SELECTED OPEN PROBLEMS IN MATCHING UNDER PREFERENCES

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Abstract

The House Allocation problem, the Stable Marriage problem and the Stable Roommates problem are three fundamental problems in the area of matching under preferences. These problems have been studied for decades under a range of optimality criteria, but despite much progress, some challenging questions remain open. The purpose of this article is to present a range of key open questions for each of these problems, which will hopefully stimulate further research activity in this area.

1 Introduction

Matching markets involve allocating a set of agents to a set of resources (e.g., pupils to schools, students to projects), or allocating one set of agents to a different

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set of agents (e.g., school leavers to universities, junior doctors to hospitals), or allocating a set of agents to one another (e.g., forming teams for groupwork). Typically agents declare which other agents / resources they find acceptable, and may rank their acceptable potential matches in order of preference.

Matching problems involving ordinal preferences over outcomes have received a great deal of attention in the literature from computer scientists, mathematicians and economists, as evidenced by a range of research monographs on the topic spanning these disciplines [61, 45, 89, 68]. The study of these problems dates back to the seminal paper of Gale and Shapley [39], whose efficient algorithm for the so-called *Stable Marriage problem*, known as the Gale–Shapley algorithm (or the Deferred Acceptance mechanism), has been deployed in many real-world matching markets [20].

One of the earliest practical applications of the Gale–Shapley algorithm was in the allocation of graduating medical students to their first posts as junior doctors in US hospitals [85]. This algorithm has been used as part of the National Resident Matching Program (NRMP) since 1952, thus predating Gale and Shapley's paper by 10 years [89]. Nowadays, over 40,000 applicants to the NRMP are handled by an extension of the Gale–Shapley algorithm [78].

Other practical settings where algorithms are deployed to clear matching markets include [20]:

- school placement in Boston [1, 3] and New York [2]
- higher education admission in China [103, 104], Germany [22] and Hungary [21];
- university faculty recruitment in France [15, 16];
- placing military cadets in branches [96];
- assigning kidney patients to donors through kidney exchanges [86, 87, 88];
- allocating students to projects in a university department [7, 32].

Some of these markets are very large (for example, in China, over 10 million students apply for admission to higher education annually through a centralised process [103]) and thus it is of paramount importance that the clearing algorithms are efficient. Moreover, the outcomes for the agents involved may impact heavily on their quality of life, which motivates the design of algorithms producing matchings that are optimal in a precise sense according to the agents' preferences.

The importance of this research area was recognised in 2012 through the award of the Nobel Prize in Economic Sciences to Al Roth and Lloyd Shapley "for the theory of stable allocations and the practice of market design" [79]. This has sparked renewed interest in matching problems: subsequently several book chapters on the topic have been published [60, 20, 33, 23], another edited volume is in progress [37], and the second edition of the Encyclopedia of Algorithms [57] includes many entries that survey algorithms for matching problems. See also [24] for a very recent survey. As far as meetings are concerned, the International Workshop on Matching Under Preferences (MATCH-UP) continues to flourish [70], established conferences such as COMSOC, EC, ICALP, IJCAI, SAGT, SODA and WINE have included many papers on matching problems in recent years, and in July 2020 the first Dagstuhl seminar entirely devoted to matching under preferences will take place [93].

The time is right, therefore, for an updated list of some of the key open problems in matching under preferences. Twelve open problems in this area were presented by Gusfield and Irving [45], and updates on these were given by Manlove [68], who in turn posed a number of additional open questions on related problems. The purpose of this article is to select some of the problems from [45] and [68] that are still open, giving updates on progress to date, and to describe some new open problems of the authors' choosing.

We categorise matching problems involving preferences according to whether the market is (i) bipartite or (ii) non-bipartite. Further, those problems in category (i) can be further classified according to whether the market involves (a) a set of agents and a set of resources (so agents have preferences over resources, but not vice-versa) or (b) two disjoint sets of agents, where each agent from one set has preferences over a subset of agents from the other set. The archetypal problem in class (i)(a) is called the *House Allocation problem* [50, 105, 4] in view of its application to campus housing allocation in universities [27, 81]. The fundamental problem in class (i)(b) is the *Stable Marriage problem* [39, 45], which has applications to junior doctor allocation [85] and higher education admission [21]. Finally the key problem in class (ii) is the *Stable Roommates problem* [51, 45] which has applications to P2P networking [65], and team allocation, for example in chess tournaments [62].

We present open questions relating to each of these fundamental problems. The remainder of this article is structured as follows. In Section 2, we provide formal definitions of the problems and solution concepts studied in this article. Then in Section 3 we present open questions relating to the House Allocation problem, the Stable Marriage problem and the Stable Roommates problem.

2 Problem definitions

2.1 House Allocation problem

An instance I of the *House Allocation problem* (HA), also known as the Assignment problem, comprises a set $A = \{a_1, a_2, \ldots, a_{n_1}\}$ of *applicants* and a set $H =$

$\{h_1, h_2, \ldots, h_{n_2}\}$ of *houses*. There is a set $E \subseteq A \times H$ of *acceptable* applicant–house pairs. Let $m = |E|$. Each applicant $a_i \in A$ has an *acceptable* set of houses $A(a_i)$, where $A(a_i) = \{h_j \in H : (a_i, h_j) \in E\}$. Similarly each house $h_j \in H$ has an acceptable set of applicants $A(h_j)$, where $A(h_j) = \{a_i \in A : (a_i, h_j) \in E\}$.

