ON SIGN-SYMMETRIC SIGNED GRAPHS

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Abstract. A signed graph is said to be sign-symmetric if it is switching isomorphic to its negation. Bipartite signed graphs are trivially sign-symmetric. We give new constructions of non-bipartite sign-symmetric signed graphs. Sign-symmetric signed graphs have a symmetric spectrum but not the other way around. We present constructions of signed graphs with symmetric spectra which are not sign-symmetric. This, in particular answers a problem posed by Belardo, Cioabă, Koolen, and Wang (2018).

1. Introduction

Let G be a graph with vertex set V and edge set E. All graphs considered in this paper are undirected, finite, and simple (without loops or multiple edges).

A *signed graph* is a graph in which every edge has been declared positive or negative. In fact, a signed graph Γ is a pair (G, σ), where $G = (V, E)$ is a graph, called the underlying graph, and $\sigma : E \to \{-1, +1\}$ is the sign function or signature. Often, we write $\Gamma = (G, \sigma)$ to mean that the underlying graph is G. The signed graph $(G, -\sigma) = -\Gamma$ is called the *negation* of Γ. Note that if we consider a signed graph with all edges positive, we obtain an unsigned graph.

Let v be a vertex of a signed graph Γ. *Switching* at v is changing the signature of each edge incident with v to the opposite one. Let $X \subseteq V$. Switching a vertex set X means reversing the signs of all edges between X and its complement. Switching a set X has the same effect as switching all the vertices in X, one after another.

Two signed graphs $\Gamma = (G, \sigma)$ and $\Gamma' = (G, \sigma')$ are said to be *switching equivalent* if there is a series of switching that transforms Γ into Γ'. If Γ' is isomorphic to a switching of Γ, we say that Γ and Γ' are *switching isomorphic* and we write $\Gamma \simeq \Gamma'$. The signed graph $-\Gamma$ is obtained from Γ by reversing the sign of all edges. A signed graph $\Gamma = (G, \sigma)$ is said to be *sign-symmetric* if Γ is switching isomorphic to $(G, -\sigma)$, that is: $\Gamma \simeq -\Gamma$.

For a signed graph $\Gamma = (G, \sigma)$, the adjacency matrix $A = A(\Gamma) = (a_{ij})$ is an $n \times n$ matrix in which $a_{ij} = \sigma(v_i v_j)$ if v_i and v_j are adjacent, and 0 if they are not. Thus A is a symmetric matrix with entries $0, \pm 1$ and zero diagonal, and conversely, any such matrix is the adjacency matrix of a signed graph. The spectrum of Γ is the list of eigenvalues of its adjacency matrix with their multiplicities. We say that Γ has a

2010 *Mathematics Subject Classification.* Primary: 05C22; Secondary: 05C50.

Key words and phrases. Signed graph, Spectrum.

symmetric spectrum (with respect to the origin) if for each eigenvalue λ of Γ, $-\lambda$ is also an eigenvalues of Γ with the same multiplicity.

Recall that (see [4]), the *Seidel adjacency matrix* of a graph G with the adjacency matrix A is the matrix S defined by

$$S_{uv} = \begin{cases} 0 & \text{if } u = v \\ -1 & \text{if } u \sim v \\ 1 & \text{if } u \nsim v \end{cases}$$

so that $S = J - I - 2A$. The Seidel adjacency spectrum of a graph is the spectrum of its Seidel adjacency matrix. If G is a graph of order n, then the Seidel matrix of G is the adjacency matrix of a signed complete graph Γ of order n where the edges of G are precisely the negative edges of Γ.

