Lemma 2. *For any graph G we have:*

$$\alpha(G) \leq \Gamma(G) \leq \max\left\{\alpha(G), \frac{n}{2} + \frac{\alpha(G)}{2} - 1\right\}. \tag{1}$$

Proof. We consider a graph G with upper dominating set D with an associated partition (F, I, P, O). The left inequality comes from the fact that any maximal independent set is a minimal dominating set. For the right inequality, we examine separately the following two cases.

1. $\Gamma(G) = \alpha(G)$. Then we trivially have $\Gamma(G) \leq \alpha(G)$.
2. $\Gamma(G) > \alpha(G)$.
   From $|F| = |P|$ (Property 3) we have $|F| = \frac{1}{2}(n - |I| - |O|) \leq \left\lfloor \frac{n - |I|}{2} \right\rfloor$ and thus

$$\Gamma(G) = |F| + |I| \le \left\lfloor \frac{n + |I|}{2} \right\rfloor .$$

From the above and Lemma 1 we have

$$\Gamma(G) \ \le\ \left\lfloor \frac{n + |I|}{2} \right\rfloor \ \le\ \left\lfloor \frac{n + \alpha(G) - 2}{2} \right\rfloor \ \le\ \frac{n}{2} + \frac{\alpha(G)}{2} - 1 .$$

This concludes the proof of the claim. □

Lemma 3. *For any graph G of minimum degree δ and maximum degree Δ, we have:*

$$\alpha(G) \ \le\ \Gamma(G) \ \le\ \max \left\{ \alpha(G),\ \frac{n}{2} + \frac{\alpha(G)(\Delta - \delta)}{2\Delta} - \frac{\Delta - \delta}{\Delta} \right\} . \tag{2}$$

Proof. Let G be a graph of maximum degree Δ, minimum degree δ and let D be an upper dominating set for G with an associated partition (F, I, P, O). Our argument is similar to the one in Lemma 2: The left inequality comes from the fact that any maximal independent set is a minimal dominating set. For the right inequality, we examine separately the following two cases.

1. $\Gamma(G) = \alpha(G)$. Then we trivially have $\Gamma(G) \le \alpha(G)$.
2. $\Gamma(G) > \alpha(G)$. Again, we obtain:

   $$\Gamma(G) = |F| + |I| = \frac{n + |I| - |O|}{2} .$$

   We next derive an improved lower bound on $|O|$. Let e be the number of edges adjacent to vertices from I. As G is of minimum degree δ, we have $e \ge \delta|I|$. As the vertices in I are only adjacent to vertices in O, there are at least e edges that have exactly one end vertex in O. Since G has maximum degree Δ, we have that $|O| \ge \lceil \frac{e}{\Delta} \rceil \ge \lceil \frac{\delta|I|}{\Delta} \rceil$.
   From the above and Lemma 1 we have

   $$\Gamma(G) \ \le\ \left\lfloor \frac{n + |I| - \left\lceil \frac{\delta|I|}{\Delta} \right\rceil}{2} \right\rfloor \ \le\ \frac{n + |I| - \frac{\delta|I|}{\Delta}}{2} = \frac{n + \frac{(\Delta - \delta)|I|}{\Delta}}{2}$$

   $$\le \frac{n + \frac{(\Delta - \delta)}{\Delta}(\alpha(G) - 2)}{2} = \frac{n}{2} + \frac{\Delta - \delta}{2\Delta}\alpha(G) - \frac{\Delta - \delta}{\Delta} . \quad \square$$

Note that Lemma 3 generalises the earlier result of Henning and Slater on upper bounds on $\mathrm{IR}(G)$ (and hence on $\Gamma(G)$) for Δ-regular graphs G [33]. Observe also that all bounds derived in this section are also valid for the upper irredundance number $\mathrm{IR}(G)$ instead of $\Gamma(G)$.

3. Classical complexity

In this section, we strengthen the known NP-hardness result for UPPER DOMINATION and consider exact algorithms for graphs of bounded path- and treewidth. We further discuss the problem of computing minimal dominating sets with a different perspective by considering the problem of extending partial solutions. Throughout this section, we consider UPPER DOMINATION as the following classical decision problem:

UPPER DOMINATION
Input: A graph $G = (V, E)$, integer k.
Question: Is $\Gamma(G) \ge k$?

It has long been known that UPPER DOMINATION is NP-complete in general [19], and even for graphs of maximum degree 6 [1]. Some polynomial-time solvable graph classes are also known. This is mainly due to the fact that on certain graph classes (like bipartite graphs) the independence number and upper domination number coincide and for those graph classes, the independence number can be computed in polynomial-time. In particular, UPPER DOMINATION is polynomial for bipartite graphs [20], chordal graphs [37], generalised series-parallel graphs [31] and graphs with bounded clique-width [21]. We refer to the textbook on domination [32] for further details. Recently, the complexity of UPPER DOMINATION in monogenic graph classes (*i.e.,* classes of graphs defined by a single forbidden induced subgraph) has led to a complexity dichotomy: if the unique forbidden induced subgraph is a P_4 or a $2K_2$ (or an induced subgraph of these), then UPPER DOMINATION is polynomial; otherwise, it is NP-complete [1].

3.1. Hardness on cubic planar graphs

Upper Domination is known to be NP-hard on planar graphs of maximum degree 6 [1]. We strengthen this result to maximum degree 3. Given that Upper Domination is trivial for graphs of maximum degree 2, this result hence completely characterises the classical complexity of Upper Domination with respect to maximum degree.

**Theorem 4.** Upper Domination *is NP-hard on cubic planar graphs.*

Proof. We present a reduction from Maximum Independent Set restricted to cubic planar graphs, which is known remain NP-hard [29]. Let $G = (V, E)$ be a cubic planar input graph for Maximum Independent Set. Construct a subcubic planar graph G' from G by adding for every $(u, v) \in E$ six new vertices $u_v, u_v^1, u_v^2, v_u, v_u^1, v_u^2$ and replacing the edge (u, v) by the graph illustrated below.

We claim that there exists an independent set of cardinality k for G if and only if there exists an upper dominating set of cardinality at least $k + 3|E|$ for G'. If IS is an independent set of cardinality k for G, the corresponding vertex-set in G' can be extended to an upper dominating set S of cardinality at least $k + 3|E|$ in the following way: For every edge (u, v) with $v \notin IS$ add $\{v_u, u_v^1, u_v^2\}$ to S. Since IS is independent, this procedure chooses three vertices for each edge-gadget in G' and creates an independent set S of cardinality $|IS| + 3|E|$. If some vertex is not dominated by S we add it to S and finally arrive at a maximal independent and consequently minimal dominating set.
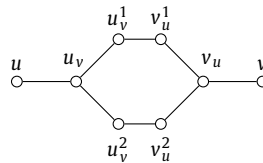
Let S be an upper dominating set of cardinality $k + 3|E|$ for G'. We claim that for every edge $(u, v) \in E$ at most three of the vertices $u_v, u_v^1, u_v^2, v_u, v_u^1, v_u^2$ can be in S. First, observe that $u_v^1, v_u^1 \in S$ is impossible because S has to contain a vertex from $N[v_u^2] = \{v_u, v_u^2, u_v^2\}$ in order to dominate v_u^2 and hence either u_v^1 or v_u^1 has no private neighbour. Also, u_v and v_u together already dominate all vertices added for the edge (u, v), hence a minimal dominating set contains three vertices from $\{u_v, u_v^1, u_v^2, v_u, v_u^1, v_u^2\}$ if it contains exactly one vertex from each of the sets $\{u_v, v_u\}$, $\{u_v^1, v_u^1\}$ and $\{u_v^2, v_u^2\}$. Now, if $u, v \in S$ for some edge (u, v), then S contains at most two vertices from $\{u_v, u_v^1, u_v^2, v_u, v_u^1, v_u^2\}$; observe that if $u_v \in S$ (or $v_u \in S$), minimality requires either u_v^1 or u_v^2 to be a private neighbour for u_v and hence $\{u_v^h, v_u^h\} \cap S = \emptyset$ for either $h = 1$ or $h = 2$. Consider $S' = S \cap V$ as potential independent set for the original graph G. If there are two vertices $u, v \in S'$ such that $(u, v) \in E$, the set S can only contain two vertices from the edge-gadget corresponding to (u, v). By successively deleting vertices from S' as long as there is a conflict with respect to independence, we arrive at an independent set of cardinality at least $|S| - 3|E| \geq k$.

So far, G' is only subcubic, as for each edge $(u, v) \in E$ the vertices u_v^h, v_u^h for $h \in \{1, 2\}$ only have degree 2. Create a graph G'' from G' by adding five vertices connected as in the graph illustrated below for every vertex w of degree 2.
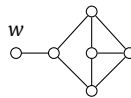
A minimal dominating set for G'' contains at most two vertices from each group of five new vertices. In case w chooses its new neighbour as private neighbour in some dominating set, minimality only allows one more vertex from the new set; in all other cases two vertices from the new subgraph can be included in the dominating set. As we add this subgraph exactly four times for each edge in the original graph, G has an independent set of cardinality k if and only if G'' has an upper dominating set of cardinality $k + 11|E|$. □

For classes defined by finitely many forbidden induced subgraphs, the boundary separating difficult instances of the problem from polynomially solvable ones consists of the so called boundary classes. Very recently, a boundary class for Upper Domination is given by the graphs such that the vertex set can be partitioned into two (possibly empty) cliques U and W such that (*a*) every vertex in W has at most two neighbours in U, (*b*) if x and y are two vertices of W each of which has exactly two neighbours in U, then their neighbourhood in U are distinct. Alternatively, this class can be defined as the class of graphs which does not contain eleven graphs as minimal forbidden induced subgraphs, see [1,2].

3.2. Exact algorithms

We will first construct an exact algorithm for graphs with a given path-decomposition, so let us first recall one important result on the pathwidth of subcubic graphs from [28].

Theorem 5. *Let $\epsilon > 0$ be given. For any subcubic graph G of order $n > n_\epsilon$, a path decomposition proving $pw(G) \leq n/6 + \epsilon$ is computable in polynomial time.*

This result immediately gives an $O^*(1.2010^n)$-algorithm for solving MINIMUM DOMINATION on subcubic graphs. We will take a similar route to prove moderately exponential-time algorithms for UPPER DOMINATION.

Proposition 6. UPPER DOMINATION *on graphs of pathwidth p can be solved in time $O^*(7^p)$, given a corresponding path decomposition.*

Proof. We are considering all partitions of each bag of the path decomposition into 6 sets: F^*, F, I, P, O, O^*, where:

- F^* is the set of vertices that belong to the upper dominating set and still need to be matched to a private neighbour;
- F is the set of vertices that belong to the upper dominating set and have already been matched to a private neighbour;
- I is the set of vertices that belong to the upper dominating set and is independent in the graph induced by the upper dominating set;
- P is the set of private neighbours already matched to vertices in F;
- O is the set of vertices that are not belonging neither to the upper dominating set nor to the set of private neighbours but are already dominated;
- O^* is the set of vertices that are not dominated yet.

(Sets within the partition can be also empty.) For each such partition, we determine the largest minimal dominating set in the situation described by the partition, assuming optimal settings in the part of the graph already forgotten.

We can assume that we are given a nice path decomposition. So, we only have to describe the table initialisation (the situation in a bag containing only one vertex) and the table updates necessary when we introduce a new vertex into a bag and when we finally forget a vertex. In the following, $-\infty$ always signals an error case when we try to introduce partitions which are not allowed.

initialisation We have six cases to consider:

- $T[\{v\}, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset] \leftarrow 1$,
- $T[\emptyset, \{v\}, \emptyset, \emptyset, \emptyset, \emptyset] \leftarrow -\infty$,
- $T[\emptyset, \emptyset, \{v\}, \emptyset, \emptyset, \emptyset] \leftarrow 1$,
- $T[\emptyset, \emptyset, \emptyset, \{v\}, \emptyset, \emptyset] \leftarrow -\infty$,
- $T[\emptyset, \emptyset, \emptyset, \emptyset, \{v\}, \emptyset] \leftarrow -\infty$,
- $T[\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \{v\}] \leftarrow 0$.

forget Assume that we want to update table T to table T' for the partition F^*, F, I, P, O, O^*, eliminating a vertex v:

- $T'[F^* \setminus \{v\}, F, I, P, O, O^*] \leftarrow -\infty$,
- $T'[F^*, F \setminus \{v\}, I, P, O, O^*] \leftarrow T[F^*, F, I, P, O, O^*]$,
- $T'[F^*, F, I \setminus \{v\}, P, O, O^*] \leftarrow T[F^*, F, I, P, O, O^*]$,
- $T'[F^*, F, I, P \setminus \{v\}, O, O^*] \leftarrow T[F^*, F, I, P, O, O^*]$,
- $T'[F^*, F, I, P, O \setminus \{v\}, O^*] \leftarrow T[F^*, F, I, P, O, O^*]$,
- $T'[F^*, F, I, P, O, O^* \setminus \{v\}] \leftarrow -\infty$.

  Clearly, it is not feasible to eliminate vertices which are not dominated or are supposed to be in F but do not have a private neighbour.

introduce We are now introducing a new vertex v into the bag. The neighbourhood N refers to the situation in the new bag, *i.e.*, to the corresponding induced graph. T' is the new table and T the old one.