Each applicant $a_i \in A$ has a *preference list* which is a strict linear order $\prec_{a_i}$ over $A(a_i)$.[1] Given an applicant $a_i \in A$, and given two houses $h_j, h_k \in A(a_i)$, a_i is said to *prefer* h_j to h_k if $h_j \prec_{a_i} h_k$. For a given acceptable applicant–house pair (a_i, h_j), define $rank(a_i, h_j)$ to be 1 plus the number of houses that a_i prefers to h_j.

An *assignment* M is a subset of E. If $(a_i, h_j) \in M$, a_i and h_j are said to be *assigned to* one another. For each $p_k \in A \cup H$, the set of assignees of p_k in M is denoted by $M(p_k)$. If $M(p_k) = \emptyset$, p_k is said to be *unassigned*, otherwise p_k is *assigned*. A *matching* M is an assignment such that $|M(p_k)| \leq 1$ for each $p_k \in A \cup H$. For notational convenience, if p_k is assigned in M then where there is no ambiguity the notation $M(p_k)$ is also used to refer to the single member of the set $M(p_k)$. Let $\mathcal{M}$ denote the set of matchings in I.

The preferences of an applicant extend to $\mathcal{M}$ as follows. Given two matchings $M, M' \in \mathcal{M}$, we say that an applicant $a_i \in A$ *prefers* M' *to* M if either (i) a_i is assigned in M' and unassigned in M, or (ii) a_i is assigned in both M and M', and a_i prefers $M'(a_i)$ to $M(a_i)$.

Given this definition, we may define a relation $\lhd$ on $\mathcal{M}$ as follows: if $M, M' \in \mathcal{M}$ then $M' \lhd M$ if no applicant prefers M to M', and some applicant prefers M' to M. If $M' \lhd M$ then M' is called a *Pareto improvement* of M. It is straightforward to establish that $\lhd$ is a partial order on $\mathcal{M}$. A matching $M \in \mathcal{M}$ is defined to be *Pareto optimal* if M is $\lhd$-minimal. Equivalently, M is Pareto optimal if and only if there is no other matching M' in I such that (i) some applicant prefers M' to M, and (ii) no applicant prefers M to M'.

Another optimality criterion for an HA instance I is *popularity*. Let $M, M' \in \mathcal{M}$, and let $P(M, M')$ denote the set of applicants who prefer M to M'. Define a "more popular than" relation $\blacktriangleleft$ on $\mathcal{M}$ as follows: if $M, M' \in \mathcal{M}$, then M' is *more popular than* M, denoted $M' \blacktriangleleft M$, if $|P(M', M)| > |P(M, M')|$. (Note that $\blacktriangleleft$ is not in general a transitive relation on $\mathcal{M}$.) Define a matching $M \in \mathcal{M}$ to be *popular* if M is $\blacktriangleleft$-minimal (i.e., there is no other matching M' such that $M' \blacktriangleleft M$). Thus, put simply, M is popular if there is no other matching that is preferred by a majority of the applicants who are not indifferent between the two matchings.

A further notion of optimality is based on the *profile* of a matching. Given a matching $M \in \mathcal{M}$, define the *degree* of M, denoted $d(M)$, to be the maximum rank of an applicant's partner in M. Formally define

$$d(M) = \max\{rank(a_i, h_j) : (a_i, h_j) \in M\}.$$

[1] That is, $\prec_{a_i}$ is an irreflexive, transitive and linear binary relation over $A(a_i)$.

The *profile* of M, denoted by $p(M)$, is a vector $\langle p_1, \ldots, p_d \rangle$, where $d = d(M)$ and for each k ($1 \leq k \leq d$), $p_k = |\{(a_i, h_j) \in M : rank(a_i, h_j) = k\}|$. Intuitively, p_k is the number of applicants who have their kth-choice house in M.

A matching M is *rank-maximal* if $p(M)$ is lexicographically maximum, taken over all matchings in $\mathcal{M}$. Intuitively, in such a matching, the maximum number of applicants are assigned to their first-choice house, and subject to this, the maximum number of applicants are assigned to their second-choice house, and so on.

A natural extension of HA arises when applicants are permitted to have ties in their preference lists—in this case each applicant $a_i \in A$ has a weak linear order $\preceq_{a_i}$ over $A(a_i)$.[2] This gives rise to the *House Allocation problem with Ties* (HAT). In this case each of the definitions of a Pareto optimal, popular and rank-maximal matching, as defined above for the HA case, carry over to HAT without alteration.

2.2 Stable Marriage problem

The *Stable Marriage problem with Incomplete lists* (SMI) can be regarded as a variant of HA in which houses have preferences over applicants. In the SMI context, applicants and houses are more commonly referred to as *men* and *women* respectively. Formally, an instance I of SMI comprises a set $U = \{m_1, m_2, \ldots, m_{n_1}\}$ of men and a set $W = \{w_1, w_2, \ldots, w_{n_2}\}$ of women. The definitions of E, *acceptable* man–woman pairs, m, $A(m_i)$ and $A(w_j)$ for each $m_i \in U$ and $w_j \in W$ are analogous to the HA case.

For each $m_i \in U$, the definitions of *preference list*, *prefer* and *rank* are analogous to the HA case. Additionally, each woman $w_j \in W$ has a *preference list* in which she ranks $A(w_j)$ in strict order. The definitions of *prefer* and *rank* for w_j are analogous to the definitions for the men. Likewise, the definitions of *assignment*, *assigned to*, $M(p_k)$ for any $p_k \in U \cup W$, *assigned*, *unassigned* and *matching* are analogous to the HA case.