Proposition 1.1. *Suppose S is a Seidel adjacency matrix of order n. If n is even, then S is nonsingular, and if n is odd,* $\operatorname{rank}(S) \geq n-1$. *In particular, if n is odd, and S has a symmetric spectrum, then S has an eigenvalue* 0 *of multiplicity 1.*

Proof. We have $\det(S) \equiv \det(I-J) (\bmod 2)$, and $\det(I-J) = 1-n$. Hence, if n is even, $\det(S)$ is odd. So, S is nonsingular. Now, if n is odd, any principal submatrix of order $n-1$ is nonsingular. Therefore, $\operatorname{rank}(S) \geq n-1$. □

The goal of this paper is to study sign-symmetric signed graphs as well as signed graphs with symmetric spectra. It is known that bipartite signed graphs are sign-symmetric. We give new constructions of non-bipartite sign-symmetric graphs. It is obvious that sign-symmetric graphs have a symmetric spectrum but not the other way around (see Remark 4.1 below). We present constructions of graphs with symmetric spectra which are not sign-symmetric. This, in particular answers a problem posed in [2].

2. Constructions of sign-symmetric graphs

We note that the property that two signed graphs Γ and Γ' are switching isomorphic is equivalent to the existence of a 'signed' permutation matrix P such that $PA(\Gamma)P^{-1} = A(\Gamma')$. If Γ is a bipartite signed graph, then we may write its adjacency matrix as

$$A = \begin{bmatrix} O & B \\ B^\top & O \end{bmatrix}.$$

It follows that $PAP^{-1} = -A$ for

$$P = \begin{bmatrix} -I & O \\ O & I \end{bmatrix},$$

which means that bipartite graphs are 'trivially' sign-symmetric. So it is natural to look for non-bipartite sign-symmetric graphs. The first construction was given in [1] as follows.

Theorem 2.1. *Let n be an even positive integer and V_1 and V_2 be two disjoint sets of size $n/2$. Let G be an arbitrary graph with the vertex set V_1. Construct the complement of G, that is G^c, with the vertex set V_2. Assume that $\Gamma = (K_n, \sigma)$ is a signed complete graph in which $E(G) \cup E(G^c)$ is the set of negative edges. Then the spectrum of Γ is sign-symmetric.*

Theorem 2.1 says that for an even positive integer n, let B be the adjacency matrix of an arbitrary graph on $n/2$ vertices. Then, the complete signed graph in which the negatives edges induce the disjoint union of G and its complement, is sign-symmetric.

2.1. **Constructions for general signed graphs.** Let $\mathcal{M}_{r,s}$ denote the set of $r \times s$ matrices with entries from $\{-1, 0, 1\}$. We give another construction generalizing the one given in Theorem 2.1:

Theorem 2.2. *Let $B, C \in \mathcal{M}_{k,k}$ be symmetric matrices where B has a zero diagonal. Then the signed graph with the adjacency matrices*

$$A = \begin{bmatrix} B & C \\ C & -B \end{bmatrix}$$

is sign-symmetric on $2k$ vertices.

Proof.

$$\begin{bmatrix} O & -I \\ I & O \end{bmatrix} \begin{bmatrix} B & C \\ C & -B \end{bmatrix} \begin{bmatrix} O & I \\ -I & O \end{bmatrix} = \begin{bmatrix} -B & -C \\ -C & B \end{bmatrix} = -A$$

□

Note that Theorem 2.2 shows that there exists a sign-symmetric graph for every even order.

We define the family $\mathcal{F}$ of signed graphs as those which have an adjacency matrix satisfying the conditions given in Theorem 2.2. To get an impression on what the role of $\mathcal{F}$ is in the family of sign-symmetric graphs, we investigate small complete signed graphs. All but one complete signed graphs with symmetric spectra of orders $4, 6, 8$ are illustrated in Fig. 6 (we show one signed graph in the switching class of the signed complete graphs induced by the negative edges). There is only one sign-symmetric complete signed graph of order 4. There are four complete signed graphs with symmetric spectrum of order 6, all of which are sign-symmetric, and twenty-one complete signed graphs with symmetric spectrum of order 8, all except the last one are sign-symmetric, and together with the negation of the last signed graph, Fig. 6 gives all complete signed graphs with symmetric spectrum of order 4, 6 and 8. Interestingly, all of the above sign-symmetric signed graphs belong to $\mathcal{F}$.

