Nowhere-zero flow on some products of signed graphs

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ABSTRACT

In this paper, we show that the Cartesian product of two signed nontrivial connected graphs has a nowhere-zero 4-flow. Also, we prove that signed biwheel B_n admits a nowhere zero k-flow where $k \in \{3, 4\}$.

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1. Introduction

A *signed graph* is a graph with each edge labelled with a sign, $+$ or $-$. An orientation of a signed graph is obtained by dividing each edge into two half-edges each of which receives its own direction. A positive edge has one half-edge directed from and the other half-edge directed to its end-vertex. Hence, a negative edge has both half-edges directed either towards or from their respective end-vertices. So, the directions on a negative edge are extroverted or introverted. A cycle in a signed graph is called *balanced* if it contains an even number of negative edges, otherwise it is called *unbalanced*. A signed graph is called *balanced* if all its cycles are balanced. Let v be a vertex of a signed graph G. The *switching* v is changing the sign of each edge incident with v to the opposite one. Let $X \subseteq V$. Switching a vertex set X means reversing the signs of all edges between X and its complement. Switching a set X has the same effect as switching all the vertices in X, one after another. A signed graph is balanced if and only if using some switchings, it is equivalent to the all-positive signature. If a signed graph can be switched into an isomorphic copy of another signed graph, the two signed graphs are called *switching isomorphic*.

A *nowhere-zero k-flow* on a signed graph G is an assignment of an orientation and a value from $\{\pm 1, \pm 2, \dots, \pm(k-1)\}$ to each edge in such a way that for each vertex of G the sum of incoming values equals the sum of outgoing values (Kirchhoff's law). We call such graphs flow-admissible.

Consider a signed graph G carrying a k-flow ϕ and let $P = e_1 e_2 \dots e_r$ be an $u - v$ trail in G. By sending a value $b \in \{\pm 1, \pm 2, \dots, \pm(k-1)\}$ from u to v along P we mean reversing the orientation of the edge e_1 so that it leaves u, adding b to $\phi(e_1)$, and adding $\pm b$ to $\phi(e_i)$ for all other edges of P in such a way that Kirchhoff's law is fulfilled at each inner vertex of P. Let for a graph G, $\Phi(G)$ be the smallest positive integer for which G admits a nowhere-zero $\Phi(G)$-flow.

For graphs G and H, the *join* of G and H is the graph $G \vee H$ with the vertex set $V = V(G) \cup V(H)$ and the edge set $E = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. A signed eulerian graph G is *triply odd* if it has a decomposition into three eulerian subgraphs G_1, G_2, and G_3, with odd number of negative edges each, that share a vertex.

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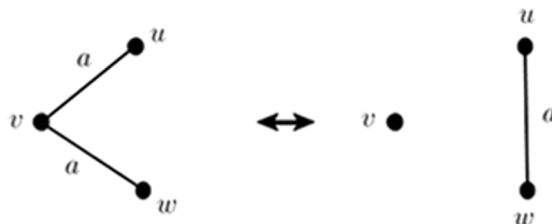
Fig. 1. Vertex-Splitting at v.

The following theorem classifies nowhere-zero flow on signed eulerian graphs.

Theorem 1.1 ([4, *Theorem 2.5*]). *Let G be a connected signed eulerian graph. Then*

   (i) *G has no nowhere-zero flow if and only if G is unbalanced and $G - e$ is balanced for some edge e;*
  (ii) *$\Phi(G) = 2$ if and only if G has even number of negative edges;*
 (iii) *$\Phi(G) = 3$ if and only if G is triply odd;*
 (iv) *$\Phi(G) = 4$ otherwise.*

We have the following theorem due to Xu and Zhang in [9].

Theorem 1.2. *Let G be a 2-edge connected signed graph. Then G admits a nowhere-zero 3-flow if and only if G admits a nowhere-zero $\mathbb{Z}_3$-flow.*

In this paper, we investigate the problem of nowhere-zero flow for some products of two signed graphs. In Section 2, we show that for two nontrivial and connected signed graphs G and H, the Cartesian product $G \square H$ admits a nowhere-zero 4-flow. Also, in Section 3, we are going to solve this problem for signed friendship, F_n, and signed biwheel B_n. In this paper dashed lines or bidirected (introverted or extroverted) edges denote the negative edges.