- If $v \notin N(I \cup O^* \cup P)$ then set $T'[F^* \cup \{v\}, F, I, P, O, O^*]$ to $\max\{T[F^*, F, I, P, O \setminus R, O^* \cup R] : R \subseteq N(v) \setminus N(F^* \cup F \cup I)\} + 1$.
- If $v \notin N(I \cup O^*)$ and $\{w\} = N(v) \cap P$ set $T'[F^*, F \cup \{v\}, I, P, O, O^*]$ to $\max\{T[F^*, F, I, P \setminus \{w\}, O \setminus R, O^* \cup \{w\} \cup R] : R \subset N(v) \setminus N(I \cup F \cup F^*)\} + 1$.
- If $N(v) \cap (F^* \cup F \cup I \cup P \cup O^*) = \emptyset$ set $T'[F, I \cup \{v\}, P, O, O^*]$ to $\max\{T[F^*, F, I, P, O \setminus R, O^* \cup R] : R \subseteq N(v) \setminus N(I \cup F)\} + 1$.
- If $v \notin N(F^* \cup I)$ and $\{w\} = N(v) \cap F$ set $T'[F^*, F, I, P \cup \{v\}, O, O^*]$ to $T[F^* \cup \{w\}, F \setminus \{w\}, I, P, O, O^*]$.
- If $v \in N(F^* \cup F \cup I)$ set $T'[F^*, F, I, P, O \cup \{v\}, O^*]$ to $T[F^*, F, I, P, O, O^*]$.
- If $v \notin N(F^* \cup F \cup I)$ set $T'[F^*, F, I, P, O, O^* \cup \{v\}]$ to $T[F^*, F, I, P, O, O^*]$.

  For all other cases set the entry for T' to $-\infty$.

The formal induction proof showing the correctness of the algorithm is an easy standard exercise. As to the running time, observe that we cycle only in one case potentially through all subsets of $O \cap N(v)$ for some vertex v, so that the running time follows by applying the binomial formula:

$$\sum_{i=0}^{p} \binom{p}{i} 5^i 2^{p-i} = 7^p . \quad \square$$

The upper bound of $O^*(7^p)$ on the running time of the algorithm described in the proof of Proposition 6 can be improved for graphs of fixed constant maximum degree; in this case, considering all subsets of $O \cap N(v)$ requires only constant effort and hence gives a bound of $O^*(6^p)$ on the running time. With this we can especially conclude the following result with Theorem 5.

Corollary 7. Upper Domination *on subcubic graphs of order n can be solved in time* $O^*(1.348^n)$, *using the same amount of space.*

Since this result will be used later to develop an approximation scheme for planar graphs, we like to point out that and idea similar to the pathwidth algorithm above can be used for treewidth.

Corollary 8. Upper Domination *on graphs of treewidth p can be solved in time* $O^*(10^p)$, *given a corresponding nice tree decomposition.*

Proof. For a given nice tree decomposition use the same partition F^*, F, I, P, O, O^* and initialisation, forget and introduce for table entries just like for Proposition 6. For a join bag, we further have the following rule:

join To create the new table entry $T'[F^*, F, I, P, O, O^*]$ from existing tables T_1 and T_2, consider all partitions $F_1 \cup F_2$ of $F \setminus N(P)$ and $P_1 \cup P_2$ of $P \setminus N(F)$ and $O_1 \cup O_{12} \cup O_2$ of $O \setminus N(F^* \cup F \cup I)$ and choose the partitions for which $v_1 := T_1[F^* \cup F_2, F \setminus F_2, I, P \setminus P_2, O \setminus O_2, O^* \cup P_2 \cup O_2] \neq -\infty$ and $v_2 := T_2[F^* \cup F_1, F \setminus F_1, I, P \setminus P_1, O \setminus O_1, O^* \cup P_1 \cup O_1] \neq -\infty$ such that $v_1 + v_2$ is maximised. For these numbers, set $T'[F^*, F, I, P, O, O^*]$ to $v_1 + v_2 - |F^* \cup F \cup I|$.

From a computational point of view, we consider all partitions of the vertices belonging to a bag into ten sets, namely, into $F_1, F \setminus F_1, F^*, I, P_1, P \setminus P_1, O_1, O_{12}, O_2, O^*$. Observe that $F_1, F \setminus F_1, P_1,$ and $P \setminus P_1$ together describe the partitions $F_1 \cup F_2$ of $F \setminus N(P)$ and $P_1 \cup P_2$ of $P \setminus N(F)$. After first initialising T' to $-\infty$ for all entries, we can then proceed updating the entries, considering the maximum of the current entry and the one that can be computed by looking up the tables T_1 and T_2 as described above.

As the join step is the most expensive one, the claimed running time follows. $\square$

3.3. On Minimal Dominating Set Extension

Algorithms working on combinatorial graph problems often try to look at local parts of the graph and then extend some part of the (final) solution that was found and fixed so far. This type of strategy is at least difficult to implement for Upper Domination, as the following example shows. First, consider a graph G_n that consists of two cliques with vertices $V_n = \{v_1, \ldots, v_n\}$ and $W_n = \{w_1, \ldots, w_n\}$, where the only edges connecting both cliques are (v_i, w_i) for $1 \leq i \leq n$. Observe that G_n has as minimal dominating sets $V_n, W_n,$ and $\{v_i, w_j\}$ for all $1 \leq i, j \leq n$. For $n \geq 3$, the first two are upper dominating sets, while the last n^2 many are minimum dominating sets. If we now add a vertex v_0 to G_n, arriving at graph G'_n, and make v_0 adjacent to all vertices in V_n, then V_n is still a minimal dominating set, but W_n is no longer a dominating set. Now, we have $\{v_i, w_j\}$ for all $0 \leq i \leq n$ and all $1 \leq j \leq n$ as minimum dominating sets. But, if we add one further vertex, w_0 to G'_n to obtain G''_n and make w_0 adjacent to all vertices in W_n, then all upper dominating sets are also minimum dominating sets and vice versa. This shows that we cannot consider vertices one by one, but must rather look at the whole graph. For many maximisation problems, like Upper Irredundance or Maximum Independent Set, it is trivial to obtain a feasible solution that extends a given vertex set by some greedy strategy, or to know that no such extension exists. This is not true for Upper Domination, as we show next. Formally, we want to discuss the following problem:

Minimal Dominating Set Extension
Input: A graph $G = (V, E)$, a set $S \subseteq V$.
Question: Does G have a minimal dominating set S' with $S' \supseteq S$?

Notice that this problem is trivial on some input with $S = \emptyset$. If S is an independent set in G, it is also always possible to extend S to a minimal dominating set, simply by greedily extending it to a maximal independent set. If S however contains two adjacent vertices, we arrive at the problem of fixing at least one private neighbour for these vertices. This problem of preserving irredundance of the vertices in S while extending S to dominate the whole graph turns out to be a quite difficult task.

In [12] it is shown that this kind of extension of partial solutions is NP-hard for the problem of computing prime implicants of the dual of a Boolean function; a problem which can also be seen as the problem of finding a minimal hitting set for the set of prime implicants of the input function. Interpreted in this way, the proof from [12] yields NP-hardness for the minimal extension problem for 3-HITTING SET. The standard reduction from HITTING SET to DOMINATING SET however does not transfer this result to MINIMAL DOMINATING SET EXTENSION; observe that if we represent the hitting-set input-hypergraph $H = (V, F)$ with partial solution $S \subset V$ (w.l.o.g. irredundant) by $G = (V \cup F, E)$ with $E = \{(v, f): v \in V, f \in F, v \in f\} \cup (V \times V)$, the set S can always be extended to a minimal dominating set by simply adding all edge-vertices which are not dominated by S. One can repair this by adjusting this construction to forbid the edge-vertices in minimal solutions in the following way: for each edge-vertex f, add three new a_f, b_f, c_f with edges $(f, a_f), (a_f, b_f), (b_f, c_f)$ and include a_f and b_f in S. This way, f is the only choice for a private neighbour for a_f.

We will show that MINIMAL DOMINATING SET EXTENSION remains hard even for very restricted cases. Our proof is based on a reduction from the NP-complete 4-BOUNDED PLANAR 3-CONNECTED SAT problem (4P3C3SAT for short) [40], the restriction of 3-satisfiability to clauses in C over variables in V, where each variable occurs in at most four clauses and the associated bipartite graph $(C \cup X, \{(c, x) \in C \times X: (x \in c) \vee (\neg x \in c)\})$ is planar.

Theorem 9. MINIMAL DOMINATING SET EXTENSION *is NP-complete, even when restricted to planar cubic graphs.*

Proof. Membership in NP is obvious. NP-hardness can be shown by reduction from 4-BOUNDED PLANAR 3-CONNECTED SAT (4P3C3SAT) [40]: Consider an instance of 4P3C3SAT with clauses $c_1, \ldots, c_m$ and variables $v_1, \ldots, v_n$. By definition, the graph $G = (V, E)$ with $V = \{c_1, \ldots, c_m\} \cup \{v_1, \ldots, v_n\}$ and $E = \{(c_j, v_i): v_i$ or $\bar{v}_i$ is literal of $c_j\}$ is planar. Replace every variable-vertex v_i by six new vertices $f_i^1, x_i^1, t_i^1, t_i^2, x_i^2, f_i^2$ with edges $(f_i^j, x_i^j), (t_i^j, x_i^j)$ for $j = 1, 2$. If v_i (positive) is a literal in more than two clauses, add the edge (f_i^1, f_i^2), else add the edge (t_i^1, t_i^2).

By definition of the problem 4P3C3SAT, each variable appears in at most four clauses and this procedure of replacing the variable-vertices in G by a P_6 preserves planarity. To see this, consider any fixed planar embedding of G and any variable v_i which appears in clauses c_1, c_2, c_3, c_4, in the embedding placed like in the picture below.
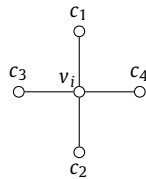

Depending on whether v_i appears negated or non-negated in these clauses, we differentiate between the following cases; in the pictures, vertices plotted in black are the ones to be put into the vertex set S predetermined to be in the minimal dominating set.

$v_i \in c_1, c_2, c_3, \bar{v}_i \in c_4$　　　$v_i \in c_2, c_4, \bar{v}_i \in c_1, c_3$　　　$v_i \in c_1, c_2, \bar{v}_i \in c_3, c_4$
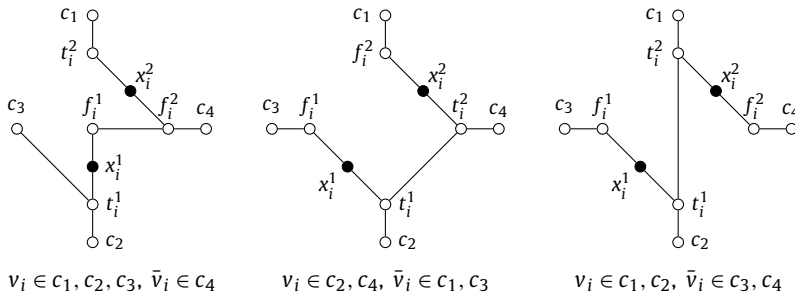
All other cases are rotations of the above three cases and/or invert the roles of v_i and $\bar{v}_i$. Also, if a variable only appears positively (or negatively), it can be deleted along with the clauses which contain it. The maximum degree of the vertices which replace v_i is 3.

Replace each clause-vertex c_j by the subgraph below. The vertices c_j^1, c_j^2 somehow take the role of the old vertex c_j regarding its neighbours: c_j^1 is adjacent to two of the literals of c_j and c_j^2 is adjacent to the remaining literal. This way, all vertices have degree at most 3 and the choices of literals to connect to c_j^1 and c_j^2 can be made such that planarity is preserved.
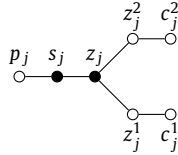
Let G' be the graph obtained from G by the procedure described above. We claim that G' with $S := \{x_i^1, x_i^2 : 1 \leq i \leq n\} \cup \{s_j, z_j : 1 \leq j \leq m\}$ is a "yes"-instance for MINIMAL DOMINATING SET EXTENSION if and only if $c_1 \wedge c_2 \wedge \cdots \wedge c_m$ is a "yes"-instance for 4P3C3SAT.