The following proposition shows that $\mathcal{F}$ is closed under switching.

Proposition 2.3. *If $\Gamma \in \mathcal{F}$ and Γ' is obtained from Γ by switching, then $\Gamma' \in \mathcal{F}$.*

Proof. Let $\Gamma \in \mathcal{F}$. It is enough to show that if Γ' is obtained from Γ by switching with respect to its first vertex, then $\Gamma' \in \mathcal{F}$. We may write the adjacency matrix of

Γ as follows:

$$A = \left[\begin{array}{c|c|c|c} 0 & \mathbf{b}^\top & c & \mathbf{c}^\top \\ \hline \mathbf{b} & B' & \mathbf{c} & C' \\ \hline c & \mathbf{c}^\top & 0 & -\mathbf{b}^\top \\ \hline \mathbf{c} & C' & -\mathbf{b} & -B' \end{array}\right].$$

After switching with respect to the first vertex of Γ, the adjacency matrix of the resulting signed graph is

$$\left[\begin{array}{c|c|c|c} 0 & -\mathbf{b}^\top & -c & -\mathbf{c}^\top \\ \hline -\mathbf{b} & B' & \mathbf{c} & C' \\ \hline -c & \mathbf{c}^\top & 0 & -\mathbf{b}^\top \\ \hline -\mathbf{c} & C' & -\mathbf{b} & -B' \end{array}\right].$$

Now by interchange the 1st and $(k+1)$-th rows and columns we obtain

$$\left[\begin{array}{c|c|c|c}
0 & \mathbf{c}^\top & -c & -\mathbf{b}^\top \\ \hline
\mathbf{c} & B' & -\mathbf{b} & C' \\ \hline
-c & -\mathbf{b}^\top & 0 & -\mathbf{c}^\top \\ \hline
-\mathbf{b} & C' & -\mathbf{c} & -B'
\end{array}\right]$$

which is a matrix of the form given in Theorem 2.2 and thus Γ' is isomorphic with a signed graph in $\mathcal{F}$. □

In the following we present two constructions for complete sign-symmetric signed graphs using self-complementary graphs.

2.2. **Constructions for complete signed graphs.** In the following, the meaning of a self-complementary graph is the same as defined for unsigned graphs. Let G be a self-complementary graph so that there is a permutation matrix P such that $PA(G)P^{-1} = A(\overline{G})$ and $PA(\overline{G})P^{-1} = A(G)$. It follows that if Γ is a complete signed graph with $E(G)$ being its negative edges, then $A(\Gamma) = A(\overline{G}) - A(G)$, (in other words, $A(\Gamma)$ is the Seidel matrix of G). It follows that $PA(\Gamma)P^{-1} = -A(\Gamma)$. So we obtain the following:

Observation 2.4. If Γ is a complete signed graph whose negative edges induce a self-complementary graph, then Γ is sign-symmetric.

We give one more construction of sign-symmetric signed graphs based on self-complementary graphs as a corollary to Observation 2.4. We remark that a self-complementary graph of order n exists whenever $n \equiv 0$ or $1 \pmod 4$.

Proposition 2.5. *Let G, H be two self-complementary graphs, and let Γ be a complete signed graph whose negative edges induce the join of G and H (or the disjoint union of G and H). Then Γ is sign symmetric. In particular, if G has n vertices, and if H is a singleton, then the complete signed graph Γ of order $n+1$ with negative edges equal to $E(G)$ is sign-symmetric.*

In the following remark we present a sign-symmetric construction for non-complete signed graphs.

Remark 2.6. Let Γ', Γ'' be two signed graphs which are isomorphic to $-\Gamma', -\Gamma''$, respectively. Consider the signed graph Γ obtained from joining Γ' and Γ'' whose negative edges are the union of negative edges in Γ' and Γ''. Then, Γ is sign-symmetric.