A *blocking pair* of matching M in I is a man–woman pair $(m_i, w_j) \in E$ such that m_i is unmatched or prefers w_j to $M(m_i)$, and w_j is unmatched or prefers m_i to $M(w_j)$. Matching M is *stable* if it admits no blocking pair.

The *Stable Marriage problem* (SM) is the special case of SMI in which $n_1 = n_2$ and $E = U \times W$. We refer to $n = n_1 = n_2$ as the *size* of the given instance I, and it is assumed that any matching M in I has size n. When preference lists may include ties, each of SM and SMI may be generalised to the *Stable Marriage problem with Ties* (SMT) and the *Stable Marriage problem with Ties and Incomplete lists* (SMTI) respectively, without modification to the stability definition.

[2]That is, $\preceq_{a_i}$ is a reflexive, transitive and linear binary relation over $A(a_i)$. Given two houses $h_j, h_k \in A(a_i)$, a_i is said to *prefer* h_j to h_k if $h_j \preceq_{a_i} h_k$ and $h_k \npreceq_{a_i} h_j$, whilst a_i is said to be *indifferent between* h_j and h_k if $h_j \preceq_{a_i} h_k$ and $h_k \preceq_{a_i} h_j$; in the latter case, h_j and h_k are said to belong to a *tie* in a_i's list.

2.3 Stable Roommates problem

The *Stable Roommates problem with Incomplete lists* (SRI) is a non-bipartite generalisation of SMI. An instance I of SRI comprises a set $A = \{a_1, a_2, \dots, a_n\}$ of *agents*. We refer to n as the *size* of I. There is a set $E \subseteq \{X \subseteq A : |X| = 2\}$ of *acceptable* agent pairs. The definitions of m and $A(a_i)$ for each $a_i \in A$ are analogous to the HA case. For each $a_i \in A$, the definitions of *preference list*, *prefer* and *rank* are also analogous to the HA case. Likewise, the definitions of *assignment*, *assigned to*, $M(a_i)$ for any $a_i \in A$, *assigned*, *unassigned* and *matching* are analogous to the HA case.

A *blocking pair* of matching M in I is a pair of agents $\{a_i, a_j\} \in E$ such that a_i is unmatched or prefers a_j to $M(a_i)$, and a_j is unmatched or prefers a_i to $M(a_j)$. Matching M is *stable* if it admits no blocking pair.

The *Stable Roommates problem* (SR) is the special case of SRI in which $|E| = n(n-1)/2$. When preference lists may include ties, each of SR and SRI may be generalised to the *Stable Roommates problem with Ties* (SRT) and the *Stable Roommates problem with Ties and Incomplete lists* (SRTI) respectively, without modification to the stability definition.

3 Open problems

We present open problems relating to HA, SM and SR in Sections 3.1, 3.2 and 3.3 respectively.

3.1 House Allocation problem

In this section we present open problems relating to the approximability of a relaxed notion of popular matchings called "least unpopular matchings" in HA (Section 3.1.1), popular and rank-maximal matchings under three models of uncertainty in applicants' preferences (Section 3.1.2), and finally HA organised via house exchanges in a so-called social network (Section 3.1.3).

3.1.1 Approximability of least unpopular matchings

In HA, the existence of a popular matching is not guaranteed. Figure 1 depicts an instance equivalent to the famous voting paradox of Condorcet [31], where none of the matchings is popular. In this context, the following result answers the most striking algorithmic question of the topic.

Theorem 1 ([6]). *There is algorithm that outputs either a (largest cardinality) popular matching or a proof for its nonexistence. For* HAT, *the algorithm runs in* $O(\sqrt{n}m)$ *time, which is reduced to* $O(n + m)$ *for* HA.

$a_1 \; : h_1 \quad h_2$
$a_2 \; : h_1 \quad h_2$
$a_3 \; : h_1 \quad h_2$

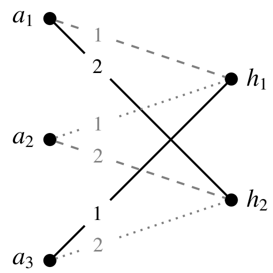

Figure 1: No popular matching exists in this instance. The lists on the left and the numbers on the edges in the figure on the right are the preferences of each applicant. All applicants prefer h_1 to h_2. The dotted gray matching $\{(a_2, h_1), (a_3, h_2)\}$ is more popular than the dashed gray matching $\{(a_1, h_1), (a_2, h_2)\}$, because both a_2 and a_3 prefer it. Similarly, the black matching $\{(a_1, h_2), (a_3, h_1)\}$ defeats the dotted gray, and the dashed gray defeats the black.

Having established in Theorem 1 that we can distinguish between instances with and without popular matchings in polynomial time, the relaxation of popularity is the next intuitive move.

First, the notion of *least unpopular* matchings was proposed to deal with instances that have no popular matchings [71]. Assume that M_1 and M_2 are two matchings in the same instance. We say that M_2 dominates M_1 by a factor of $\frac{u}{v}$, if $|P(M_2, M_1)| = u$ and $|P(M_1, M_2)| = v$. To be precise, this factor is $\frac{u}{v}$ if $v \neq 0$, it is 1 if $u = v = 0$, and it is ∞ if $v = 0$ and $u > 0$. For instance, matching $\{(a_2, h_1), (a_3, h_2)\}$ in Figure 1 dominates matching $\{(a_1, h_1), (a_2, h_2)\}$ by a factor of 2. The *unpopularity factor* of a matching M is the maximum factor by which it is dominated by any other matching. According to this definition, a matching is popular if and only if its unpopularity factor is exactly 1.