2. Nowhere-zero flow on signed Cartesian product of two graphs

The *Cartesian product* $G \square H$ of two signed graphs (G, σ_1) and (H, σ_2) is a generalization of the Cartesian product of ordinary graphs, see [1]. It is defined as the signed graph $(G \square H, \sigma)$ with the vertex set $V(G \square H) = V(G) \times V(H)$, where two distinct vertices (u, v) and (x, y) of $G \square H$ are adjacent if either

$$u = x \text{ and } vy \in E(H) \text{ or } v = y \text{ and } ux \in E(G).$$

Also, the signature of the edge $(u, v)(x, y)$ in $G \square H$ is defined by (see Fig. 1).

$$\sigma((u, v)(x, y)) = \begin{cases} \sigma_2((v, y)) & \text{if } u = x \\ \\ \sigma_1((u, x)) & \text{if } v = y. \end{cases}$$

Given a graph G, let v be a vertex of G with degree at least 3. Consider two edges $e = uv$ and $f = vw$ incident with v. We obtain a graph $G_{[v;e,f]}$ by deleting e and f from G and adding a new edge g joining u to w. That is $G_{[v;e,f]} = (G \backslash \{e, f\}) \cup \{uw\}$, see [8]. Now, suppose that (G, σ) is a signed graph. Consider $(G_{[v;e,f]}, \sigma')$ with the following condition:

$$\sigma'(g) = \begin{cases} \sigma(e)\sigma(f) & \text{if } g = uw \\ \\ \sigma(g) & \text{otherwise.} \end{cases}$$

One can check that if $(G_{[v;e,f]}, \sigma')$ has a nowhere-zero k-flow, then so does (G, σ).

Lemma 2.1 ([6, *Lemma 3.1*]). *Let G be a nontrivial connected graph. Then there exists a series of vertex-splittings which converts G into*

   (i) *one circuit, if G is eulerian, or into*
  (ii) *a disjoint union of m nontrivial paths, if G has 2m vertices of odd valency.*

Authors in [3] and [6] investigated the effect of vertex-splitting on the structure of the Cartesian product of two graphs. Now, we need to discuss the effect of vertex-splitting on the structure of the Cartesian product of two signed graphs. Let G and H be two signed graphs. Consider $(G \square H, \sigma)$ and form the graph $G' = G_{[v;e,f]}$. It is easy to check that if $(G' \square H, \sigma')$ admits a nowhere-zero k-flow, then $(G \square H, \sigma)$ also admits a nowhere-zero k-flow.

The following theorem is due to Imrich and Škrekovski [3]. Then in 2012, Rollová and Škoviera in [6], gave a much shorter proof for this theorem.

Theorem 2.2. *The Cartesian product of two graphs without isolated vertices has a nowhere-zero 4-flow.*

Now, we are going to deal with the Cartesian product of two signed graphs. One can check that using switching operation, an unbalanced cycle is equivalent to the same cycle with a negative edge. Also, a balanced cycle is equivalent with the same cycle with all positive edges.

Remark 2.3. The following fact is obtained by a series of switchings. If (P_m, σ_1) and (C_n, σ_2) are signed graphs, then signed graph $C_n \square P_m$, is equivalent to product of (C_n, σ_2) and (P_m, σ_1), where (C_n, σ_2) is a signed graph with all edges positive or with just one negative edge, and (P_m, σ_1) is a signed graph with all edges positive. For abbreviation, let C_n and C_n^- be a balanced and an unbalanced n-cycle, respectively.

In order to prove the existence a nowhere-zero flow on the Cartesian product of two signed graphs, we need the following theorem.

Theorem 2.4. *Let m and n be two positive integers and C_n^- be an unbalanced n-cycle. Then $C_n^- \square P_m$ admits*

(i) *a nowhere-zero 3-flow if n is odd,*
(ii) *a nowhere-zero 4-flow if n is even.*