If the formula $c_1 \wedge c_2 \wedge \cdots \wedge c_m$ is a "yes"-instance for 4P3C3SAT, consider any satisfying assignment ϕ for it and the corresponding vertex-set $W := \{t_i^1, t_i^2 : \phi(v_i) = 1\} \cup \{f_i^1, f_i^2 : \phi(v_i) = 0\}$ in G'. Let W' be an arbitrary inclusion-minimal subset of W such that $\{c_j^1, c_j^2\} \cap N_{G'}(W') \neq \emptyset$ for all $j \in \{1, \ldots, m\}$; W itself has this domination-property since ϕ satisfies the formula $c_1 \wedge c_2 \wedge \cdots \wedge c_m$. By the inclusion-minimality of W', the set $S \cup W'$ is irredundant: Each vertex in W' has at least one of the c_j^k as private neighbour, the vertices x_i^k have either t_i^k or f_i^k as a private neighbour, $pn(s_j, S \cup W') = \{p_j\}$ and $pn(z_j, S \cup W') = \{z_j^1, z_j^2\}$. The set $S \cup W$ might however not dominate all vertices c_j^k. Adding the set $Y := \{z_j^k : c_j^k \notin N_{G'}(W)\}$ to $S \cup W$ creates a dominating set. Since for each clause c_j either $c_j^1 \in N_{G'}(W')$ or $c_j^2 \in N_{G'}(W')$, either z_j^1 or z_j^2 remains in the private neighbourhood of z_j. Other private neighbourhoods are not affected by Y. At last, each vertex $z_j^k \in Y$ has the clause-vertex c_j^k as private neighbour, by the definition of Y, so overall the set $S \cup W' \cup Y$ is a minimal dominating set for G'.

Conversely, if the input (G', S) is a "yes"-instance for MINIMAL DOMINATING SET EXTENSION, the set S can be extended to a set S' which especially dominates all vertices c_j^k and has at least one private neighbour for each z_j. The latter condition implies that $S' \cap \{z_j^k, c_j^k\} = \emptyset$ for $k = 1$ or $k = 2$ for each $j \in \{1, \ldots, m\}$. A vertex c_j^k for which $S' \cap \{z_j^k, c_j^k\} = \emptyset$ has to be dominated by a variable-vertex, which means that $t_i^k \in S'$ ($f_i^k \in S'$) for some variable v_i which appears positively (negatively) in c_j. Minimality of S' requires at least one private neighbour for each x_i^k which, by construction of the variable-gadgets, means that either $\{f_i^1, f_i^2\} \cap S' = \emptyset$ or $\{t_i^1, t_i^2\} \cap S' = \emptyset$, so each variable can only be represented either positively or negatively in S'. Overall, the assignment ϕ with $\phi(v_i) = 1$ if $\{t_i^1, t_i^2\} \cap S' \neq \emptyset$ and $\phi(v_i) = 0$ otherwise satisfies $c_1 \wedge c_2 \wedge \cdots \wedge c_m$.

Finally, G' can be transformed into a cubic planar graph, by adding the subgraph illustrated below once to every vertex v of degree 2, and twice for each vertex of degree 1. For these new subgraphs add the new black vertices to the set S.
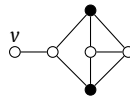
With these alterations, the resulting graph is cubic and all new vertices are dominated and adding another one of them to the dominating set violates irredundance. The original vertex is not dominated, adding it to the dominating set does not violate irredundance within the new vertices and the new vertices can never be private neighbours to any original vertex so the structure of G' in the above argument does not change. $\square$

4. Approximation perspective

We now want to discuss the properties of UPPER DOMINATION with respect to approximability. From this perspective we formally consider the following maximisation problem which we will also denote with UPPER DOMINATION for the sake of simplicity.

UPPER DOMINATION
Instance: A graph $G = (V, E)$.
Feasible Solutions: $S \subseteq V$ s.t. $N[S] = V$ and $pn(v, S) \neq \emptyset$ for all $v \in S$.
Objective: Maximise $|S|$.

A typical pattern that often shows up is that MaxMin versions of classical problems turn out to be much harder than the originals, especially when one considers approximation. For example, MAXIMUM MINIMAL VERTEX COVER does not admit any $n^{\frac{1}{2} - \epsilon}$ approximation, while VERTEX COVER admits a 2-approximation [11]; LAZY BUREAUCRAT is APX-hard while KNAPSACK admits a PTAS [4]; and though MINIMUM MAXIMAL INDEPENDENT SET and INDEPENDENT SET share the same (inapproximable) status, the proof of inapproximability of the MinMax version is considerably simpler, and was known long before the corresponding hardness results for INDEPENDENT SET [30].

We will show that this pattern also holds for Upper Domination: while Dominating Set admits a greedy $\ln n$ approximation, Upper Domination does not admit an $n^{1-\epsilon}$ approximation for any $\epsilon > 0$, unless P = NP. Further, we discuss special graph classes and find that Upper Domination remains APX-hard on cubic graphs but can be approximated within some factor with respect to the maximum degree. Also, despite the hardness of the extension problem, we find that Upper Domination admits a PTAS on k-outerplanar graphs.

4.1. General graphs

We show that Upper Domination is hard to approximate in two steps: first, we show that a related natural problem, Maximum Minimal Hitting Set, is hard to approximate, and then we show that this problem is essentially equivalent to Upper Domination. Formally we consider the problem:

Maximum Minimal Hitting Set
Instance: A hypergraph $G = (V, F)$, $F \subseteq 2^V$.
Feasible Solutions: $H \subseteq V$ s.t. $e \cap H \neq \emptyset$ for all $e \in H$ (hitting set) and $\{e \in F : e \cap H = \{v\}\} \neq \emptyset$ for all $v \in H$ (minimality).
Objective: Maximise $|H|$.

This problem generalises Upper Domination: given a graph $G = (V, E)$, we can produce a hypergraph by keeping the same set of vertices and creating a hyperedge for each closed neighbourhood $N[v]$ of G. An upper dominating set of the original graph is now exactly a minimal hitting set of the constructed hypergraph. We will also show that Maximum Minimal Hitting Set can be reduced to Upper Domination.

Let us note that Maximum Minimal Hitting Set, as defined here, also generalises Maximum Minimal Vertex Cover. We recall that for this problem there exists a $n^{1/2}$-approximation algorithm, while it is known to be $n^{1/2-\varepsilon}$-inapproximable [11]. Here, we generalise this result for arbitrary $d \geq 2$ to d-uniform hypergraphs, i.e., hypergraphs $G = (V, F)$ with $|e| = d$ for all $e \in F$.

Theorem 10. *For all $\varepsilon > 0$, $d \geq 2$, there is no $n^{\frac{d-1}{d} - \varepsilon}$-approximation for Maximum Minimal Hitting Set on d-uniform hypergraphs of order n in polynomial time, unless $P = NP$. This statement still holds for the restriction to hypergraphs with $O(n)$ hyperedges.*

Proof. Fix some constant hyperedge size $d \geq 2$. We will present a reduction from Maximum Independent Set, which is known to be inapproximable [34]. Specifically, for all $\varepsilon > 0$, it is known to be NP-hard to distinguish for an n-vertex graph G if $\alpha(G) > n^{1-\varepsilon}$ or $\alpha(G) < n^\varepsilon$.

Take an instance $G = (V, E)$ of Maximum Independent Set. If $d > 2$ we begin by turning G into a d-uniform hypergraph $G' = (V, F)$ such that any (non-trivial) hitting set of G' is a vertex cover of G and vice-versa (for $d = 2$ we simply set $G' = G$). We proceed as follows: for every edge $e \in E$ and every $S \subseteq V \setminus e$ with $|S| = d - 2$ we add to F the hyperedge $e \cup S$ (with size exactly d). Thus, $|F| = O(n^d)$. Any vertex cover of G is also a hitting set of G'. For the converse, we only want to prove that any hitting set of G' of size at most $n - d$ is also a vertex cover of G (this is without loss of generality, since d is a constant, so we will assume $\alpha(G) > d$). Take a hitting set H of G' with at most $n - d$ vertices; take any edge $e \in E$ and a set S with $S \subseteq V \setminus (H \cup e)$ and $|S| = d - 2$ (such a set S exists since $|V \setminus H| \geq d$). Now, $(e \cup S) \in F$, therefore H must contain a vertex of e. We thus conclude that the maximum size of $V \setminus H$, where H is a hitting set of G' is either at least $n^{1-\varepsilon}$ or at most n^ε, that is, the maximum size of $V \setminus H$ is $\alpha(G)$.

We now add some vertices and hyperedges to G' to obtain a hypergraph G''. For every set $S \subseteq V$ such that $|S| = d - 1$ and $V \setminus S$ is a hitting set of G', we add to G'' n new vertices, call them $u_{S,i}$, $1 \leq i \leq n$. Also, for each such vertex $u_{S,i}$ we add to G'' the hyperedge $S \cup \{u_{S,i}\}$, $1 \leq i \leq n$. This completes the construction. It is not hard to see that G'' has hyperedges of size exactly d, and its vertex and hyperedge set are both of size $O(n^d)$.

Let us analyse the approximability gap of this reduction. First, suppose that there is a minimal hitting set H of G' with $|V \setminus H| > n^{1-\varepsilon}$. Then, there exists a minimal hitting set of G'' with size at least $n^{d-O(d\varepsilon)}$. To see this, consider the set $H \cup \{u_{S,i} : S \subseteq V \setminus H, 1 \leq i \leq n\}$. This set is a hitting set, since H hits all the hyperedges of G', and for every new hyperedge of G'' that is not covered by H we select $u_{S,i}$. It is also minimal, because H is a minimal hitting set of G', and each $u_{S,i}$ selected has a private hyperedge. To calculate its size, observe that for each $S \subseteq V \setminus H$ with $|S| = d - 1$ we have n vertices. There are at least $\binom{n^{1-\varepsilon}}{d-1}$ such sets.

For the converse direction, we want to show that if $|V \setminus H| < n^\varepsilon$ for all minimal hitting sets H of G', then any minimal hitting set of G'' has size at most $n^{1+O(d\varepsilon)}$. Consider a hitting set H' of G''. Then, $H' \cap V$ is obviously a hitting set of G'. Let $S \subset V$ be a set of vertices such that $S \cap H' \neq \emptyset$. Then $u_{S,i} \notin H'$ for all i, because the (unique) hyperedge that contains $u_{S,i}$ also contains some other vertex of H', contradicting minimality. Now, because $V \cap H'$ is a hitting set of G' we have $|V \setminus H'| \leq n^\varepsilon$. Thus, the maximum number of different sets $S \subseteq V$ such that some $u_{S,i} \in H'$ is $\binom{n^\varepsilon}{d-1}$ and the total size of H' is at most $|H' \cap V| + n^{\varepsilon(d-1)+1} \leq n^{1+O(d\varepsilon)}$. $\square$

Corollary 11. *For any $\varepsilon > 0$ there is no polynomial $n^{1-\varepsilon}$-approximation for* Maximum Minimal Hitting Set *on hypergraphs of order n, unless P = NP. This statement still holds restricted to hypergraphs with $O(n)$ hyperedges.*

Proof. Assume there were, for some $\varepsilon > 0$, a polynomial approximation algorithm A with ratio $n^{1-\varepsilon}$ for Maximum Minimal Hitting Set. Then, choose d such that $1/d \le \varepsilon/2$ and hence $(d-1)/d \ge 1 - \varepsilon/2$. Then, A would be a polynomial $n^{(d-1)/d - \varepsilon/2}$-approximation algorithm for Maximum Minimal Hitting Set restricted to d-uniform hypergraphs, contradicting Theorem 10. □

A graph is called co-bipartite if its complement is bipartite. Using Corollary 11 and the reduction of [38] from Minimum Hitting Set to Minimum Dominating Set, the following holds.

Theorem 12. *For any $\varepsilon > 0$* Upper Domination *restricted to co-bipartite graphs of order n, is not $n^{1-\varepsilon}$-approximable in polynomial time, unless P = NP.*

Proof. We construct an approximation-preserving reduction from Maximum Minimal Hitting Set. Given a hypergraph $G = (V, F)$ as an instance of Maximum Minimal Hitting Set, we define a graph $G' = (V', E')$ as an instance of Upper Domination as follows: V' contains a vertex v_i associated to any vertex i from V, a vertex u_e for any edge $e \in F$ and a new vertex v. The set E' contains edges such that $G'[V]$ and $G'[F]$ are cliques. Moreover, v is adjacent to every vertex $v_i \in V$, and $(v_i, u_e) \in E'$ if and only if $i \in e$ in G.

First we show that given a solution S that is a minimal hitting set in G, S is also a minimal dominating set in G'. Indeed if S is a hitting set in G then S is a dominating set in G'. If S is minimal, that is, any proper subset $S' \subset S$ is no longer a hitting set, then it is also the case that S' is no longer a dominating set in G'. That implies that $\text{opt}(G') \ge \text{opt}(G)$.

Consider now an upper dominating set S for G'. To dominate the vertex v, S has to contain at least one vertex $w \in V \cup \{v\}$. If S contains one vertex $u_e \in E$, then the set $\{w, u_e\}$ is already dominating. If we want a solution of cardinality more than 2, then $S \subseteq V$. If $S \subseteq V$ is a minimal dominating set in G', S is also a minimal hitting set in G since S covers all hyperedges in G if and only if it dominates all edge-vertices in G'. So starting with any minimal dominating set S of G' of cardinality larger than 2, S is also a minimal hitting set of G. The result now follows from Corollary 11. □

Note that, in fact, the inapproximability bound given in Theorem 10 is tight, for every fixed d, a fact that we believe may be of independent interest. This is shown in the following theorem, which also generalises results on Maximum Minimal Vertex Cover [11]. To simplify presentation, we assume that we are given a hypergraph without *isolates*, meaning that every vertex appears in at least one hyperedge, and also such that there are no two hyperedges $h, h' \in F$ such that $h \subset h'$. Observe that in this case h' can be deleted without affecting the computation of a minimal hitting set, we will therefore call such a hyperedge *redundant*.