McCutchen [71] also defined an alternative concept to measure the degree of popularity, called the *unpopularity margin*. Using the same u and v as above, M_2 dominates M_1 by a margin of $u - v$, instead of $\frac{u}{v}$. Returning to the same example in Figure 1, we can state that $\{(a_2, h_1), (a_3, h_2)\}$ dominates $\{(a_1, h_1), (a_2, h_2)\}$ by a margin of 1. The *unpopularity margin* of M is the maximum margin by which M is dominated by any other matching. According to this definition, a matching is popular if and only if its unpopularity margin is exactly 0.

Theorem 2 ([71], [68])**.** *For* HAT*, there is an $O(m\sqrt{n})$ time algorithm to find the unpopularity factor of a matching and there is an $O(m\sqrt{n} \cdot \log n)$ time algorithm to find the unpopularity margin of a matching.*

Theorem 3 ([71])**.** *Each of the problems of finding a least unpopularity factor matching and a least unpopularity margin matching is NP-hard in* HA*. Also, there is no approximation algorithm for the problem of finding a least unpopularity* *factor matching in a given* HA *instance with performance guarantee* $\frac{3}{2} - \varepsilon$*, for any* $\varepsilon > 0$*, unless P=NP.*

Huang et al. [48] presented a polynomial-time iterative algorithm for finding a matching with low unpopularity factor or margin in random instances. It constructs a sequence of k graphs such that the original instance admits a popular matching with unpopularity factor at most $k - 1$ and unpopularity margin at most $n_1 \left(1 - \frac{2}{k}\right)$ if the k-th graph in the sequence admits a perfect matching. However, future work could explore least unpopular matchings further by finding a better approximation algorithm or proving a tighter inapproximability bound.

3.1.2 Popular and rank-maximal matchings under uncertain preferences

Applicants' preferences may not be completely known because of a lack of information or communication. The reason for an applicant in an HAT instance placing two houses in the same tie might be that she has not gained sufficiently detailed information about these two options to differentiate between them. Also, she might not be able to receive all the information she requires about each available house, because it would take too many rounds of communication. However, later the applicant might refine her preferences, if she gains extra information.

Aziz et al. examined Pareto optimal matchings in variants of HA [14, 11], and also stable matchings in SM [13] under uncertain preferences. They set up three different models to capture the uncertainty of agents. In the *lottery model*, a probability distribution over strict preference lists is given for each agent. In the *compact indifference model*, each agent reports a single, weakly ordered preference list (i.e., a preference list with ties as in SMTI). Each complete linear order extension of this weak order is assumed to be equally likely. Finally, in the *joint probability model*, a probability distribution over the possible sets of preference lists is specified.

Besides other, more restricted problems, the following key computational problems are formulated.

1. PO / SM PROBABILITY: What is the probability that a given matching is Pareto optimal / stable?

2. PO / SM HIGHESTPROBABILITY: Compute a matching with the highest probability of being Pareto optimal / stable.

We summarise the most important findings from [13, 14, 11] in Table 1.

In another paper [12], similar complexity problems are studied, but instead of the three uncertainty models, the input has uncertain pairwise preferences. This means that each agent only expresses a probability of preferring one agent over

	lottery	compact indifference	joint probability
PO Probability	#P-complete	#P-complete	polynomial
PO HighestProbability	NP-hard	NP-hard	NP-hard
SM Probability	#P-complete	?	polynomial
SM HighestProbability	?	NP-hard	NP-hard

Table 1: The complexity table of the two most central questions (rows) in the three different uncertainty models (columns) from [13, 14, 11]. Question marks denote the open cases.

another for all possible pairs. If this probability is 0 or 1, then the agent has expressed *certain preferences*. In this framework, the complexity of each problem mainly depends on the transitivity of certain preferences.

One obvious direction forward is to fill the gaps in Table 1. After Pareto optimal and stable matchings, one might consider to investigate other fairness notions under uncertain preferences, such as popular or rank-maximal matchings.

3.1.3 House allocation via exchanges in a social network

In a well-studied and realistic variant of HA, each applicant a_i initially owns a house $M(a_i)$, and the goal of the applicants is to exchange these houses among themselves [95]. These exchanges can happen either in a centralised [95, 5, 84, 96, 10] or a decentralised manner [92, 29, 34, 43].

The latter case was first studied in a social network by Gourvés et al. [43]. Two applicants in a social network either know each other, and they are capable of exchanging their houses, or they do not know each other, in which case no exchange can happen. More formally, each applicant a_i has an acceptable set $A(a_i)$ of houses, as in the other variants of HA discussed above, but in addition a_i has an acceptable set $A_a(a_i)$ of applicants. These $A_a(a_i)$ sets trivially define a *social network* N, which is a graph $N = (A, E^*)$, where $\{a_i, a_j\} \in E^*$ if and only if $a_j \in A_a(a_i)$ and $a_i \in A_a(a_j)$. As in the case of HA, applicant a_i ranks $A(a_i)$, including her initial endowment, in strict order, and she is only inclined to participate in an exchange if the house she receives is better than the one she currently owns.