Proof. (i). For any odd n, $C_n^- \square P_2$ admits a nowhere-zero $\mathbb{Z}_3$-flow. Since $C_n^- \square P_2$ is a cubic graph, it is sufficient to assign value 1 to all edges, and for each vertex v, the direction of the edges incident with v is the same. Hence, one can find a nowhere-zero $\mathbb{Z}_3$-flow for $C_n^- \square P_2$. So, by Theorem 1.2, $C_n^- \square P_2$ admits a nowhere-zero 3-flow. Let $m \geqslant 3$. We claim that $C_n^- \square P_m$ admits a nowhere-zero $\mathbb{Z}_3$-flow. Assign just value 1 to all edges of $C_n^- \square P_m$. Set the direction of all edges incident with any vertex of degree 3, the same. In this way, one can find a nowhere-zero $\mathbb{Z}_3$-flow on $C_n^- \square P_m$ for $m = 3, 4$, see Graphs (a) and (b) given in Fig. 2. Note that the value of each edge in Fig. 2, is 1. By a similar method, one can conclude that $C_n^- \square P_m$ for any $m \geqslant 5$ has a nowhere-zero $\mathbb{Z}_3$-flow. Hence by Theorem 1.2, for any odd n, $C_n^- \square P_m$ admits a nowhere-zero 3-flow. (ii). Assume that n is even. Since $C_n^- \square P_2$ is not antibalanced, by [5, Theorem 3.1], it does not have a nowhere-zero 3-flow. We claim that it has a nowhere-zero 4-flow. One can find a certain algorithm showing a nowhere-zero 4-flow on $C_n^- \square P_2$ for even n, see Graph (a) given in Fig. 3. At the first, assign value 1 to both negative edges (extroverted and introverted). Then, send values 2 and 3 to the edges along the outer n-cycle, alternatively and counterclockwise. And, send values 2 and 3 to the edges along the inner n-cycle, alternatively and clockwise. In this way, one can determine the value and the direction of the edges between two n-cycles. All of such edges have value 1, see Graph (a) in Fig. 3. Now, consider $m = 3$. It is not hard to check that $C_n^- \square P_3$ does not have a nowhere-zero $\mathbb{Z}_3$-flow, so by Theorem 1.2, it does not have a nowhere-zero 3-flow. We show that $C_n^- \square P_3$ admits a nowhere-zero 4-flow. Send values 2 and 3 to the edges along the third cycle, alternatively and clockwise. Then for the second cycle, assign 1 to the next two edges, near to the negative edge, see Graph (b) given in Fig. 3. Now, send values 3 and 2 to the edges along the mid-cycle, alternatively and counterclockwise. For the edges between the first cycle and the second one, assign 2, 1 and 3 to the next three edges, after negative edge, see Graph (b) in Fig. 3. Then, assign value 2 to the edges between the first cycle and the mid-cycle. All of the remaining edges have value 1.

Now, based on the basic cases, $C_n^- \square P_2$ and $C_n^- \square P_3$, we can conclude the assertion (ii). $\square$

Theorem 2.5. *Let G and H be two nontrivial and connected signed graphs. Then, $G \square H$ admits a nowhere-zero 4-flow.*

Proof. Without loss of generality we can assume that $G \square H$ is connected. This implies that G is connected and that H has no isolated vertices. Apply vertex-splitting for G and H. By Lemma 2.1, it is sufficient to investigate the existence of a nowhere-zero 4-flow for the Cartesian products $P_n \square P_m$, $C_n \square P_m$, $C_n^- \square P_m$, $C_n \square C_m$, $C_n^- \square C_m$ and $C_n^- \square C_m^-$. It is not hard to check that $P_n \square P_m$, $C_n \square P_m$, $C_n \square C_m$, $C_n^- \square C_m$ and $C_n^- \square C_m^-$ have a nowhere-zero 4-flow, see [6]. Note that $C_n \square C_m$, $C_n^- \square C_m$ and $C_n^- \square C_m^-$ are eulerian graphs, so by Theorem 1.1, each of them has a nowhere-zero 4-flow. Also by Theorem 2.4, $C_n^- \square P_m$ admits a nowhere-zero 4-flow. Hence, we can conclude that $G \square H$ has a nowhere-zero 4-flow. $\square$

3. Nowhere-zero flow on the friendship F_n and biwheel B_n

A *friendship graph* F_n is a collection of n triangles with a common vertex, see Fig. 4. A *biwheel* B_n is a graph on $n + 2$ vertices which is obtained by joining a cycle on n vertices and K_2, $C_n \vee K_2$. A wheel $W_n = C_n \vee K_1$ on $n + 1$ vertices is a planar graph for all values of n, but the biwheel B_n is non-planar for $n \geqslant 3$. Authors in [2], proved a nowhere-zero flow on signed wheels. Let in biwheel B_n, $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_2) = \{u, u'\}$.

Lemma 3.1. *Let n be a positive integer and T be a balanced triangle. F_n has a nowhere-zero k-flow if and only if $F_{n-1} = F_n \setminus T$ has a nowhere-zero k-flow for some $k \geqslant 2$.*