Theorem 13. *For any $d \ge 1$ and hypergraph of order n with maximum edge-size d, without isolates or redundant hyperedges, a minimal hitting set of size $\Omega(n^{1/d})$ can be computed in polynomial-time. This shows an $O(n^{\frac{d-1}{d}})$-approximation for* Maximum Minimal Hitting Set *on hypergraphs with maximum edge-size d.*

Proof. We will in fact prove a slightly stronger statement. For a vertex u of a hypergraph $G = (V, F)$ without isolates, we define its *rank* as the size of the smallest hyperedge that contains it, formally $\min\{|e|: e \in F, u \in e\}$. Clearly, in any hypergraph where all hyperedges have size at most d, all vertices also have rank at most d. We will establish that in any hypergraph $G = (V, F)$, without isolates, where all vertices have rank at most d and no hyperedge is redundant, we can construct in polynomial time a minimal hitting set of size $\Omega(n^{1/d})$. The statement of the theorem will then follow.

The proof is by induction on d. For $d = 1$, if all vertices have rank at most d, all vertices belong in an edge of size 1, and since the graph contains no isolates, the only feasible hitting set is V. This yields a solution for Maximum Minimal Hitting Set of size $\Omega(n)$.

If all vertices have rank at most d, for some $d > 1$, we do the following: first, greedily construct a maximal set $M \subseteq F$ of pair-wise disjoint hyperedges. If $|M| \ge n^{1/d}$ then we know that any hitting set of G must contain at least $n^{1/d}$ vertices. So, we simply produce an arbitrary feasible solution by starting with V and deleting redundant vertices until our hitting set becomes minimal.

Suppose then that $|M| < n^{1/d}$. Let H be the set of all vertices contained in M, so $|H| < d|M| \in O(n^{1/d})$. Clearly, H is a hitting set of G (otherwise M is not maximal), but it is not necessarily minimal. Let us now consider all sets $S \subseteq H$ with the following two properties: $|S| \le d - 1$ and all edges $e \in F$ have an element in $V \setminus S$ (in other words, $V \setminus S$ is a hitting set of G). For such a set S and a vertex $u \in V \setminus H$ we will say that u *is seen by S*, and write $u \in B(S)$, if there exists $e \in F$ such that $e \cap H = S$, $u \in e$ and for all $e' \in F$ such that $u \in e'$ we have $|e'| \ge |e|$.

Intuitively, what we are trying to do here is finding a set S that will *not* be included in our hitting set. Vertices seen by S are then vertices which are more likely to be contained in a maximum minimal hitting set. We observe here that all vertices of $V \setminus H$ are seen by some set S with the above properties. To see this, let $u \in V \setminus H$ and consider an edge e of minimal size that contains u. Then the set $S_u := e \cap H$ has size at most $d - 1$ (since it does not contain u). Furthermore $V \setminus S_u$ is a

hitting set, since otherwise there would exist a hyperedge $e' \in F$ with $e' \subseteq S_u$, which would give $e' \subset e$, a contradiction to the assumption that no hyperedge is redundant. Hence, S_u is one of the sets that will be considered, and $u \in B(S_u)$.

Let B_i be the union of all $B(S)$ for the sets S with $|S| = i$. As argued above, all vertices of $V \setminus H$ are seen by at least one set S, and therefore belong to some B_i. Therefore, the union of all B_i has size at least $|V \setminus H| \geq n - O(n^{1/d}) = \Omega(n)$. The largest of these sets, then, has size at least $\frac{|V \setminus H|}{d} = \Omega(n)$. Consider then the largest such set and let s be its index. By definition, this set B_s is build by the union of sets $B(S)$ with $|S| = s$ and there are at most $\binom{|H|}{s} = O(n^{s/d})$ different sets S of cardinality s. Since all together they see $\Omega(n)$ vertices of $V \setminus H$, one of them must see at least $\Omega(n^{1-\frac{s}{d}})$ vertices. Call this set S_m.

Consider now the following hypergraph: we start with the hypergraph induced by $S_m \cup B(S_m)$ and delete the vertices of S_m from every hyperedge; then we remove all redundant hyperedges. Call the resulting hypergraph G'. We can see that in G' every vertex has rank at most $d - s$, because for all $u \in B(S_m)$, there exists a smallest hyperedge containing u which also contains all vertices in S_m. Furthermore, if we consider such a smallest hyperedge e incident to $u \in B(S_m)$ that contains all of S_m, we see that e does not contain any other hyperedge in the new graph, hence the new hypergraph has no isolates. We can therefore proceed by induction. By induction hypothesis, we can in polynomial time find a minimal hitting set of G' with at least $\Omega((n^{1-\frac{s}{d}})^{\frac{1}{d-s}}) = \Omega(n^{1/d})$ vertices. Call this set H'.

We will now build our solution. Start with the set $V \setminus (S_m \cup B(S_m))$ and add to it the vertices of H'. First, this is a hitting set, because any hyperedge not hit by $V \setminus (S_m \cup B(S_m))$ is induced by $S_m \cup B(S_m)$, and H' hits all such hyperedges. We now proceed to make this set minimal by arbitrarily deleting redundant vertices. The crucial point here is that no vertex of H' is deleted, since this would contradict the minimality of H' as a hitting set of the hypergraph induced by $S_m \cup B(S_m)$. Thus, the resulting solution has size $\Omega(n^{1/d})$.

Finally, we note that we can obtain the approximation ratio in $O(n^{\frac{d-1}{d}})$ on any hypergraph with maximum edge-size d, by first removing all isolates and all redundant hyperedges (the ones that contain), and then using the arguments above. □

4.2. Bounded-degree graphs

Unlike the general case, Upper Domination admits a simple constant factor approximation when restricted to graphs of maximum degree Δ. This follows from the fact that any dominating set in such a graph has size at least $\frac{n}{\Delta+1}$. We will see that this factor can be improved however not arbitrarily as the following result shows.

Corollary 14. Upper Domination *is APX-hard on cubic graphs.*

Proof. We use the same construction as in the proof of Theorem 4, this time from Maximum Independent Set restricted to cubic graphs which remains APX-hard [42]. The input graph G is cubic which means that $\alpha(G) \in \Theta(|V|)$ and $|E| \in \Theta(|V|)$. The reduction in the proof of Theorem 4 preserves APX-hardness, since it creates a graph G'' which has an upper dominating set of cardinality $k + 11|E| \in \Theta(k)$ if G has an independent set of cardinality k. □

On the positive side, we can make use of approximation algorithms for Maximum Independent Set to derive approximations for Upper Domination.

Theorem 15. *Consider some graph-class $\mathcal{G}(p, \rho)$ with the following properties:*

- *Every $G \in \mathcal{G}(p, \rho)$ can be properly coloured with p colours in polynomial time.*
- *For any $G \in \mathcal{G}(p, \rho)$, Maximum Independent Set is ρ-approximable in polynomial time.*

Then, for every $G \in \mathcal{G}(p, \rho)$, Upper Domination is approximable in polynomial time within ratio at most

$$\max \left\{ \rho, \frac{\Delta \rho p + \Delta - 1}{2\rho\Delta} \right\}. \tag{3}$$

Proof. The approximation algorithm consists of running two independent set algorithms, by greedily augmenting solutions computed in order to become maximal for inclusion and of returning the best among them, denoted by U. Recall that any maximal independent set is a feasible upper dominating set.

The algorithms used are:

(i) the ρ-approximation algorithm for Maximum Independent Set assumed for $\mathcal{G}(p, \rho)$ and
(ii) the (also assumed) algorithm that colours the vertices of the input graph with p colours and takes the largest colour-class as solution.

Recall that $\Gamma(G) \leq \max \left\{ \alpha(G), \frac{n}{2} + \frac{\alpha(G)}{2} - 1 \right\}$ by Lemma 2. If the maximum on the right is realised by the first term $\alpha(G)$, then we are done since the ρ-approximation for Maximum Independent Set-algorithm, also achieves ratio ρ for Upper Domination.

Suppose now that $\alpha(G) < \frac{n}{2} + \frac{\alpha(G)}{2} - 1$ and set $\alpha(G) = n/t$, for some $t \geq 1$ that will be fixed later. In order to simplify calculations we will use the following bounds for $\Gamma(G)$, easily derived from Lemma 3:

$$\Gamma(G) \leq \frac{n}{2} + \frac{\alpha(G)(\Delta - 1)}{2\Delta} = \frac{\Delta(t+1) - 1}{2t\Delta}n = \frac{\Delta(t+1) - 1}{2\Delta}\alpha(G). \tag{4}$$

Expression (4) yields:

$$\alpha(G) \geq \frac{2\Delta}{\Delta(t+1) - 1}\Gamma(G). \tag{5}$$

If the solution U returned by the algorithm is the maximal independent set computed by approximation algorithm (i), Equation (5) yields:

$$|U| \geq \rho\alpha(G) \geq \frac{2\rho\Delta}{\Delta(t+1) - 1}\Gamma(G),$$

$$\frac{\Gamma(G)}{|U|} \leq \frac{\Delta(t+1) - 1}{2\rho\Delta}. \tag{6}$$

Assume now that the solution U is the one computed by approximation algorithm (ii) and note that the largest colour computed is assigned to at least n/p vertices of the input graph which, obviously, form an independent set. So, in this case, $|U| \geq n/p$ and using (4), the ratio achieved is:

$$\frac{\Gamma(G)}{|U|} \leq \frac{\frac{\Delta(t+1)-1}{2t\Delta}n}{\frac{n}{p}} = \frac{p(\Delta(t+1) - 1)}{2t\Delta}. \tag{7}$$

Equality of ratios in (6) and (7) implies $t = \rho p$ and setting it to either one of those leads to the second term of the max expression in (3). □

Any connected graph of maximum degree Δ, except a complete graph or an odd cycle, can be coloured with at most Δ colours [41]; furthermore, the class $\mathcal{G}(\Delta, (\Delta + 3)/5)$ contains all graphs of maximum degree Δ, as for these graphs MAXI-MUM INDEPENDENT SET is approximable within ratio $(\Delta + 3)/5$.

Proposition 16. UPPER DOMINATION *is approximable in polynomial time within a ratio of* $(6\Delta^2 + 2\Delta - 3)/10\Delta$ *in graphs of maximum degree* Δ.

Theorem 15 can be improved for regular graphs where $\Gamma(G) \leq \frac{n}{2}$ [33].

Proposition 17. UPPER DOMINATION *in regular graphs is approximable in polynomial time within ratio* $\Delta/2$.

4.3. Planar graphs

In this section we present an EPTAS (a PTAS with running time $f(\frac{1}{\epsilon}) \cdot poly(|I|)$) for UPPER DOMINATION on planar graphs. We use techniques based on the ideas of Baker [5]. As we shall see, some complications arise in applying these techniques, because of the hardness of extending solutions to this problem.

We use the notion of outerplanar graphs. An *outerplanar* (or 1-outerplanar) graph G is a graph such that there is a planar embedding of G, where all vertices are incident to the outer face of G. For $k > 1$, graph G is a k-outerplanar graph if there is a planar embedding of G, such that when all vertices, incident to the outer face are removed, G is a $(k - 1)$-outerplanar graph. Removing stepwise the vertices that are incident to the outer face, the vertices of G can be partitioned into levels $L_1, \ldots, L_k$. We write $|L_i|$ for the number of vertices in level L_i (if $i < 1$ or $i > k$ we write $|L_i| = 0$). Bodlaender [10] proved that every k-outerplanar graph has treewidth of at most $3k - 1$. Together with Corollary 8, this implies the following:

Proposition 18. *An upper dominating set of a k-outerplanar graph G can be computed in time $f(k)n$.*

To obtain the EPTAS, we use that every planar graph is k-outerplanar for some k. By removing some of the levels L_i we split the graph G into several ℓ-outerplanar subgraphs G_i of some small $\ell < k$. A maximum minimal dominating set of size $\Gamma(G_i)$ can be computed using the corollary from above. Finally, the partial solutions of G_i are merged to obtain a minimal dominating set for G. In the following theorem, we analyse how the upper domination number of the subgraphs G_i correlates to $\Gamma(G)$.

Theorem 19. *Let $G = (V, E)$ be a k-outerplanar graph with levels $L_1, \ldots, L_k \subseteq V$. For some $i \leq k$, let G_1 be the subgraph which is induced by levels $L_1, \ldots, L_{i-1}$ and let G_2 be the subgraph induced by levels $L_{i+1}, \ldots, L_k$. Then,*

$$\Gamma(G_1) + \Gamma(G_2) \geq \Gamma(G) - \sum_{j=i-3}^{i+3} |L_j|.$$

Proof. Consider a given maximum minimal dominating set D of G with associated vertex set partition (F, I, P, O) where $\Gamma(G) = |F| + |I|$. We construct minimal dominating sets for G_1 and G_2 from (F, I, P, O) as follows. We perform the following operations on a graph $G' = (V', E')$ with dominating set D' with associated partition (F', I', P', O') to remove the vertices from L_i, where initially $G' = G$ and $D' = D$. After each step, we obtain a minimal dominating set for the graph induced by vertices $V' \setminus \{v\}$, $v \in L_i$, until $V' \cap L_i = \emptyset$ and we obtain minimal dominating sets for G_1 and G_2 by choosing the corresponding subsets.