In a social network it can happen that an exchange between applicants a_i and a_j that was infeasible earlier becomes feasible later. If $a_j \in A_a(a_i)$, and a_i finds $M(a_j)$ worse than her own house $M(a_i)$, then a_i will not accept $M(a_j)$ in any feasible exchange. However, if a_j participates in an exchange following which she receives a house $M'(a_j)$ that a_i ranks higher than her current house, then a_i suddenly becomes interested in accepting a_j's new house $M'(a_j)$.

Gourvés et al. [43], Saffidine and Wilczynski [91], and Bentert et al. [18] restricted their attention to *swaps* in social networks. Swaps are exchanges of length 2, involving two applicants only, who swap their houses with each other, changing $\{(a_i, M(a_i)), (a_j, M(a_j))\}$ to $\{(a_i, M(a_j)), (a_j, M(a_i))\}$ in the new matching. If a matching M' is reachable from the initial matching M by a sequence of such swaps, then we call it a *reachable matching*. Similarly, houses that an applicant a_i can receive via swaps belong to the set of *reachable houses*. Pareto optimal matchings are also defined based on swaps. A matching M' is considered to be Pareto optimal if it is reachable from M and there is no other reachable matching M'' that Pareto-dominates M'. We summarise the main results from [43, 18] in the following three theorems.

Theorem 4 ([43, 18])**.** *The problem of deciding whether house h_j is reachable for applicant a_i is NP-complete even if the network N is a tree, a clique, a generalised caterpillar*[3]*, or if each agent finds at most 4 houses acceptable. The problem becomes polynomially solvable if N is a path, a star, or if each agent finds at most 3 houses acceptable.*

Theorem 5 ([43])**.** *The problem of deciding whether matching M' is reachable from matching M is NP-complete, but it becomes polynomially solvable if N is a tree.*

Theorem 6 ([43])**.** *The problem of finding a Pareto optimal matching is NP-hard even if the network N is a tree, but it becomes polynomially solvable if N is a star.*

Table 2 contains a structured interpretation of the above results for the most relevant graph classes. On the reachability of a house (Theorem 4), further results were derived in a parameterised complexity setting, focusing on budget constraints such as the number of exchanges an agent may be involved in or the total duration of the process [91].

	path	star	tree	general
h REACHABLE	polynomial	polynomial	NP-complete	NP-complete
M REACHABLE	polynomial	polynomial	polynomial	NP-complete
FIND PO MATCHING	polynomial	polynomial	NP-complete	NP-complete

Table 2: The complexity table summarising the three problems first posed in [43], corresponding to the three rows, and also to Theorems 4, 5, and 6, in this order. The columns are the type of graph for which the complexity results hold.

The most striking open question regarding exchanges in social networks involves longer cycles instead of swaps only. Strategyproofness is another topic worthy of investigation. Also, the hard cases in Theorems 5 and 6 could be tackled from a parameterised complexity viewpoint.

[3] A *generalised caterpillar* is a tree in which all vertices of degree more than two are on a path.

3.2 Stable Marriage problem

In this section we present open problems relating to finding the maximum number of stable matchings admitted by an SM instance of a given size (Section 3.2.1), determining whether there is a path from an arbitrary matching to a stable matching in the so-called "divorce digraph" (Section 3.2.2), finding a maximum cardinality stable matching in an instance of SMTI (Section 3.2.3) and finally determining whether there is an efficient algorithm to list all stable matchings in a given instance of SMT (Section 3.2.4).

3.2.1 Maximum number of stable matchings

This open problem, first posed by Gusfield and Irving [45], concerns finding the maximum number x_n of stable matchings admitted by any SM instance of size n. Formally, for a given SM instance I, let $\mathcal{S}_I$ denote the set of stable matchings in I, and define

$$x_n = \max\{|\mathcal{S}_{I_n}| : I_n \text{ is an SM instance of size } n\}.$$

This problem is still open. However some progress has been made, which we now summarise.

Knuth [61, p.56] and Eilers [38] showed that $x_4 = 10$. More generally, if n is a power of 2, Irving and Leather [52] and Knuth (personal communication, reported in [45]) proved that $x_n > 2.28^n/(1+\sqrt{3})$ [52, 45]. Thurber [100] showed that x_n is a strictly increasing function of n, and also that $x_n > 2.28^n/(1+\sqrt{3})^{(\log n+1)}$ for each $n \geq 1$.

As far as upper bounds are concerned, Stathopoulos [97] showed that, for each $n \geq 4$, $x_n \leq n!/2^{n-3}$. A major advance was obtained recently by Karlin et al. [58], who showed that, for each $n \geq 1$, $x_n \leq c^n$ for some constant c (the authors report that, at the time of writing, $c \leq 2^{17}$). Nevertheless, clearly there remains a large gap between the best known lower and upper bounds for x_n.