Proof. It is sufficient to assign $+1$ to each edge of triangle T. It is not hard to see that we can ignore T from F_n. $\square$
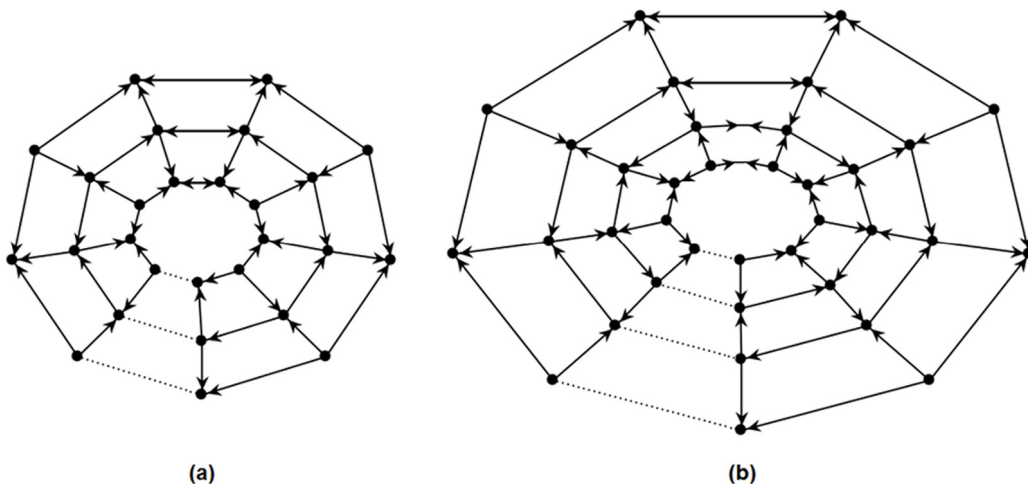
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Fig. 2. Nowhere-zero $\mathbb{Z}_3$-flow on $C_n^- \square P_3$ and $C_n^- \square P_4$.
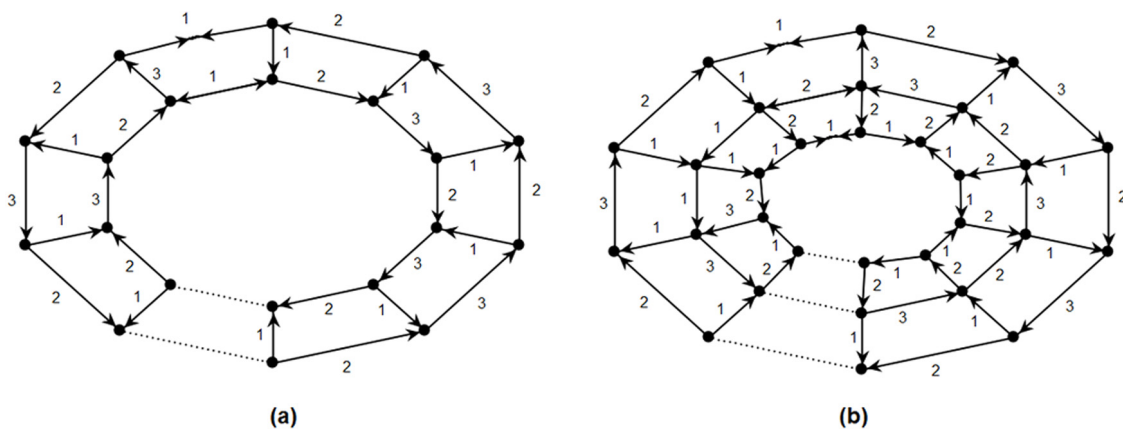

Fig. 3. Nowhere-zero 4-flow on $C_n^- \square P_2$ and $C_n^- \square P_3$.

Fig. 4. Friendship graph F_3.

Remark 3.2. Without loss of generality by doing switching operation in some vertices, we can consider all negative edges of F_n on the edges which are not incident with the central vertex. Hence, we can consider all triangles of F_n unbalanced. Focus on F_n in which all edges not incident with the central vertex are negative.

Theorem 3.3. *For a signed friendship graph F_n, the following statements hold:*

 (i) *For even n, F_n admits a nowhere-zero 2-flow,*
 (ii) *For odd n, F_n admits a nowhere-zero 3-flow.*

Proof. Use Theorem 1.1 and Lemma 3.1. □

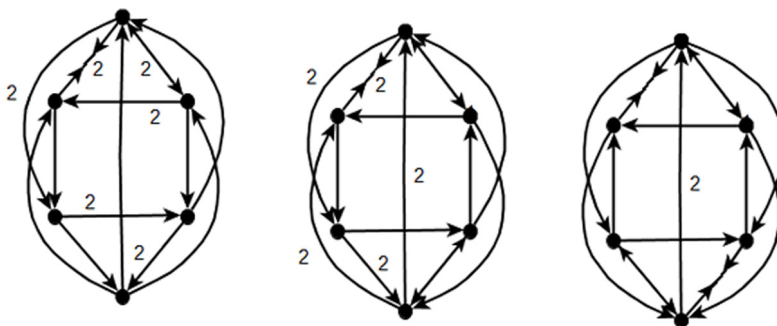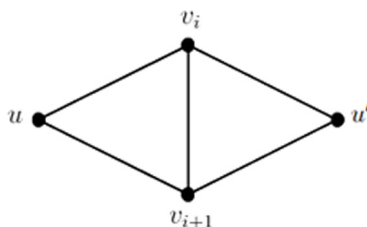**Fig. 5.** Nowhere-zero 3-flow on B_4.
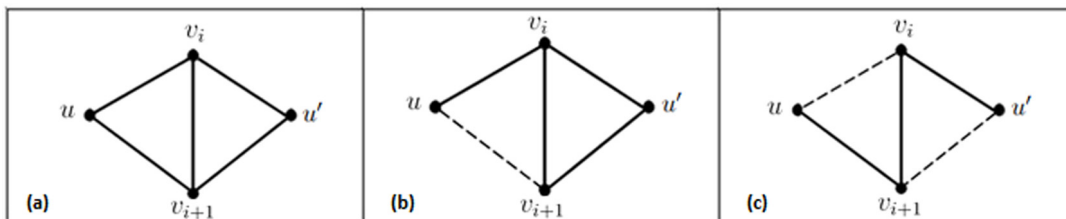

Fig. 6. A subgraph of B_n.