Case 1: $v \in O'$. In this case v can be removed from G without violating any constraint of (F, I, P, O).

Case 2: $v \in P'$. Consider vertex $u \in F$ such that $v \in pn(u, D')$. Since this especially implies $u \in N(v)$ we know that $u \in L_j$ for some $i - 1 \leq j \leq i + 1$. If $pn(u, D') = \{v\}$, we add u to O' and remove v. Otherwise, we select a vertex $v' \in pn(u, D') \setminus \{v\}$ and add v' to P' and remove v.

Case 3: $v \in I' \cup F'$. First, in case $v \in F$ and after deleting v there exists a vertex $u \in F'$ such that $N(u) \cap F' = \emptyset$, we move u to I' and its private neighbour to O'.

Consider the private neighbourhood $W = pn(v, D')$. Then $N[W] \subseteq P' \cup O'$. Our goal is to extend D' to dominate W. We select a maximal independent set I_W of the subgraph of G' induced by W (which is also a minimal dominating set of the same subgraph) and we add I_W to I'. Dominating W this way might lead to conflicts as a vertex $w \in W$ that is added to the dominating set D' might be adjacent to some vertex $w' \in P'$. We solve those conflicts in the same way as in Case 2.

Note that since all vertices in W belong to levels L_{i-1}, L_i, L_{i+1}, w' belongs to levels $L_{i-2}, \ldots, L_{i+2}$ and hence $pn(w')$ belongs to levels $L_{i-3}, \ldots, L_{i+3}$.

Using the above construction yields a minimal dominating set for the graph induced by vertices $V \setminus \{L_i\}$. As vertices are moved from I or F to O, the dominating set for the resulting graph may be reduced. In each case we only modify the dominating set of vertices in $L_{i-3}, \ldots, L_{i+3}$ and hence the dominating set is reduced by at most $\sum_{j=i-3}^{i+3} |L_j|$. □

Using the above theorem iteratively for several levels $L_{i_1}, \ldots, L_{i_s-1}$ yields the following corollary.

Corollary 20. *Let $G = (V, E)$ be a k-outerplanar graph with levels $L_1, \ldots, L_k \subseteq V$. For indices $0 = i_0 < i_1 < \ldots \leq i_s = k$, let G_j be the subgraph which is induced by levels $L_{i_j}, \ldots, L_{i_{j+1}}$. Then,*

$$\sum_{j=0}^{s-1} \Gamma(G_j) \geq \Gamma(G) - \sum_{k=0}^{s} \sum_{j=i_k-3}^{i_k+3} |L_j|.$$

The following algorithm shows how partial solutions of subgraphs can be used to obtain a minimal dominating set for the whole graph G.

Algorithm 1: GreedyUD(G_1, D_1, G_2, D_2, L_i).

input : *Subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of $G = (V, E)$ separated by level L_i such that $V_1 \cup L_i \cup V_2 = V$ and minimal dominating sets D_1 and D_2 of G_1 and G_2 respectively.*

output: *A minimal dominating set for G.*

$D = D_1 \cup D_2$

repeat

    choose $v \in L_i \setminus N[D]$

    $D = D \cup \{v\}$

    Remove vertices in $N[N[v]]$ from D until D is inclusion minimal

until $L_i \subseteq N[D]$

return D

Theorem 21. *Let $G = (V, E)$ be a k-outerplanar graph with levels $L_1, \ldots, L_k \subseteq V$. For some $i \leq k$, let G_1 be the subgraph which is induced by levels $L_1, \ldots, L_{i-1}$ and let G_2 be the subgraph induced by levels $L_{i+1}, \ldots, L_k$. Let S_1 and S_2 be a minimal dominating set of G_1 and G_2, respectively. Then Algorithm 1 returns a minimal dominating set S with*

$$|S| \geq |S_1| + |S_2| - |L_{i-1}| - |L_{i+1}|.$$

Proof. By definition of the algorithm, vertices $v \in L_i$ that are not covered by vertices in L_{i-1} or L_{i+1} are added to the dominating set. This may lead to conflicts that are resolved as in the proof of the Theorem 19.

Since we remove for every vertex $w \in N[v]$ in L_{i-1} or L_{i+1} at most one vertex from the dominating set, at most $|L_{i-1}| + |L_{i+1}|$ vertices are removed from the dominating set by Algorithm 1. $\square$

We now state our EPTAS for planar UPPER DOMINATION.

Algorithm 2: `ComputeUD(G, k, ε).`

input : *A k-outerplanar graph $G = (V, E)$ for some $k \in \mathbb{N}$ with layers $L_1, \ldots, L_k$ and parameter ϵ.*
output: *A minimal dominating set for G.*

$\mu = \lceil \frac{36}{\epsilon} \rceil$

$x = \text{argmin}\{\sum_{j \in \mathbb{N}}((\sum_{i=-3}^{3}|L_{j\mu+x+i}|) + |L_{j\mu+x-1}| + |L_{j\mu+x+1}|) : x < \mu\}$

for $0 \leq i \leq \lceil \frac{k}{\mu} \rceil$ **do**
$\quad G_i = G[L_{(i-1)\mu+x+1} \cup \ldots \cup L_{i\mu+x-1}]$
$\quad$ Compute an upper dominating set D_i for G_i with Proposition 18. $H_i = G[L_1 \cup \ldots \cup L_{i\mu+x-1}]$

/* note that L_i with $i < 1$ or $i > k$ are empty sets   */

$\bar{D}_0 = D_0$
for $0 \leq i \leq \lceil \frac{k}{\mu} \rceil$ **do**
$\quad \bar{D}_{i+1} = \text{GreedyUD}(H_i, \bar{D}_i, G_{i+1}, D_{i+1}, L_{i\mu+x})$
return $(\bar{D}_{\lceil \frac{k}{\mu} \rceil})$

Theorem 22. *Algorithm 2 returns a minimal dominating set S of cardinality at least $(1 - \epsilon)\Gamma(G)$ in time bounded by $f(\frac{1}{\epsilon})n + O(n^2)$.*

Proof. Claim 1.

$$\sum_{j \in \mathbb{N}}\left(\left(\sum_{i=-3}^{3}|L_{j\mu+x+i}|\right) + |L_{j\mu+x-1}| + |L_{j\mu+x+1}|\right) = \frac{9|V|}{\mu}.$$

Proof of Claim 1. The following term equals $9|V|$ as every level of G is counted exactly 9 times by the inner sum:

$$\sum_{x=0}^{\mu-1}\sum_{j \in \mathbb{N}}\left(\left(\sum_{i=-3}^{3}|L_{j\mu+x+i}|\right) + |L_{j\mu+x-1}| + |L_{j\mu+x+1}|\right) = 9|V|.$$

Since x is chosen minimally over all $0, \ldots, \mu - 1$ we obtain Claim 1 by the pigeonhole principle.

Claim 2.

$$\Gamma(G) \geq \frac{|V|}{4}.$$

Proof of Claim 2. Since every planar graph is 4-colourable, there exists an independent set of size $\alpha(G) \geq \frac{|V|}{4}$ (by choosing the set of vertices with the most frequent appearing colour). Using $\alpha(G) \leq \Gamma(G)$ we obtain Claim 2.

Proof of the main theorem. The minimal dominating sets D_i for the subgraphs G_i have cardinality $\Gamma(G_i)$. Then Algorithm 1 is used to obtain a minimal dominating set for G. By Theorem 21, the algorithm returns a solution of value:

$$\sum_{j=0}^{\lfloor \frac{k}{\mu} \rfloor + 1} \Gamma(G_j) - \sum_{j=0}^{\lfloor \frac{k}{\mu} \rfloor}(|L_{j\mu+x-1}| + |L_{j\mu+x+1}|)$$

$$\geq \sum_{j=0}^{\lfloor \frac{k}{\mu} \rfloor + 1} \Gamma(G_j) - \sum_{j \in \mathbb{N}}(|L_{j\mu+x-1}| + |L_{j\mu+x+1}|)$$

$$\geq \Gamma(G) - \sum_{j \in \mathbb{N}}((\sum_{i=-3}^{3}|L_{j\mu+x+i}|) + |L_{j\mu+x-1}| + |L_{j\mu+x+1}|) \qquad \text{(Cor. 20)}$$

$$\geq \Gamma(G) - \frac{9|V|}{\mu} \qquad\qquad\qquad\qquad\qquad\qquad \text{(Claim 1)}$$

$$\geq \Gamma(G) - \frac{9 \cdot 4\Gamma(G)}{\mu} \qquad\qquad\qquad \text{(Claim 2)}$$

$$= \Gamma(G)(1 - \epsilon). \quad \square$$

5. Fixed parameter tractability

In this section we will investigate the fixed parameter tractability of UPPER DOMINATION as a parameterised problem. We mainly refer to a recent textbook [23] in the area of parameterised complexity. Important notions that we will make use of include the parameterised complexity classes FPT, W[1] and W[2], parameterised reductions and kernelisation. We discuss the following dual pair of parameterised problems:

UPPER DOMINATION **Instance:** A graph $G = (V, E)$. **Parameter:** $k \in \mathbb{N}$. **Question:** Is $\Gamma(G) \geq k$?	CO-UPPER DOMINATION **Instance:** A graph $G = (V, E)$. **Parameter:** $\ell \in \mathbb{N}$. **Question:** Is $\Gamma(G) \geq |V| - \ell$?

As we will only consider this natural parameterisation, we refrain from explicitly mentioning the parameter and again reuse the name UPPER DOMINATION to refer to the parameterised problem throughout this section. Notice that CO-UPPER DOMINATION could also be addressed as MINIMUM MAXIMAL NONBLOCKER or as MINIMUM MAXIMAL STAR FOREST; see [3] for further discussion.

5.1. General graphs

The problems MINIMUM DOMINATION, MINIMUM INDEPENDENT DOMINATION and MAXIMUM INDEPENDENT SET were among the first problems conjectured not to be in FPT [22]. In fact, aside from UPPER DOMINATION, all other problems from the domination chain (see [32]) are now known to be complete for either W[1] or W[2] (see [9] and [24] for UPPER and LOWER IRREDUNDANCE respectively). It is perhaps not very surprising that UPPER DOMINATION is also unlikely to belong to FPT, and it looks rather unexpected that this question has been open for such a long time. We show that UPPER DOMINATION is W[1]-hard by a reduction from MULTICOLOURED CLIQUE, a problem introduced in [26,43] to facilitate W[1]-hardness proofs:

MULTICOLOURED CLIQUE **Input:** A graph $G = (V, E)$ with k colour-classes $V = V_1 \cup V_2 \cup \cdots \cup V_k$. **Parameter:** $k \in \mathbb{N}$. **Question:** Is there a $C \subseteq V$ s.t. $(v, w) \in E$ for all $v, w \in C$ and $|V_i \cap C| = 1$ for all $i \in \{1, \ldots, k\}$?

For this problem, one can assume that each set V_i is an independent set in G, since edges between vertices of the same colour-class have no impact on the existence of a solution. MULTICOLOURED CLIQUE is known to be W[1]-complete. While the construction used in our reduction itself is not very complicated, proving its correctness turns out to be quite complex and technical.

Theorem 23. UPPER DOMINATION *is W[1]-hard.*

Proof. Let $G = (V, E)$ be a graph with k different colour-classes given by $V = V_1 \cup V_2 \cup \cdots \cup V_k$. We construct a graph G' which has an upper dominating set of cardinality (at least) $k + \frac{1}{2}(k^2 - k)$ if and only if G is a "yes"-instance for MULTI-COLOURED CLIQUE which proves W[1]-hardness for UPPER DOMINATION, parameterised by $\Gamma(G')$.

Consider $G' = (V', E')$ given by: $V' := V \cup \{v_e : e \in E\}$ and

$$E' := \bigcup_{i=1}^{k} V_i \times V_i \ \cup \ \bigcup_{i=1}^{k} \bigcup_{j=1}^{k} \left\{ (v_e, v_{e'}) : e, e' \in (V_i \times V_j) \cap E \right\}$$

$$\cup \bigcup_{i=1}^{k} \bigcup_{j=1}^{k} \left\{ (v_{(u,w)}, x) : (u, w) \in (V_i \times V_j) \cap E, x \in \left((V_i \cup V_j) \setminus \{u, w\} \right) \right\}.$$

If $C \subset V$ is a (multi-coloured) clique of cardinality k in G, the set $S' := C \cup \{v_{(u,v)} : u, v \in C\}$ is an upper dominating set for G' of cardinality $k + \frac{1}{2}(k^2 - k)$: First of all, $\{v_{(u,v)} : u, v \in C\} \subset V'$ since $(u, v) \in E$ for all $u, v \in C$. Further, by definition of the edges E', $u, v \notin N_{G'}(v_{(u,v)})$ and $u \notin N_{G'}(v)$ for u and v from different colour classes so S' is an independent set in

G' and hence a minimal dominating set. It can be easily verified that S' is also dominating for G'; observe that it contains exactly one vertex for each clique in the graph.