Remark 2.7. By Proposition 2.5, we have a construction of sign-symmetric complete signed graphs of order $n \equiv 0, 1$ or $2 \pmod 4$. All complete sign-symmetric signed graphs of order 5 and 9 (depicted in Fig. 7) can be obtained in this way. There is just one sign-symmetric signed graph of order 5 which is obtained by joining a vertex to a complete signed graph of order 4 whose negative edges form a path of length 3 (which is self-complementary). Moreover, there exist sixteen complete signed graphs of order 9 with symmetric spectrum of which ten are sign-symmetric; the first three are not sign-symmetric, and when we include their negations we get them all. All of these ten complete sign-symmetric signed graphs can be obtained by joining a vertex to a complete signed graph of order 8 whose negative edges induce a self-complementary graph. Note that there are exactly ten self-complementary graphs of order 8.

Theorem 2.8. *There exists a complete sign-symmetric signed graph of order n if and only if $n \equiv 0, 1$ or $2 \pmod 4$.*

Proof. Using the previous results obviously one can construct a sign-symmetric signed graph of order n whenever $n \equiv 0, 1$ or $2 \pmod 4$. Now, suppose that there is a complete sign-symmetric signed graph Γ of order n with $n \equiv 3 \pmod 4$. By [7, Corollary 3.6], the determinant of the Seidel matrix of Γ is congruent to $1-n \pmod 4$. Since $n \equiv 3 \pmod 4$, the determinant of the Seidel matrix (obtained from the negative edges of Γ) is not zero. Hence, we can conclude that all eigenvalues of Γ are nonzero. Therefore, Γ cannot have a symmetric spectrum, and also it cannot be sign-symmetric. $\square$

In [9] all switching classes of Seidel matrices of order at most seven are given. There is a error in the spectrum of one of the graphs on six vertices in [9, Table 4.1] (2.37 should be 2.24), except for that, the results in [9] coincide with ours.

3. Positive and negative cycles

A graph whose connected components are K_2 or cycles is called an *elementary graph*. Like unsigned graphs, the coefficients of the characteristic polynomial of the adjacency matrix of a signed graph Γ can be described in terms of elementary subgraphs of Γ.

Theorem 3.1 ([3, Theorem 2.3])**.** *Let $\Gamma = (G, \sigma)$ be a signed graph and*

$$P_\Gamma(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \tag{1}$$

be the characteristic polynomial of the adjacency matrix of Γ. Then

$$a_i = \sum_{B \in \mathcal{B}_i} (-1)^{p(B)} 2^{|c(B)|} \sigma(B),$$

where $\mathcal{B}_i$ is the set of elementary subgraphs of G on i vertices, $p(B)$ is the number of components of B, $c(B)$ the set of cycles in B, and $\sigma(B) = \prod_{C \in c(B)} \sigma(C)$.

Remark 3.2. It is clear that Γ has a symmetric spectrum if and only if in its characteristic polynomial (1), we have $a_{2k+1} = 0$, for $k = 1, 2, \ldots$.

In a signed graph, a cycle is called *positive* or *negative* if the product of the signs of its edges is positive or negative, respectively. We denote the number of positive and negative ℓ-cycles by c_ℓ^+ and c_ℓ^-, respectively.

Observation 3.3. For sign-symmetric signed graph, we have

$$c_{2k+1}^+ = c_{2k+1}^- \text{ for } k = 1, 2, \ldots.$$

Remark 3.4. If in a signed graph Γ, $c_{2k+1}^+ = c_{2k+1}^-$ for all $k = 1, 2, \ldots$, then it is not necessary that Γ is sign-symmetric. See the complete signed graph given in Fig. 3. For this complete signed graph we have $c_{2k+1}^+ = c_{2k+1}^-$ for all $k = 1, 2, \ldots$, but it is not sign-symmetric. Moreover, one can find other examples among complete and non-complete signed graphs. For example, the signed graph given in Fig. 2 is a non-complete signed graph with the property that $c_{2k+1}^+ = c_{2k+1}^-$ for all $k = 1, 2, \ldots$, but it is not sign-symmetric.