Fig. 7. All types of the signed subgraph given in Fig. 6.

Now, we are going to discuss the existence a nowhere-zero flow for signed biwheel B_n. For unsigned version, we have the following theorem due to Shahmohamad in [7].

Theorem 3.4. *Let B_n be the unsigned biwheel graph or the signed biwheel graph with all positive edges. Then the following conditions hold:*

(i) $\Phi(B_{2m-1}) = 2$ *for $m \geqslant 2$,*
(ii) $\Phi(B_{2m}) = 3$ *for $m \geqslant 2$.*

By switching operation, we may assume that a signed cycle has at most one negative edge. Denote the number of negative edges incident with vertex v, and the number of negative edges in a graph G with $n(v)$ and $n(G)$, respectively. Then we can assume that $n(v) \leqslant \dfrac{d(v)}{2}$, $d(v)$ is the degree of v, for any $v \in V(G)$. Now, if the cycle C_n is balanced, then $n(B_n) \leqslant n$.

Lemma 3.5. *If a signed biwheel B_4 is flow-admissible, then it has a nowhere-zero 3-flow.*

Proof. All types of flow-admissible signed biwheel B_4 are listed in Figs. 11 and 12. It is not hard to check that each of them admits a nowhere-zero 3-flow. For example, we show the existence a nowhere-zero 3-flow on three types of signed biwheel B_4 in Fig. 5. Note that all edges in Fig. 5 with no value having value 1. $\square$

Remark 3.6. Let the cycle C_n in B_n be balanced, so one can consider all edges of C_n positive. In this case, for a subgraph given in Fig. 6, up to switching isomorphism we have three types presented in Fig. 7.
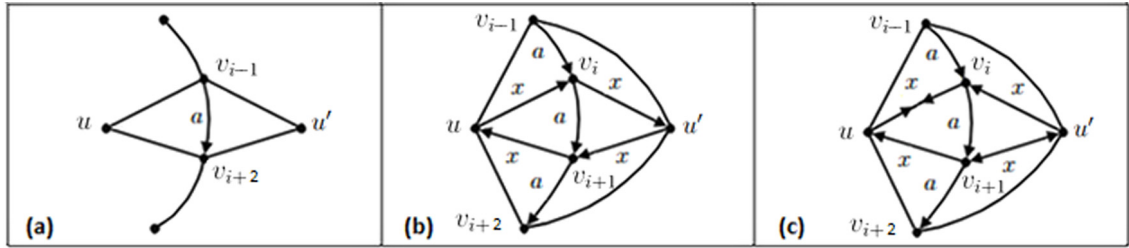
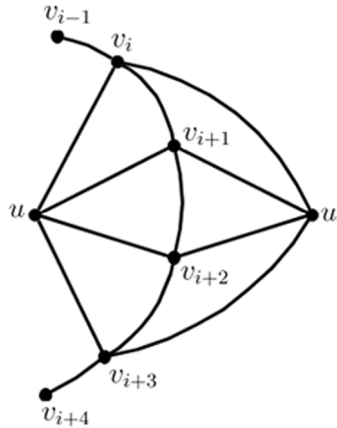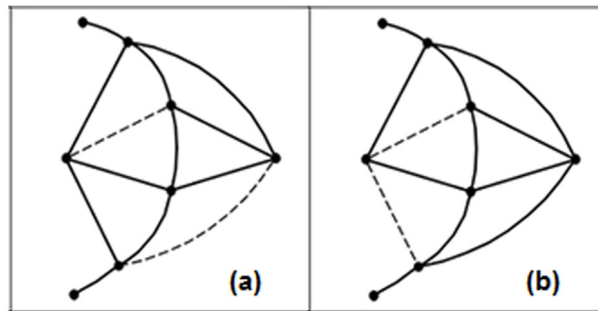**Fig. 8.** Types of signed Subgraphs of B_n.

Fig. 9. Graph H.

Fig. 10. Types of signed subgraph H.

In the two following lemmas assume that the cycle C_n in B_n is balanced. So, we can consider the sign of all edges of C_n positive.