3.2.2 Divorce digraph

Let I be an SM instance and let $\mathcal{M}_I$ be the set of matchings in I. The *divorce digraph* of I is a digraph $D_I = (V, A)$, where D_I contains a vertex for each matching in $\mathcal{M}_I$ (so $|V| = n!$, where n is the size of I), and the edges in D_I are defined as follows. Given two matchings M, M' in $\mathcal{M}_I$, we say that M' can be obtained from M by a *divorce operation* (referred to as a *b-interchange* in [98]) if

$$M' = (M \backslash \{(m, M(m)), (M(w), w)\}) \cup \{(m, w), (M(w), M(m))\}$$

for some blocking pair (m, w) of M. (Thus in M', the man and woman involved in the blocking pair are matched together, and the "divorcees" are also matched together.) Given two vertices v_M, $v_{M'}$ in V, corresponding to matchings M and M' in $\mathcal{M}_I$ respectively, $(v_M, v_{M'}) \in A$ if and only if M' can be obtained from M by a divorce operation. It follows that $v_M \in V$ is a sink vertex of D_I if and only if the corresponding matching $M \in \mathcal{M}_I$ is stable. Knuth [61, pp.2–3] showed that D_I could contain cycles. Gusfield and Irving [45] posed the question as to whether, given an arbitrary matching $M_0 \in \mathcal{M}_I$, we can always find a path in D_I from v_{M_0} to a sink vertex.

Tamura [98] solved this problem by constructing an SM instance I_4 and identifying a set $\mathcal{M}_0 \subseteq \mathcal{M}_{I_4}$ such that (i) I_4 has size 4, (ii) five of the $4! = 24$ matchings in $\mathcal{M}_{I_4}$ are stable, (iii) $|\mathcal{M}_0| = 16$, and (iv) given any matching $M_0 \in \mathcal{M}_0$, there is no path in D_{I_4} from v_{M_0} to a sink vertex. Tan and Su [99] independently solved this problem by constructing an SM instance I'_4 of size 4 and a set of stable matchings $\mathcal{M}'_0$ in I'_4 with similar properties (we remark that $I'_4 \neq I_4$).

Tamura [98] also gave an algorithm to produce a stable matching from an initial matching M_0 using a combination of divorce operations and an additional type of step. More formally, he gave an algorithm that traverses a path from v_{M_0} in D_I until either the path reaches a sink vertex or it cycles; in the latter case a matching M'_0 is constructed such that the set of blocking pairs of M'_0 is a strict subset of those of M_0. The process repeats from $v_{M'_0}$ and eventually we must reach a stable matching.

Tamura was not able to determine whether this algorithm terminates in polynomial time. Moreover his algorithm might not exclusively use divorce operations even when there is a path from v_{M_0} to a sink vertex in D_I, due to a cycle being traversed in D_I instead. This leads to a natural open question, namely to resolve the complexity of the following decision problem: given an SM instance I and a matching M_0, is there a path in D_I from v_{M_0} to a sink vertex? Very recently, Chen [25] showed that this problem is NP-complete if I is an instance of SMTI.

Note that, if we drop the insistence, as in this subsection, that the "divorcees" marry one another as part of the divorce operation, then the landscape changes dramatically — see [90], [68, Section 2.6], and [28] for more details.

3.2.3 Approximability of MAX SMTI

Given an instance I of SMTI, the stable matchings in I can have different sizes [69], and MAX SMTI, the problem of finding a maximum size stable matching in I, is NP-hard [55, 69]. This result holds even if, simultaneously, each man's list is strictly ordered, each women's list is either strictly ordered or is a tie of length 2 [69], and each person's list is of length at most 3 [53, 74].

The NP-hardness of MAX SMTI also holds in the presence of *master lists*. A *master list* L of women (respectively men) is a uniform ranking of all women (men), possibly involving ties, such that the preference list of each individual man (woman) is derived from L by deleting his/her unacceptable partners (thus maintaining the ordering from L over his/her acceptable partners). When the preference lists on both sides are derived from a master list L_U of men and a master list L_W of women, it remains NP-hard to find a maximum size stable matching, and this remains true even if L_U is strictly ordered, whilst either (i) L_W contains a single tie or (ii) each tie in L_W is of length 2 [54].

Approximability results for MAX SMTI up to 2016 were surveyed in [73, 102]. At the time of writing those surveys, the best upper bound for the approximability of unrestricted MAX SMTI was 3/2 [72, 59, 80], whilst the best lower bound was 33/29, even if each tie is of length 2 [101]. This lower bound was improved to $4/3 - \varepsilon$, for any $\varepsilon > 0$ (for the same restriction), assuming the Unique Games Conjecture (UGC) [101]. These upper and lower bounds for general MAX SMTI remain the best known to date, and it is an open problem to close the gap between them.

The 3/2 barrier has been broken for special cases of MAX SMTI. For 1S-MAX SMTI, the restriction of MAX SMTI in which ties occur in the preference lists on one side only, the best upper bound up to 2016 reported in [73, 102] was 19/13 [35, 56], whilst the best lower bound was $21/19-\varepsilon$, for any $\varepsilon > 0$, which holds even if each woman's list is strictly ordered or is a tie of length 2 [101]. This lower bound was improved to $5/4-\varepsilon$, for any $\varepsilon > 0$ (for the same restriction), assuming UGC [101].

Bauckholt et al. [17] tightened the analysis of an earlier approximation algorithm for 1S-MAX SMTI [47] to improve the upper bound derived in [47] from 22/15 to 13/9. Very recently, Lam and Plaxton gave new approximation algorithms that improved the latter bound to $\ln 4 \approx 1.3863$ [63] and then to $(1 + 1/e) \approx 1.3679$ [64]. It is thus an open problem as to whether this latter upper bound can be reduced to $5/4$ in the case of 1S-MAX SMTI.

For the special case of 1S-MAX SMTI in which each tie occurs at the tail of some woman's list, Huang et al. [46] gave an approximation algorithm with performance guarantee 5/4, thus matching the known lower bound (assuming UGC). For the restriction of MAX SMTI in which ties can occur on both sides, but each tie is of length 2, Huang and Kavitha [47] gave an approximation algorithm with performance guarantee 10/7. A tighter analysis of this approximation algorithm was given by Chiang and Pashkovich [30], resulting in an improved upper bound of 4/3. Again, this matches the known lower bound (assuming UGC).