By Theorem 3.1, we have that $a_3 = 2(c_3^- - c_3^+)$. By Theorem 3.1 and Remark 3.2 for signed graphs having symmetric spectrum, we have $c_3^+ = c_3^-$. Further, for each complete signed graph with a symmetric spectrum, it can be seen that $c_5^+ = c_5^-$. However, the equality $c_{2k+1}^+ = c_{2k+1}^-$ does not necessarily hold for $k \geq 3$. The complete signed graph in Fig. 1 has a symmetric spectrum for which $c_7^+ \neq c_7^-$.

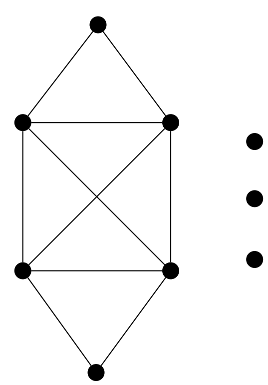

FIGURE 1. The graph induced by negative edges of a complete signed graph on 9 vertices with a symmetric spectrum but $c_7^+ \neq c_7^-$

Remark 3.5. There are some examples showing that for a non-complete signed graph we have $c_{2k+1}^+ = c_{2k+1}^-$ for all $k = 1, 2, \ldots$, but their spectra are not symmetric. As an example see Fig. 2, (dashed edges are negative; solid edges are positive).

Now, we may ask a weaker version of the result mentioned in Remark 3.4 as follows.

Question 3.6. Is it true that if in a complete signed graph Γ, $c_{2k+1}^+ = c_{2k+1}^-$ for all $k = 1, 2, \ldots$, then Γ has a symmetric spectrum?

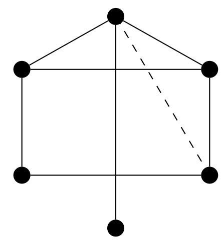

FIGURE 2. A signed graph with $c^+_{2k+1} = c^-_{2k+1}$ for $k = 1, 2, \ldots$, but its spectrum is not symmetric

4. Sign-symmetric vs. symmetric spectrum

Remark 4.1. Consider the complete signed graph whose negative edges induces the graph of Fig. 3. This graph has a symmetric spectrum, but it is not sign-symmetric. Note that this complete signed graph has the minimum order with this property. Moreover, for this complete signed graph we have the equalities $c^+_{2k+1} = c^-_{2k+1}$ for $k = 1, 2, 3$.

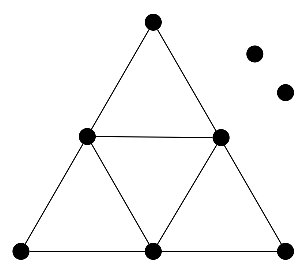

FIGURE 3. The graph induced by negative edges of a complete signed graph on 8 vertices with a symmetric spectrum but not sign-symmetric

Remark 4.2. A *conference matrix* C of order n is an $n\times n$ matrix with zero diagonal and all off-diagonal entries ± 1, which satisfies $CC^\top = (n-1)I$. If C is symmetric, then C has eigenvalues $\pm\sqrt{n-1}$. Hence, its spectrum is symmetric. Conference matrices are well-studied; see for example [4, Section 10.4]. An important example of a symmetric conference matrix is the Seidel matrix of the Paley graph extended with an isolated vertex, where the *Paley graph* is defined on the elements of a finite field $\mathbf{F_q}$, with $q \equiv 1 \pmod 4$, where two elements are adjacent whenever the difference is a nonzero square in $\mathbf{F_q}$. The Paley graph is self-complementary. Therefore, by Proposition 2.5, C is the adjacency matrix of a sign-symmetric complete signed graph. However, there exist many more symmetric conference matrices, including several that are not sign-symmetric (see [5]).

In [2], the authors posed the following problem on the existence of the non-complete signed graphs which are not sign-symmetric but have symmetric spectrum.