Lemma 3.7. *Let* $n \geqslant 6$ *be an even positive integer and* $(B_n, \sigma) = (C_n \vee K_2, \sigma)$. *Suppose that* v_i *and* v_{i+1} *are two adjacent vertices of* C_n, *and* $B_{n-2} = (B_n \setminus \{v_i, v_{i+1}\}) \cup \{v_{i-1}v_{i+2}\}$ *where the signature on the common edges of* B_n *and* B_{n-2} *is the same and also* $\sigma(v_{i-1}v_{i+2}) = \sigma(v_{i-1}v_i)\sigma(v_{i+1}v_{i+2})$. *Then* B_n *has a nowhere-zero 3-flow provided that* B_{n-2} *has a nowhere-zero 3-flow and the induced subgraph* $\langle u, u', v_i, v_{i+1} \rangle$ *is one of the Graphs (a) or (c) given in* Fig. 7.

Proof. Assume that there is a nowhere-zero 3-flow on B_{n-2}. Consider Graph (a) given in Fig. 8 as a subgraph of B_{n-2} in which $a \in \{\pm 1, \pm 2\}$ is the assigned value of the edge $v_{i-1}v_{i+2}$. We can find a nowhere-zero 3-flow on B_n by adding two vertices v_i and v_{i+1} to B_{n-2}. Let induced subgraph $\langle u, u', v_i, v_{i+1} \rangle$ be one of the Graphs (a) or (c) given in Fig. 7. It is sufficient to assign the value a to the edges $v_{i-1}v_i$, v_iv_{i+1} and $v_{i+1}v_{i+2}$ as well as choose a value x with $x \in \{\pm 1, \pm 2\}$, and then assign value x to the edges v_iu, v_iu', $v_{i+1}u$ and $v_{i+1}u'$, see Graphs (b) and (c) in Fig. 8. Hence, B_n has a nowhere-zero 3-flow. $\square$
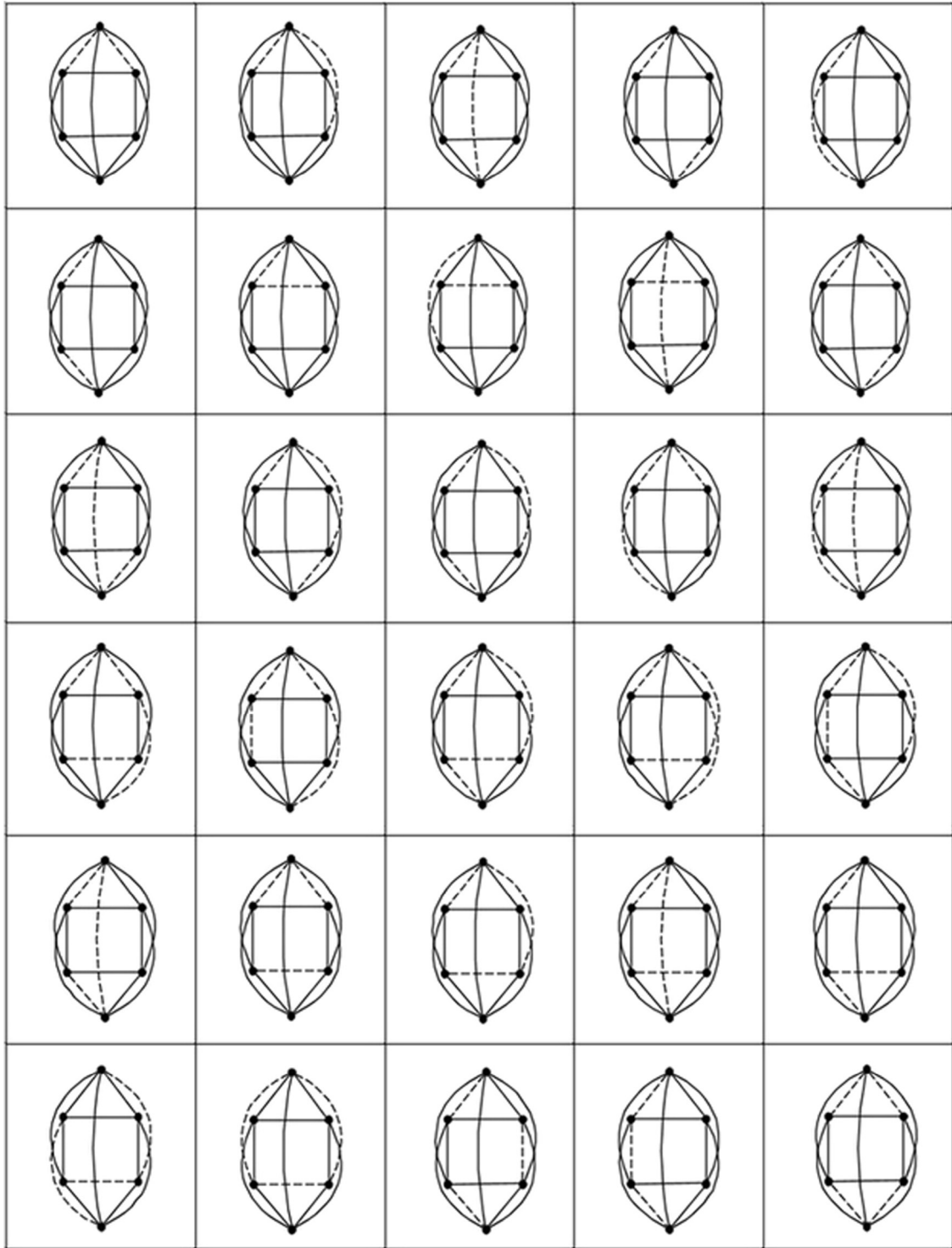
Fig. 11. Signed biwheel B_4.