A summary of the lower and upper bounds for the approximability of MAX SMTI and its various restrictions is given in Figure 2. The obvious open problems correspond to closing the gaps between the lower and upper bounds. Furthermore, it would appear that no existing work on approximation algorithms for MAX SMTI has attempted to utilise the structure that master lists give rise to, and hence it is an open question as to whether improved upper bounds exist in the presence of master lists on one or both sides.

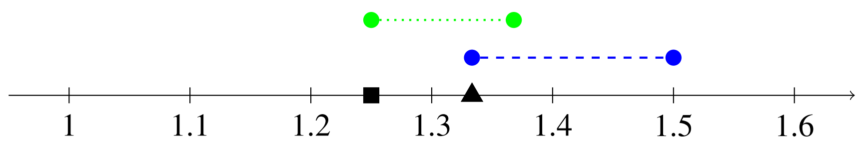

Figure 2: The figure depicts the gap between the most recent lower and upper bounds on the approximability in four cases of MAX SMTI (assuming UGC). The unrestricted case is denoted by the dashed blue segment, while the dotted green segment represents 1S-MAX SMTI. A tight approximation algorithm is known for 1S-MAX SMTI, if each tie occurs at the end of a list, which is denoted by the black square. The black triangle denotes the tight approximation for the case with ties of length at most 2.

3.2.4 Generating stable matchings in SMT

Given an instance I of SMT, let $\mathcal{S}_I$ denote the set of stable matchings in I. If I is an instance of SM of size n, it is known that all stable matchings in I can be listed in $O(n^2 + n|\mathcal{S}_I|)$ time and $O(n^2)$ space [44]. That is, the first stable matching can be output in $O(n^2)$ time, and each subsequent stable matching can be output in $O(n)$ time. On the other hand, it is an open problem as to whether there is an efficient algorithm for listing all stable matchings, given an instance I of SMT of size n. By *efficient*, we mean that the algorithm should have complexity $O(p(n) + |\mathcal{S}_I|q(n))$, where p and q are polynomial functions.

A partial result along these lines was, however, provided by Scott [94]. He showed that, given an SMTI instance I and a stable matching M in I, we can, in $O(L)$ time, find a stable matching $M' \neq M$ if one exists, or else report that M is unique, where L is the total length of the men's preference lists in I. On the other hand, if I is an instance of SMT in which the preference lists on both sides are derived from two master lists of men and women, all the stable matchings in I can be generated in $O(n + s + |\mathcal{S}_I| \log n)$ time, where s is the number of *stable pairs* (i.e., man–woman pairs who belong to some stable matching) in I [54].

3.3 Stable Roommates problem

In this section we present open questions for SR, relating to the probability that a given instance admits a stable matching (Section 3.3.1), the recently-introduced notions of robustness (Section 3.3.2), and the complexity of finding the weighted straight skeleton of a polygon (Section 3.3.3).

3.3.1 Solvability probability of random SR instances

A long-standing open problem, formulated as Problem 8 in Gusfield and Irving [45], is the computation of p_n, the probability that a random instance, chosen uniformly from all possible SR instances of size n, admits a stable matching. Of particular interest is the asymptotic behaviour of p_n, as n grows large.

The best-known lower and upper bounds are due to Pittel [82], and Pittel and Irving [83], respectively, and are as follows:

$$\frac{2e^{3/2}}{\sqrt{\pi n}} \leq p_n \leq \frac{\sqrt{e}}{2}$$

Mertens [76] derived an explicit formula for p_n. This formula is a sum over cycle types of permutations of size n, and each term in the sum is an integral with an exponential number of terms. Notice that, although the underlying ideas had already been discussed by Pittel [82], the formula itself had not been published before. Mertens [76] used code written in Mathematica to perform exact computations of p_n up to $n \leq 12$.

In addition to extensive simulations in [75] with the maximum value of n equal 20 000, ten years later Mertens [77] analysed the behaviour of Irving's algorithm for SR [51]. He showed that in random instances, Irving's algorithm only looks at $O(\sqrt{n})$ entries in each preference list, and so he was able to provide a modification that has average time and space complexity $O(n^{2/3})$. This enabled him to compute p_n for instances of size $n = n_0 2^k$ for $k = 0, \ldots, k_{max}$, and $n_0 \in \{8, 10, 12, 14\}$, where k_{max} is limited by the available memory. This helped computations for instances more than 500 times larger than previously studied ones. The obtained results support Mertens' conjecture [77] that $p_n = \Theta\left(n^{-1/4}\right)$.

Still, these results do not shed enough light on the ultimate behavior of p_n as n becomes large. It seems that exact evaluation of p_n for larger values of n is likely to be infeasible without some unexpected new approach.

3.3.2 Robust matchings

Robust matchings have been studied recently in the literature; intuitively they are stronger than stable matchings. Several different robustness notions have been introduced: they either require that a matching remains stable even if agents change their preferences slightly (perhaps because their original preferences were uncertain) or that the stability of a matching can be repaired at a small bounded cost in case some pairs in a stable matching break up.