Suppose S is a minimal dominating set for G'. Consider the partition $S = \left(\bigcup_{i=1}^{k} S_i \right) \cup \left(\bigcup_{1 \le i < j \le k} S_{\{i,j\}} \right)$ defined by: $S_i := S \cap V_i$ for $i \in \{1, \ldots, k\}$ and $S_{\{i,j\}} := S \cap \{v_e : e \in V_i \times V_j\}$ for all $1 \le i < j \le k$. The minimality of S gives the following properties for these subsets:

1. If $|S_i| > 1$ for some index $i \in \{1, \ldots, k\}$, minimality implies $|S_i| = 2$ and for all $j \ne i$ either $S_{\{i,j\}} = \emptyset$ or $S_j = \emptyset$:
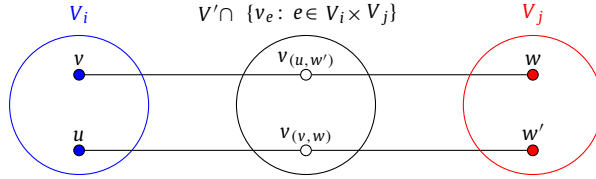
Since for every $u \in V_i$ and every j, $j \ne i$, by construction $V_i \subset N[u]$, and if there is more than one vertex in S_i, then their private neighbours have to be in $\{v_e : e \in E\}$. A vertex v_e with $e \in V_i \times V_j$ is not adjacent to a vertex $u \in V_i$ if and only if $e = (u, w)$ for some $w \in V_j$. For two different vertices $u, v \in V_i$ all v_e with $e \in V_i \times V_j$ are adjacent to either u or v, a third vertex $w \in V_i$ consequently can not have any private neighbour. This also means that any vertex $v_e \in S_{\{i,j\}}$ has to have a private neighbour in V_j, so if $S_{\{i,j\}} \ne \emptyset$ the set S_j has to be empty because one vertex from S_j dominates all vertices in V_j. These observations hold for all $j \ne i$.

2. If $|S_{\{i,j\}}| > 1$ for some indices $i, j \in \{1, \ldots, k\}$ we find that $|S_{\{i,j\}}| = 2$, $|S_i|, |S_j| \le 1$ and that $S_i \ne \emptyset$ implies $S_j = S_{\{j,l\}} = \emptyset$ for all $l \in \{1, \ldots, k\} \setminus \{i, j\}$ (and equivalently $S_j \ne \emptyset$ implies $S_i = S_{\{i,l\}} = \emptyset$ for all $l \in \{1, \ldots, k\} \setminus \{i, j\}$):
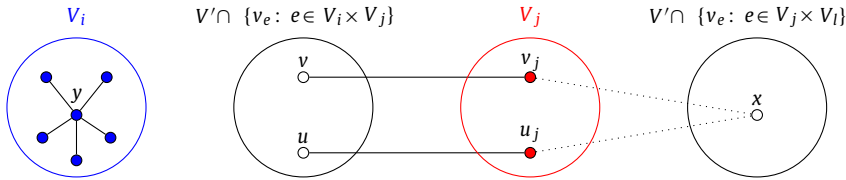
Let (u_i, u_j) and (v_i, v_j) be the edges in G corresponding to $u, v \in S_{\{i,j\}}$. We know that $pn(u, S) \subseteq \{v_i, v_j\} \setminus \{u_i, u_j\}$, so assume w.l.o.g. that $v_j \in pn(u, S)$ which implies $v_j \ne u_j$, $S_j = \emptyset$ and $V_i \subset N(u) \cup N(v)$.

For a third vertex $w \in S_{\{i,j\}}$ with corresponding edge $\{w_i, w_j\}$, we know that $w_j = v_j$, since otherwise $v_j \in N(w)$. This means that both w and v have a private neighbour in V_i which is not possible, since $V_i \setminus N(u) = \{u_i\}$. So we know that $S_{\{i,j\}} = \{u, v\}$.

If there is a vertex y in S_i, it already dominates all of V_i so $pn(v, S) = \{u_j\}$. Any $x \in V' \cap \{v_e : e \in V_j \times V_l, \ 1 \le l \le k\}$ is adjacent to at least u_j or v_j, so $S_j = S_{\{j,l\}} = \emptyset$. Dominating the vertices in $S_{\{j,l\}}$ for $l \ne i$ then requires $|S_l| = 2$ for all $l \ne i$, which leaves no possible private vertices outside V_i for vertices in V_i, so $|S_i| = 1$.

3. If $|S_i| = 2$ there exists an index $j \ne i$ such that $S_{\{i,j\}} = \emptyset$ and $|S_j| \le 1$.

   Let $u, v \in S_i$. By the structure of G', u and v share all neighbours in V_i and v_e such that $e = (x, y) \in V_i \times V_l$ with $x \notin \{u, v\}$ for all $l \ne i$, so especially the private neighbourhood of u is restricted to $pn(u, S) \subseteq \{v_e : e = (v, y) \in E\}$. Let j be an index such that there is a vertex $z \in V_j$ with $v_{(u,z)} \in pn(v, S)$ (there is at least one such index). No neighbour of $v_{(u,z)}$ beside v can be in S, which means that $S_{\{i,j\}} = \emptyset$ and $S_j \subseteq \{z\}$.

4. $|S_{\{i,l\}}| = 2$ implies $|S_{\{j,l\}}| \le 1$ for all $j \ne i$.

   Suppose $|S_{\{i,l\}}|, |S_{\{j,l\}}| \ge 2$ for some indices $i, j, l \in \{1, \ldots, k\}$. By Property 2 both sets $S_{\{i,l\}}, S_{\{j,l\}}$ have cardinality 2 so let $u_i, w_i \in S_{\{i,l\}}$ and $u_j, w_j \in S_{\{j,l\}}$. Since each set $\{v_e : e \in E \cap (V_s \times V_t)\}$ is a clique, the private neighbours for these vertices have to be in V_i, V_j, V_l. Suppose $v \in pn(u_i, S) \cap V_l$ which means that w_i, u_j, w_j are not adjacent to v. This is only possible if w_i represents some edge $(v, x) \in E \cap V_l \times V_i$ and u_j, w_j represent some edges $(v, y), (v, y') \in E \cap V_l \times V_j$. By definition of E', w_i, u_j, w_j then share their neighbourhood in V_l (namely $V_l \setminus \{v\}$) which means that $pn(w_i, S) \subset V_i$ and $pn(u_j) \cup pn(w_j) \subset V_j$ which implies $S_i = S_j = \emptyset$. So in any case, even if there is no $v \in pn(u_i, S) \cap V_l$, at least one of the sets V_i or V_j contains two vertices which are private neighbours for $S_{\{i,j\}}$ and $S_i = S_j = \emptyset$.

   Suppose V_j contains two private vertices $y \ne y'$ for u_j and w_j respectively. For any two arbitrary vertices $n_1, n_2 \in V_j$, any vertex $x \in \{v_e : e \in E \cap (V_i \times V_j)\}$ is adjacent to at least one of them, which means that any $x \in S_{\{i,j\}}$ would steal at least $y \in pn(u_j)$ or $y' \in pn(w_j)$ as private neighbour. Minimality of S hence demands $S_i = S_j = S_{\{i,j\}} = \emptyset$. A set with this property however does not dominate any of the vertices v_e with $e \in E \cap (V_i \times V_j)$. (The set $E \cap (V_i \times V_j)$ is not empty unless the graph G is a trivial "no"-instance for MULTICOLOURED CLIQUE.)

According to these properties, the indices of these subsets of S can be divided into the following six sets: $C_i := \{j : |S_j| = i\}$ and $D_i := \{(j, l) : |S_{\{j,l\}}| = i\}$ for $i = 0, 1, 2$ which then give $|S| = 2(|C_2| + |D_2|) + |C_1| + |D_1|$. If $|C_2| + |D_2| \neq 0$ and $k > 3$, we can construct an injective mapping $f : C_2 \cup D_2 \cup \{a\} \to C_0 \cup D_0$ with some $a \notin V'$ in the following way:

- For every $i \in C_2$ choose some $j \neq i$ with $(i, j) \in D_0$ and $j \notin C_2$ which exists according to property 3 and set $f(i) = (i, j)$. Since $j \notin C_2$ this f is injective.
  If $D_2 = \emptyset$ and $C_2 = \{i\}$, choose some $l \neq i$ and map a via f either to l or to (i, l), since, by property 1, one of them is in C_0 or D_0 respectively. If $D_2 = \emptyset$ and $|C_2| > 1$, choose some $i, l \in C_2$ and set $f(a) = (i, l)$ since $S_{\{i,l\}} = \emptyset$ by property 1 and neither i nor l is mapped to (i, l).
- For $(i, j) \in D_2$, property 2 implies at least i or j lies in C_0. By Property 4 we can choose one of them arbitrarily without violating injectivity. If both are in C_0 we can use one of them to map a. If for all $(i, j) \in D_2$ only one of the indices i, j is in C_0, we still have to map a, unless $f(a)$ has been already defined. Assume for $(i, j) \in D_2$ that $i \notin C_0$. By property 2 $\{(j, l) : l \notin \{i, j\}\} \subset D_0$. If we cannot choose one of these index-pairs as injective image for a, they have all been used to map C_2 which means $\{1, \ldots, k\} \setminus \{i, j\} \subseteq C_2$ and hence, by property 1, all index-pairs (l, h) with $l, h \in \{1, \ldots, k\} \setminus \{i, j\}$ are in D_0 and so far not in the image of f, so we are free to chose one of them as image of a, unless $f(a)$ has been already defined.

This injection proves that $|C_2| + |D_2| > 0$ implies that $|C_2| + |D_2| < |C_0| + |D_0|$. This means that, regardless of the structure of the original graph G, the subsets S_i and $S_{i,j}$ of S either all contain exactly one vertex or $k + \frac{1}{2}(k^2 - k) = |C_1| + |D_1| + |C_0| + |D_0| + |C_2| + |D_2| > |C_1| + |D_1| + 2(|C_2| + |D_2|) = |S|$.

So if $|S| = k + \frac{1}{2}(k^2 - k)$, the above partition into the sets $S_i, S_{i,j}$ satisfies $|S_i| = |S_{\{i,j\}}| = 1$ for all i, j. A set with this property is always dominating for G' but only minimal if each vertex has a private neighbour. For some $v_e \in S_{\{i,j\}}$ this implies that there is some private neighbour $e' = (u, v) \in V' \cap (V_i \times V_j)$ that is not dominated by the (existing) vertex u' in S_i or the vertex v' in S_j; (all vertices V_i and V_j are already dominated by $\{u', v'\} \subset S$ and cannot be private neighbours for v_e). By construction of E', this is only possible if $(u, v) = (u', v') \in E$. Since this is true for all index-pairs (i, j), the vertices $\{v : v \in S_i, 1 \leq i \leq k\}$ form a clique in the original graph G. $\quad\square$

We want to point out that the above reduction also works for the restriction of UPPER DOMINATION to solutions for which I is empty:

Corollary 24. (F, P, O)-DOMINATION*, that is the restriction of* UPPER DOMINATION *to solutions S such that $V = N(S)$, is W[1]-hard.*

Proof. The proof of Theorem 23 showed that there exists a minimal dominating set of cardinality $k + 0.5(k^2 - k)$ for G' if and only if each set S_i and $S_{i,j}$ contains exactly one vertex. If we build G'' from G' by adding k new vertices $w_1, \ldots, w_k$ and edge sets $\{(w_i, v) : v \in V_i\}$ for all $i \in \{1, \ldots, k\}$, G'' has still has the same property that $|S_i| = 2$ or $|S_{i,j}| = 2$ is not possible without the same private neighbourhood situation as for G'; in fact it immediately follows that $|S_{i,j}| \leq 1$, since dominating the new vertices requires at least one vertex from $V_i \cup \{w_i\}$ for each i, which leaves no possible private neighbours in $V_i \cup V_j$ for $v_e, v_{e'} \in E \cap (V_i \times V_j)$. For (F, P, O)-DOMINATION, we can not include any of the vertices w_i in a solution, since adding any neighbour violates minimality. This means that there can only exist a solution of cardinality $k + 0.5(k^2 - k)$ for (F, P, O)-DOMINATION for G'' if $|S_i| = 1$ or $|S_{i,j}| = 1$ for each i and j as defined in the proof of Theorem 23, which is only possible if the vertices from S_i are a clique in the original graph G.