Now, if B_n has no Graph (a) or Graph (c) given in Fig. 7, as a subgraph, then we need to consider a larger subgraph of B_n in order to find a nowhere-zero flow on B_n.
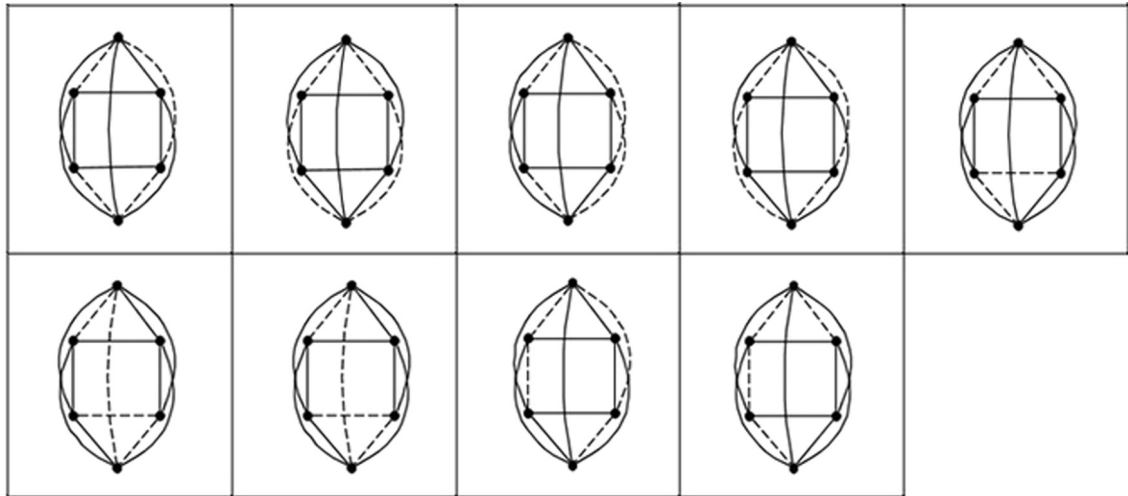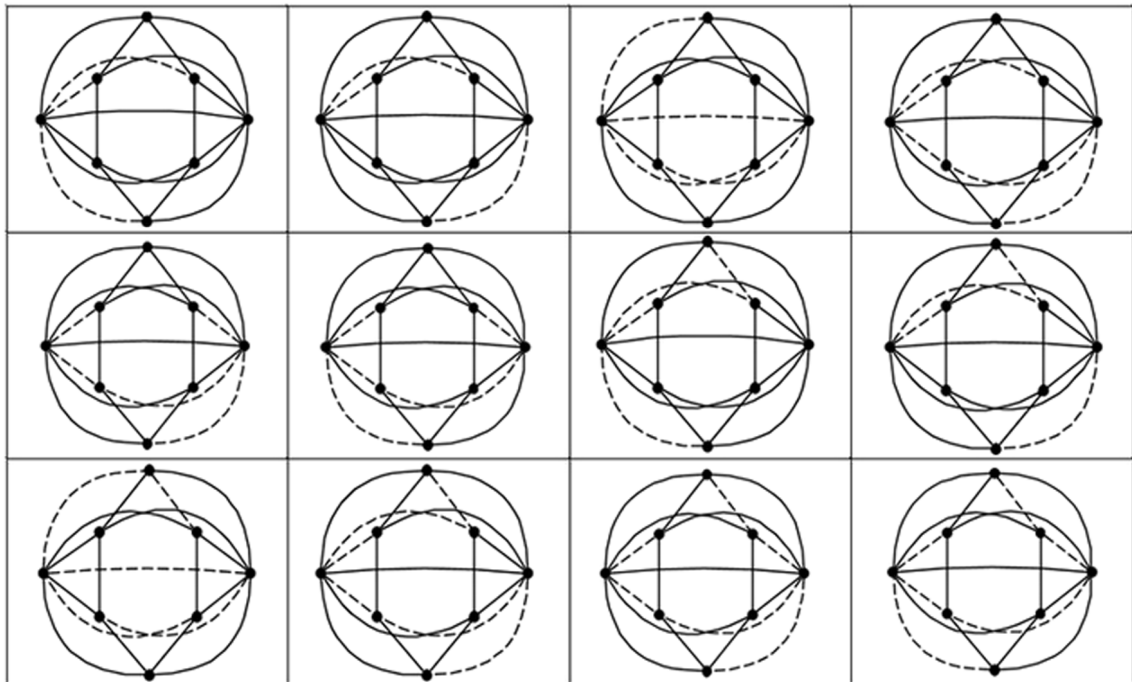
Fig. 12. Signed biwheel B_4.