Genc et al. [40] define an (a, b)-*supermatch* in an SMI or SRI instance I as a stable matching such that if any a non-fixed stable pairs[4] break up their assignments,

[4] A stable pair is *fixed* if it belongs to every stable matching in I.

it is possible to find another stable matching in I by changing the partners of the agents involved in those a pairs and by also changing at most b other pairs.

Genc et al. [42] showed that the problem of deciding if there exists a $(1, b)$-supermatch is NP-complete in SMI for any $b \geq 1$, which also implies that this problem is NP-hard in SRI. However, for the more general case of (a, b)-supermatches, it is not even known whether the problem belongs to NP. By contrast, given a stable matching M in an SRI instance, Genc et al. [41] gave a polynomial-time algorithm to verify whether M is a $(1, b)$-supermatch. The algorithm uses a deep knowledge of the structure of the set of all stable matchings, described by the complete closed subsets of the reduced rotation poset of the given SRI instance.

Genc et al. [41] also provided two metaheuristics for finding a $(1, b)$-supermatch for a given SRI instance that minimises the value of b, however, the approximability of this problem has not been studied theoretically.

Mai and Vazirani [66, 67] define robustness in a different way. Given an SM instance I, let $\mathcal{J}(I, D)$ be the set of instances that result after introducing one error from a domain D. The domains of errors may be, for example:

(i) For any agent a, swap the positions of two adjacent agents in the preference list of a.

(ii) For any agent a, shift a position of an agent in the preference list of a upwards (downwards).

(iii) For any agent a, arbitrarily permute the preference list of a.

Given a probability distribution for D, a *robust* stable matching is a matching that is stable in I and has the highest probability of being stable after introducing *one* error from D. A *fully robust* stable matching is a matching that is stable in I and in each of the instances in $\mathcal{J}(I, D)$.

For a given instance of SM, Mai and Vazirani proposed polynomial-time algorithms for finding a robust stable matching (for the domains defined by (ii) above) [66] and for checking if there is a fully robust stable matching (for the domains defined by (iii) above) [67]. Further, they proved that the set of all such matchings forms a sublattice of the lattice of all stable matchings. They used this structure to find a fully robust stable matching that maximises (or minimises) a given weight function.

Chen et al. [26] define a matching M in an instance of SMI to be *d-robust* if M is stable and remains stable after performing an arbitrary sequence of d errors of type (i), i.e., swaps. They provide an $O(n^4)$ algorithm that, given an SMI instance I and an integer $d \geq 0$, finds a matching that is d-robust if it exists. The result is also based on exploiting the structural properties of the rotations structure of I. Chen et al. [26] also provide a polynomial-time algorithm that finds a d-robust matching with minimum egalitarian cost if one exists. However, when ties are present, they show that finding a robust matching is NP-hard.

As far as we are aware, robust stable matchings in SR have not been studied previously (either in the senses of Mai and Vazirani [66, 67] or Chen et al. [26]).

3.3.3 The complexity of finding the weighted straight skeleton of a polygon

We complete our list of open problems with a seemingly unrelated problem from geometry. Aichholzer et al. [8, 9] defined the straight skeleton of a polygon. It is derived from shrinking the polygon by translating each of its edges at a fixed rate, as the edges of the polygon are moved inwards parallel to themselves at a constant speed, as illustrated in Figure 3. The vertices of the shrunk polygon trace angular bisectors, called *wavefront edges*, which build the *straight skeleton* of the polygon. In a generalized setting, the speed of moving the edges can differ for each edge—in this case we talk about a *weighted straight skeleton*. The (weighted) straight of skeleton of a polygon can be used to construct a polygonal roof over a set of walls [9], and even to solve certain origami design problems [36].

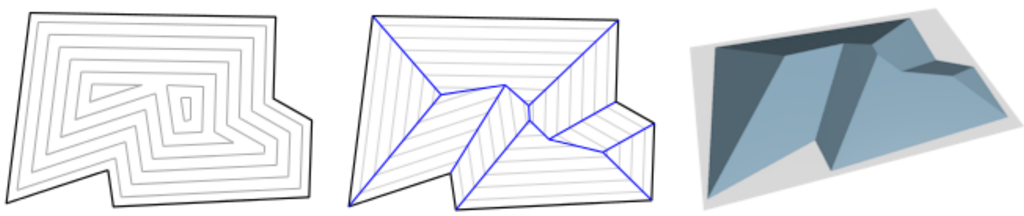

Figure 3: Constructing the straight skeleton of a polygon [49]. The first figure indicates the shrinking of the polygon. The blue edges in the second figure are the wavefront edges. The third figure illustrates how a polygonal roof can be constructed based on the straight skeleton.

Detecting combinatorial changes in the input polygon as it shrinks is essential when computing the straight skeleton of a polygon. Biedl et al. [19] computed the weighted straight skeleton by utilising stable matchings in SR. At each wavefront edge, two edges of the polygon meet, but when a wavefront collapses or splits, the new wavefront edge will correspond to a different pair of polygon edges. Biedl et al. [19] translated the problem of finding which polygon edge pairs will form a wavefront edge into a planar matching problem, which is then further translated into an instance of SR. Calculating a weighted straight skeleton can thus be reduced to computing a stable matching in the constructed SR instance, which is guaranteed to admit one.

Biedl et al. [19] gave an upper bound on the complexity of finding the weighted straight skeleton of a polygon in time $O(N^5)$, where N denotes the number of vertices of the polygon. They conjectured that this running time can be improved. A deeper insight into the structure of the underlying sr instance might lead to such an improvement in the efficiency of the algorithm.

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