It remains to show that in the "yes"-case for MULTICOLOURED CLIQUE, there exists a minimal dominating set S of cardinality $k + 0.5(k^2 - k)$ for G'' which has the (F, P, O)-structure. Let $\{v_1, \ldots, v_k\}$ be a multicoloured clique in G. We can extend this set to such a minimal dominating set of G' by adding for each pair (i, j) with $i, j \in \{1, \ldots, k\}$ and $i \neq j$ an edge-vertex v_e with $e \in E$ and $e \in (V_i \setminus \{v_i\}) \times (V_j \setminus \{v_j\})$. This way, v_e is adjacent to v_i and v_j in G' and has as private neighbour the vertex $v_{(v_i, v_j)}$ while the vertices v_i and v_j have private neighbours w_i and w_j respectively. We can assume that a vertex $e \in E \cap (V_i \setminus \{v_i\}) \times (V_j \setminus \{v_j\})$ always exists since otherwise, if $N(V_i \setminus \{v_i\}) = \{v_j\}$ for some pair (i, j), we know that each multicoloured clique for G contains either v_i or v_j. Branching on these two possibilities for all pairs (i, j) with this property yields a reduction in $O(2^k)$ to a graph which has a multicoloured k'-clique if and only if the corresponding graph G'' has an (F, P, O)-dominating set of cardinality $k' + 0.5(k'^2 - k')$ with $k' \leq k$. $\quad\square$

Corollary 24 means that if we consider somehow splitting the problem UPPER DOMINATION into the subproblems of computing the independent vertices I and (F, P, O)-DOMINATION, we end up with two W[1]-hard problems. Considering UPPER TOTAL DOMINATION, the construction in the proof of Theorem 23 is not very helpful, since unfortunately any set S with $|S \cap V_i| = 1$ for all $i \in \{1, \ldots, k\}$ and $|S \cap V_{i,j}| = 1$ for all $i \neq j$, regardless of the structure of the original graph G, is a minimal total dominating set for G'. We can however use a much simpler construction to show W[1]-hardness for UPPER TOTAL DOMINATION, a result which cannot be inferred from the known NP-hardness of the problem, see [25].

Theorem 25. Upper Total Domination *is W[1]-hard.*

Proof. We reduce from Multicoloured Independent Set, which is equivalent to Multicoloured Clique on the complement graph. Let $G = (V, E)$ be a graph with k different colour-classes given by $V = V_1 \cup V_2 \cup \cdots \cup V_k$. We construct a graph $G' = (V', E')$ as follows: Starting from G, we add k vertices $C = \{c_1, \ldots, c_k\}$ and turn each vertex set $V_j \cup \{c_j\}$ into a clique. We claim that G admits a multicoloured independent set (of size k) if and only if G' has a minimal total dominating set with $2k$ vertices.

If $K = \{v_1, \ldots, v_k\} \subseteq V$ is a multi-coloured independent set, then $D := K \cup C$ is a total dominating set. It is minimal, because removing a vertex $v \in \{v_j, c_j\}$ from D would yield $u \notin N(D)$ for $u \in \{v_j, c_j\} \setminus \{v\}$, since both v_j and c_j are not adjacent to any c_i with $i \neq j$ hence especially not to any vertex in $K \setminus \{v_j\}$.

Conversely, any dominating set must contain at least one vertex from $V_j \cup \{c_j\}$ for each j in order to dominate c_j. Let D be some minimal total dominating set for G', with $|D| \geq 2k$. If for some j, $|D \cap (V_j \cup \{c_j\})| > 2$, then, as $V_j \cup \{c_j\}$ forms a clique, all $\ell > 2$ private open neighbours $p_1, \ldots, p_\ell$ of the vertices from $\{u_1, \ldots, u_\ell\} = D \cap (V_j \cup \{c_j\})$ are from $V' \setminus (V_j \cup \{c_j\})$, so in fact from $V \setminus V_j$. Each p_i belongs to some colour class $f(i) \in \{1, \ldots, k\}$, and $f : \{1, \ldots, \ell\} \to \{1, \ldots, k\}$ is an injective mapping; namely, suppose there were $i \neq i'$ with $f(i) = f(i') = s$. The vertex c_s needs to be dominated, which is then impossible without stealing the private neighbour from either u_i or $u_{i'}$.

With the same argument, it is also clear that u_i is the only vertex from $D \cap (V_s \cup \{c_s\})$ for $s = f(i)$. Hence, $|D \cap \{x \in (V_r \cup \{c_r\}) : r = j \vee r \in f(\{1, \ldots, \ell\})\}| = 2\ell$, but this affects $\ell + 1$ colour classes. Hence, D contains less than $2k$ vertices, a contradiction. Therefore, for all j, $1 \leq |D \cap (V_j \cup \{c_j\})| \leq 2$. In order to satisfy $|D| \geq 2k$, this means that, for all j, $1 \leq |D \cap (V_j \cup \{c_j\})| = 2$. We can argue as before that all vertices from $D \cap (V_j \cup \{c_j\})$ must find their private open neighbours within $D \cap (V_j \cup \{c_j\})$. This also means that $K := D \cap V$ forms an independent set in G with $|I| \geq k$. $\square$

We do not know if Upper Domination belongs to W[1], but we can at least place it in W[2], the next level of the W-hierarchy. We obtain this result by describing a suitable multi-tape Turing machine that solves this problem, see [16].

**Proposition 26.** Upper Domination *belongs to W[2].*

Proof. We employ a strategy as similarly used for showing that Minimum Domination belongs W[2] by providing an appropriate multi-tape Turing machine [16]. First, the k vertices that should belong to the dominating set are guessed, and then this guess is verified in k further (deterministic) steps using n further tapes in parallel, where n is the order of the input graph, as in the standard proof showing that Minimum Domination belongs to W[2]. More precisely, for each vertex v guessed to be in the dominating set, all heads corresponding to vertices in $N[v]$ are moved forward. If we detect that after processing all k vertices, some head on some of the n auxiliary tapes did not move, then the guessed vertex set was not a dominating set, while it is a dominating set if all heads did move. Then, we need to make sure that the guessed set of vertices is minimal. To this end, we copy the guessed vertices k times, leaving one out each time, and we also guess one vertex for each of the $k - 1$-element sets that is not dominated by this set. This takes $O(k^2)$ time altogether. Such a guess can be tested in the same way as sketched before using parallel access to the $n + 1$ tapes. Namely, we again move all heads corresponding to vertices in $N[v]$ for all $k - 1$ vertices and then check if the head corresponding to the guessed non-dominated vertex did not move. The whole computation takes $O(k^2)$ parallel steps of the Turing machine, which shows the claim. $\square$

Let us notice that very similar proofs also show membership in W[2] and hardness for W[1] for the question whether, given some hypergraph G and parameter k, there exists a minimal hitting set of G with at least k vertices. The same argumentation as for Proposition 26 also gives the following.

Corollary 27. Upper Total Domination *belongs to W[2].*

In the context of parameterised complexity, we would like to point out another difference between Upper Domination and Minimum Domination. Despite its W[2]-hardness, there is at least a reduction-rule for Minimum Domination, which deals with vertices of degree one, as they can be assumed not to be contained in a minimum dominating set. One might suspect that any upper dominating set would conversely always choose to contain degree-one vertices.

As the example below illustrates, there can not be such a rule for Upper Domination, since the degree-one vertex v is never part of a maximum solution; in fact, the black vertices form the unique optimal solution for this graph.
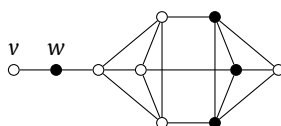
The parameterised dual of Minimum Domination, usually called Nonblocker, is known to be fixed parameter tractable, as are the duals of all other problems from the domination chain except Upper Domination. Hence, we now want to investigate the behaviour of Co-Upper Domination.

Theorem 28. Co-Upper Domination *is in FPT. More precisely, it admits a kernel of at most $\ell^2 + \ell$ vertices and at most ℓ^2 edges.*

Proof. Let $G = (V, E)$ be an input graph of order n. Consider a vertex $v \in V$ with $deg(v) > \ell$ and any minimal dominating set D with partition (F, I, P, O):

- If $v \in I$, all neighbours of v have to be in O which means $|O| \geq |N(v)| > \ell$.
- If $v \in F$, exactly one neighbour p of v is in P and $N[v] \setminus \{p\} \subseteq F \cup O$, which gives $|O| + |P| = |O| + |F| \geq |N[v] \setminus \{p\}| > \ell$.
- If $v \in P$, exactly one neighbour p of v is in F and $N[v] \setminus \{p\} \subseteq P \cup O$, so $|O| + |P| > \ell$.

We always have either $v \in O$ or $|O| + |P| > \ell$, in which case (G, ℓ) is a "no"-instance for Co-Upper Domination. Consider the graph G' built from G by deleting the vertex v and all its edges. For any minimal dominating set D for G with partition (F, I, P, O) such that $v \in O$, D is also minimal for G', since $pn(w, D) \supseteq \{w\}$ for all $w \in I$ and $|pn(u, D) \cap P| = 1$ for all $u \in F$. Also, any set $D' \subset V \setminus \{v\}$ which does not dominate v has a cardinality of at most $|V \setminus N[v]| < n - \ell$, so if G' has a dominating set D' of cardinality at least $n - \ell$, $N(v) \cap D' \neq \emptyset$; hence, D' is also dominating for G. These observations allow us to successively reduce (G, ℓ) to (G', ℓ') with $\ell' = \ell - 1$, as long as there are vertices v with $deg(v) > \ell$. Any isolated vertex in the resulting graph G' originally only has neighbours in O which means it belongs to I in any dominating set D with partition (F, I, P, O) and can hence be deleted from G' without affecting the existence of an upper dominating set with $|P| + |O| \leq \ell'$.

Let (G', ℓ') be the instance obtained after the reduction above with $G' = (V', E')$ and let $n' = |V'|$. If there is an upper dominating set D for G' with $|D| \geq n' - \ell'$, any associated partition (F, I, P, O) for D satisfies $|P| + |O| \leq \ell'$. Since G' does not contain isolated vertices, every vertex in I has at least one neighbour in O. Also, any vertex in V', and hence especially any vertex in O, has degree at most ℓ', which means that $|I| \leq |N(O)| \leq \ell'|O|$. Overall:

$$|V'| \leq |I| + |F| + |P| + |O| \leq (\ell' + 1)|O| + 2|P| \leq \max_{j=0}^{\ell'}\{j(\ell' + 1), 2(\ell' - j)\},$$

and hence $|V'| \leq \ell'(\ell' + 1)$, or (G', ℓ') and consequently (G, ℓ) is a "no"-instance. Concerning the number of edges, we can derive a similar estimate. There are at most ℓ edges incident with each vertex in O. In addition, there is exactly one edge incident with each vertex in P that has not yet been accounted for, and, in addition, there could be $\ell - 1$ edges incident to each vertex in F that have not yet been counted. This shows the claim. $\square$

We just derived a kernel result for Co-Upper Domination, in fact a kernel of quadratic size in terms of the number of vertices and edges. This poses the natural question if we can do better also with respect to the question whether the brute-force search we could perform on the quadratic kernel is the best we can do to solve Co-Upper Domination in FPT time.

Proposition 29. Co-Upper Domination *for a graph $G = (V, E)$ and a parameter ℓ can be solved with* `ComputeCoUD(G, ℓ, ∅, ∅, ∅, ℓ)` *in time $O^*(4.3077^\ell)$.*

Proof. Algorithm 3 is a branching algorithm, with halting rules (H1) and (H2), reduction rule (R1), and three branching rules (B1)–(B3). We denote by $G = (V, E)$ the input graph and by ℓ the parameter. At each call, the set of vertices V is partitioned into four sets: F, I, $\overline{D}$ and R. The set of **r**emaining vertices R is equal to $V \setminus (F \cup I \cup \overline{D})$, and thus can be obtained from G and the three former sets.

At each recursive call, the algorithm picks some vertices from R. They are either added to the current dominating set $D := F \cup I$, or to the set $\overline{D}$ to indicate that they do not belong to any extension of the current dominating set. The sets F and I are as previously described (*i.e.*, if we denote by D the dominating set we are looking for, $I := \{v \in D : v \in pn(v, D)\}$ and $F := D \setminus I$).

Note that parameter κ corresponds to our "budget", which is initially set to $\kappa := \ell$. Recall that any minimal dominating set of a graph $G = (V, E)$ can be associated with a partition (F, I, P, O). If we denote by D a minimal dominating set of G and by $\overline{D}$ the set $V \setminus D$, then by definition, F, I is a partition of D and P, O is a partition of $\overline{D}$. Also, by definition of F and P, it holds that $|F| = |P|$ and there is a perfect matching between vertices of F and P. Since each vertex of F will (finally) be matched with its private neighbour from P, we define our budget as $\kappa = \ell - \left(\frac{|F|}{2} + \frac{|P|}{2} + |O|\right)$. One can observe that if D is a minimal dominating set of size at least $n - \ell$ then $\kappa \geq 0$. Conversely, if $\kappa < 0$ then any dominating set D such that $F \cup I \subseteq D$ is of size smaller than $n - \ell$. This shows the correctness of **(H1)**. We now consider the remaining rules of the algorithm. Note that by the choice of κ, each time a vertex x is added to $\overline{D}$, the value of κ decreases by $\frac{1}{2}$ (or by 1 if we can argue that x is not matched with a vertex of F and thus belongs to O). Also, whenever a vertex x is added to F, the value of κ decreases by $\frac{1}{2}$.

Algorithm 3: `ComputeCoUD(G, ℓ, F, I, D̄, κ)`.