Fig. 13. Signed biwheel B_6.

Lemma 3.8. *Let $n \geqslant 8$ be an even positive integer and $(B_n, \sigma) = (C_n \vee K_2, \sigma)$. Assume that for any two adjacent vertices such as v_i and v_{i+1} of C_n, the induced subgraph $\langle u, u', v_i, v_{i+1} \rangle$ is Graph (b) given in* Fig. 7. *Suppose that $B_{n-4} = (B_n \setminus \{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}) \cup \{v_{i-1}v_{i+4}\}$ where the signature on the common edges of B_n and B_{n-4} is the same and also $\sigma(v_{i-1}v_{i+4}) = \sigma(v_{i-1}v_i)\sigma(v_{i+3}v_{i+4})$. Then B_n has a nowhere-zero 3-flow if B_{n-4} has a nowhere-zero 3-flow.*

Proof. Up to switching isomorphism, there are two types for the signature of the edges of a signed subgraph H given in Fig. 9, in which they do not contain Graph (a) or (c) in Fig. 7 as a subgraph. They are presented in Fig. 10. For example, consider Graph (a) given in Fig. 10. We use two values x and y such that $x, y \in \{\pm 1, \pm 2\}$. Assign x into the edges $v_i u$, $v_i u'$, $v_{i+2} u$, and $v_{i+2} u'$, and set the value y into the edges $v_{i+1}u$, $v_{i+1}u'$, $v_{i+3}u$, and $v_{i+3}u'$. Also, send the value a from v_{i-1} to v_{i+4} along the path $v_{i-1}v_iv_{i+1}v_{i+2}v_{i+3}v_{i+4}$. Note that the edges $v_{i+1}u$ and $v_{i+3}u'$ are introverted and extroverted,

respectively. Moreover for another type, Graph (b) given in Fig. 10, one can do similarly. Hence, we can conclude that if the B_{n-4} has a nowhere-zero 3-flow, then B_n has also.　□

Note that without loss of generality in Lemmas 3.7 and 3.8, one can also consider the cycle C_n unbalanced (with just one negative edge). Moreover, it is not hard to check that in Lemmas 3.7 and 3.8, if B_n is flow-admissible, then there is no case B_{n-2} and B_{n-4} with just one negative edge (otherwise, the signed biwheel is not flow-admissible). Therefore, we can conclude that for even n with $n \geqslant 8$, signed graph B_n with any signature has a nowhere-zero 3-flow, see the following theorem.

Theorem 3.9. *Let n be an even positive integer with $n \geqslant 4$. Each signed biwheel B_n admits a nowhere-zero 3-flow.*

Proof. We prove the assertion using induction on $n \geqslant 4$. By Lemma 3.5, signed biwheels B_4 admit a nowhere-zero 3-flow. Then assume that $n \geqslant 6$. If there is one of the Graphs (a) or (c) given in Fig. 7, as a subgraph of B_n, then using Lemmas 3.5 and 3.7, we can find a nowhere-zero 3-flow for B_n. Otherwise, all the subgraphs (up to switching isomorphism) are the form of Graph (b) in Fig. 7. In this case, use Lemma 3.8 and also apply the hypothesis induction to prove the existence a nowhere-zero 3-flow on B_n for $n \geqslant 8$. Now, let $n = 6$, and all subgraphs are the form of Graph (b) in Fig. 7. All types of signed biwheel B_6 with this property are listed in Fig. 13. One can check that each of them admits a nowhere-zero 3-flow.　□

Theorem 3.10. *For a signed biwheel B_n, we have the following conditions:*

  (i) *If n is odd, then B_n has a nowhere-zero 4-flow,*
 (ii) *If n is even, then B_n has a nowhere-zero 3-flow.*

Proof. For odd n, B_n is an eulerian graph, so by Theorem 1.1, B_n with any signature has a nowhere-zero 4-flow. Moreover for even n, by Theorems 3.4 and 3.9, the proof is completed.　□

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