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input : *Graph $G = (V, E)$, parameter $\ell \in \mathbb{N}$, disjoint sets $F, I, \overline{D} \subseteq V$ and $\kappa \leq \ell$.*
output: *Answer "yes" if $\Gamma(G) \geq |V| - \ell$; "no" otherwise.*

Let $R \leftarrow V \setminus (F \cup I \cup \overline{D})$
if $\kappa < 0$ **then return** "no" ;   (H1)
if *R is empty* **then**   (H2)
    if *$F \cup I$ is a minimal dominating set of G and $|F \cup I| \geq n - \ell$* **then**
        └── **return** "yes"
    else return "no"
if *there is a vertex $v \in R$ s.t. $N(v) \subseteq \overline{D}$* **then**   (R1)
    └── **return** `ComputeCoUD(G, ℓ, F, I∪{v}, D̄, κ)`
if *there is a vertex $v \in R$ s.t. $|N(v) \cap F| \geq 1$* **then**   (B1)
    return `ComputeCoUD(G, ℓ, F∪{v}, I, D̄, κ−½)` ∨
    `ComputeCoUD(G, ℓ, F, I, D̄∪{v}, κ−½)`
if *there is a vertex $v \in R$ s.t. $|N(v) \cap R| = 1$* **then**   (B2)
    Let u be the unique neighbour of v in R
    return `ComputeCoUD(G, ℓ, F∪{u,v}, I, D̄, κ−1)` ∨ `ComputeCoUD(G, ℓ, F∪{u}, I, D̄∪{v}, κ−1)` ∨
    `ComputeCoUD(G, ℓ, F, I∪{v}, D̄∪{u}, κ−1)`
else   (B3)
    Let v be a vertex of R
    return `ComputeCoUD(G, ℓ, F, I∪{v}, D̄∪N(v), κ−2)` ∨
    `ComputeCoUD(G, ℓ, F∪{v}, I, D̄, κ−½)` ∨
    `ComputeCoUD(G, ℓ, F, I, D̄∪{v}, κ−½)`

(**H2**) If R is empty, then all vertices have been decided: they are either in $D := F \cup I$ or in $\overline{D}$. It remains to check whether D is a minimal dominating set of size at least $n - \ell$.

(**R1**) All neighbours (if any) of v are in $\overline{D}$ and thus v has to be in $I \cup F$. As v will also belong to $pn(v, D)$, we can safely add v to I. Observe also that this reduction rule does not increase our budget.

(**B1**) Observe that if v has a neighbour in F, then v cannot belong to I. When a vertex v is added to F, the budget is reduced by at least $\frac{1}{2}$; when v is added to $\overline{D}$, the budget is reduced by $\frac{1}{2}$, as well. So (B1) gives a branching vector of $(\frac{1}{2}, \frac{1}{2})$.

(**B2**) If (R1) and (B1) do not apply and $N(v) \cap R = \{u\}$, then the vertex v has to either dominate itself or be dominated by u. Every vertex in F has a neighbour in F, which in this case means that $v \in F$ implies $u \in F$ (first branch). Moreover, the budget is reduced by at least $2 \cdot \frac{1}{2}$.

If v is put in I, u has to go to $\overline{D}$ (third branch). Thus u cannot be a private neighbour of some F-vertex, and the budget decreases by at least 1 ($u \in O$).

If v does not dominate itself, u has to be in $F \cup I$. In this last case it suffices to consider the less restrictive case $u \in F$, as v can be chosen as the private neighbour for u (second branch). If u is indeed in I for a minimal dominating set which extends the current $I \cup F$, there is a branch which puts all the remaining neighbours of u in $\overline{D}$. Observe that we only dismiss branches with halting rule (H2) where we check if $F \cup I$ is a minimal dominating set, we do not require the chosen partition to be correct. As for the counting in halting rule (H1): whether we count $u \in F$ and $v \in P$ (recall that $P \subseteq \overline{D}$) each with $\frac{1}{2}$ or count $v \in O$ (recall that $O \subseteq \overline{D}$) with 1 does not make a difference for κ. So the budget decreases by at least 1.

Altogether (B2) gives a branching vector of $(1, 1, 1)$.

(**B3**) The correctness of (B3) is easy as all possibilities are explored for vertex v. Observe that by (R1) and (B2), vertex v has at least two neighbours in R. When v is added to I, these two vertices are removed (and cannot be the private neighbours of some F-vertices). Thus we reduce the budget by at least 2. When v is added to F, the budget decreases by at least $\frac{1}{2}$. When v is added to $\overline{D}$, we reduce the budget by at least $\frac{1}{2}$. Thus (B3) gives a branching vector of $(2, \frac{1}{2}, \frac{1}{2})$. However, we can observe that the second branch (*i.e.*, when v is added to F) implies a subsequent application of (B1) (or rule (H1) would stops the recursion). Thus the branching vector can be refined to $(2, 1, 1, \frac{1}{2})$.

The worst-case over all branching vectors establishes the claimed running time. □

Of course, the question remains to what extent the previously presented parameterised algorithm can be improved on. In this context, we briefly discuss the issue of (parameterised) approximation for this parameter.

Theorem 30. Co-Upper Domination *is 4-approximable in polynomial time, 3-approximable with a running time in $O^*(1.0883^{\tau(G)})$ and 2-approximable in time $O^*(1.2738^{\tau(G)})$ or $O^*(1.2132^n)$.*

Proof. First of all, observe by subtracting n from Eq. (1) that $\tau(G)$ relates to the co-upper domination number in the following way:

$$\frac{\tau(G)}{2} + 1 \leq n - \Gamma(G) \leq \tau(G) \tag{8}$$

Using any 2-approximation algorithm one can compute a vertex cover V' for G, and define $S' = V \setminus V'$. Let S be a maximal independent set containing S'. $V \setminus S$ is a vertex cover of size $|V \setminus S| \leq |V'| \leq 2\tau(G) \leq 4(n - \Gamma(G))$. Moreover, S is maximal independent and hence minimal dominating set which makes $V \setminus S$ a feasible solution for Co-Upper Domination with $|V \setminus S| \leq 4(n - \Gamma(G))$. The claimed running times for the factor-2 approximation correspond to the current best parameterised and exact algorithms for Minimum Vertex Cover by [18] and [39], and the one from the factor-3 approximation corresponds to the parameterised approximation in [15]. □

5.2. Graphs of bounded degree

In contrast to the case of general graphs, Upper Domination turns out to be easy (in the sense of parameterised complexity) for graphs of bounded degree.

Proposition 31. *Fix $\Delta > 2$. Upper Domination is in FPT when restricted to graphs of maximum degree Δ. More precisely, the problem can be solved in time $O^*((\Delta + 1)^{2k})$.*

The statement of the proposition is of course also true for $\Delta \in \{0, 1, 2\}$, but then the problem is (trivially) solvable in polynomial time. In the following, we give an argument based on branching.

Proof. Consider the simple branching algorithm that branches on all at most $\Delta + 1$ possibilities to dominate a vertex that is not dominated. Once we have fixed a new vertex in the dominating set, we let follow another branch (of at most $\Delta + 1$ possibilities) to determine the private neighbour of the new vertex in the dominating set. Assuming that we are only looking for sets of size k, we can find a "yes"-instance in each branch where we needed to put k vertices in the dominating set (so far); if that set is not yet dominating, we can turn it into a minimal dominating set by a greedy approach, respecting previous choices. The overall running time of the branching algorithm is hence $O^*((\Delta + 1)^{2k})$. □

The astute reader might wonder why we have to do this unusual 2-stage branching, but recall Theorem 9 which shows that it is difficult to extend some set of vertices of size at most k to a minimal dominating set containing it. Brooks' Theorem yields the following result.

Proposition 32. *Fix $\Delta > 2$. Upper Domination has a problem kernel with at most Δk many vertices.*

Proof. First, we can assume that the input graph G is connected, as otherwise we can apply the following argument separately on each connected component. Assume G is a cycle or a clique. Then, the problem Upper Domination can be optimally solved in polynomial time, *i.e.*, we can produce a kernel as small as we want. Otherwise, Brooks' Theorem yields a polynomial-time algorithm that produces a proper colouring of G with (at most) Δ many colours. Extend the biggest colour class to a maximal independent set I of G. As I is maximal, it is also a minimal dominating set. So, there is a minimal dominating set I of size at least n/Δ, where n is the order of G. So, $\Gamma(G) \geq n/\Delta$. If $k < n/\Delta$, we can therefore immediately answer "yes". In the other case, $n \leq \Delta k$ as claimed. □

With some more combinatorial effort, we obtain:

Proposition 33. *Fix $\Delta > 2$. Co-Upper Domination has a problem kernel with at most $(\Delta + 0.5)\ell$ many vertices.*

Proof. Consider any graph $G = (V, E)$. For any partition (F, I, P, O) corresponding to an upper dominating set $D = I \cup F$ for G, isolated vertices in G always belong to I and can hence be deleted in any instance of Co-Upper Domination without changing ℓ. For any graph G without isolated vertices, the set $P \cup O$ dominating for G, since $\emptyset \neq N(v) \subset O$ for all $v \in I$ and $N(v) \cap P \neq \emptyset$ for all $v \in F$. This implies that $n = |N[P \cup O]| \leq (\Delta + 1)\ell$ as any dominating set in a graph of maximum degree Δ has a cardinality of at least $\frac{n}{\Delta + 1}$.

Since any connected component can be solved separately, we can assume that G is connected. For any $v \in P$, the structure of the partition (F, I, P, O) yields $|N[v] \cap D| = 1$, so either $|N[v]| = 1 < \Delta$ or there is at least one $w \in P \cup O$ such that $N[v] \cap N[w] \neq \emptyset$. For any $v \in O$, if $N[v] \cap F \neq \emptyset$, the F-vertex in this intersection has a neighbour $w \in P$, which means $N[w] \cap N[v] \neq \emptyset$. If $N(v) \subseteq I$ and $N[v] \neq V$, at least one of the I-vertices in $N(v)$ has to have another neighbour to connect to the rest of the graph. Since $N(I) \subseteq O$, this also implies the existence of a vertex $w \in O$, $w \neq v$ with $N[w] \cap N[v] \neq \emptyset$. Finally, if $N[v] \not\subset I \cup F$, there is obviously a $w \in P \cup O$, $w \neq v$ with $N[w] \cap N[v] \neq \emptyset$.

Assume that there is an upper dominating set with partition (F, I, P, O) such that $|P \cup O| = l \leq \ell$ and let $v_1, \ldots, v_l$ be the $l > 1$ vertices in $P \cup O$. By the above argued domination-property of $P \cup O$, we have:

$$n = |\bigcup_{i=1}^{l} N[v_i]| = \frac{1}{2} \sum_{i=1}^{l} |N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]| + \frac{1}{2} \sum_{i=1}^{l} |N[v_i] \setminus \bigcup_{j=i+1}^{l} N[v_j]|.$$

Further, by the above argument about neighbourhoods of vertices in $P \cup O$, maximum degree Δ yields for every $i \in \{1, \ldots, l\}$ either $|N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]| \leq \Delta$ or $|N[v_i] \setminus \bigcup_{j=i+1}^{l} N[v_j]| \leq \Delta$ which gives:

$$n = \frac{1}{2} \sum_{i=1}^{l} |N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]| + |N[v_i] \setminus \bigcup_{j=i+1}^{l} N[v_j]| \leq \frac{1}{2} l(2\Delta + 1) \leq (\Delta + 0.5)\ell.$$

Any graph with more than $(\Delta + 0.5)\ell$ vertices is consequently a "no"-instance which yields the stated kernelisation, as the excluded case $|P \cup O| = 1$ (or in other words $N[v] = V$ for some $v \in O$) can be solved trivially. $\quad\square$

This implies that we have a $3k$-size vertex kernel for Upper Domination, restricted to subcubic graphs, and a 3.5ℓ-size vertex kernel for Co-Upper Domination, again restricted to subcubic graphs. In [17, Theorem 3.1] it is shown that if a parameterised problem, for which the corresponding decision problem is NP-hard, admits a linear kernel of size αk and its parameterised dual admits a kernel of size $\alpha_d k_d$, then $(\alpha - 1)(\alpha_d - 1) < 1$ implies P = NP. With this, we can conclude the following:

Corollary 34. *For any $\varepsilon > 0$, Upper Domination, restricted to subcubic graphs, does not admit a kernel with less than $(1.4 - \varepsilon)k$ vertices; neither does Co-Upper Domination, restricted to subcubic graphs, admit a kernel with less than $(1.5 - \varepsilon)\ell$ vertices, unless P = NP.*

6. Summary and conclusion

The motivation to study Upper Domination (at least for some of the authors) was based on the following observation based on enumeration; see [27].

Proposition 35. Upper Domination *can be solved in time $O^*(1.7159^n)$ on general graphs of order n.*

So far there is no better algorithm (analysis) than this enumeration algorithm for Upper Domination although the minimisation counterpart can be solved in better than $O^*(1.5^n)$ time [36,44]. As this appears to be quite a tough problem, it makes a lot of sense to consider approximative and parameterised approaches and also study restricted graph classes as we did in this paper. We summarise some open problems.

- Is Upper Domination in W[1]? Or, hard for W[2]?
- Are there smaller kernels for Upper and/or Co-Upper Domination on graphs of bounded degree?
- Is it possible to improve the 4-approximation of Co-Upper Domination?
- Can we find exact (*e.g.,* branching or pathwidth-based) algorithms that beat the enumeration for Upper Domination?
- Conversely, are there better enumeration algorithms for minimal dominating sets in the degree-restricted scenario?

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