Problem 4.3 ([2])**.** Are there non-complete connected signed graphs whose spectrum is symmetric with respect to the origin but they are not sign-symmetric?

We answer this problem by showing that there exists such a graph for any order $n \geq 6$. For $s \geq 0$, define the signed graph Γ_s to be the graph illustrated in Fig. 4.

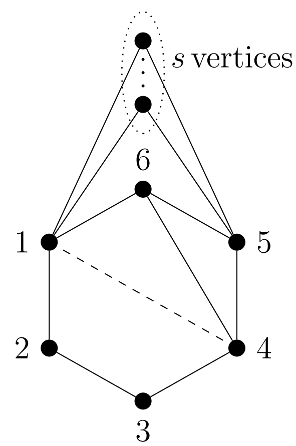

FIGURE 4. The graph Γ_s

Theorem 4.4. *For $s \geq 0$, the graph Γ_s has a symmetric spectrum, but it is not sign-symmetric.*

Proof. Let S be the set of s vertices adjacent to both 1 and 5. The positive 5-cycles of Γ_s are 123461 together with $u1645u$ for any $u \in S$, and the negative 5-cycles are $u1465u$ for any $u \in S$. Hence, $c_5^+ = s+1$ and $c_5^- = s$. In view of Observation 3.3, this shows that Γ_s is not sign-symmetric.

Next, we show that Γ_s has a symmetric spectrum. It suffices to verify that $a_{2k+1} = 0$ for $k = 1, 2, \ldots$.

The graph Γ_s contains a unique positive cycle of length 3: 4564 and a unique negative cycle of length 3: 1461. It follows that $a_3 = 0$.

As discussed above, we have $c_5^+ = s+1$ and $c_5^- = s$. We count the number of positive and negative copies of $K_2 \cup C_3$. For the negative triangle 1461, there are $s+1$ non-incident edges, namely 23 and $5u$ for any $u \in S$ and for the positive triangle 4564, there are $s+2$ non-incident edges, namely 12, 23 and $1u$ for any $u \in S$. It follows that

$$a_5 = -2((s+1) - s) + 2((s+2) - (s+1) = 0.$$

Now, we count the number of positive and negative elementary subgraphs on 7 vertices:

C_7: s positive: $u123465u$ for any $u \in S$, and no negative;

$K_2 \cup C_5$: $2s$ positive: $u5 \cup 123461$, and $23 \cup u1645u$ for any $u \in S$, and s negative: $23 \cup u1465u$ for any $u \in S$;

$2K_2 \cup C_3$: $s+1$ positive: $u1 \cup 23 \cup 4564$ for any $u \in S$, and $s+1$ negative: $u5 \cup 23 \cup 1461$ for any $u \in S$;

$C_4 \cup C_3$: none.

Therefore,

$$a_7 = -2(s-0) + 2(2s-s) - 2((s+1)-(s+1)) = 0.$$

The graph Γ_s contains no elementary subgraph on 8 vertices or more. The result now follows. □

More families of non-complete signed graphs with a symmetric spectrum but not sign-symmetric can be found. Consider the signed graphs $\Gamma_{s,t}$ depicted in Fig. 5, in which the number of upper repeated pair of vertices is $s \geq 0$ and the number of upper repeated pair of vertices is $t \geq 1$. In a similar fashion as in the proof of Theorem 4.4 it can be verified that $\Gamma_{s,t}$ has a symmetric spectrum, but it is not sign-symmetric.

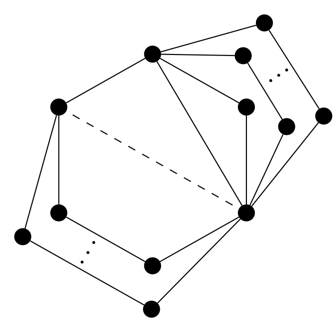

FIGURE 5. The family of signed graphs $\Gamma_{s,t}$

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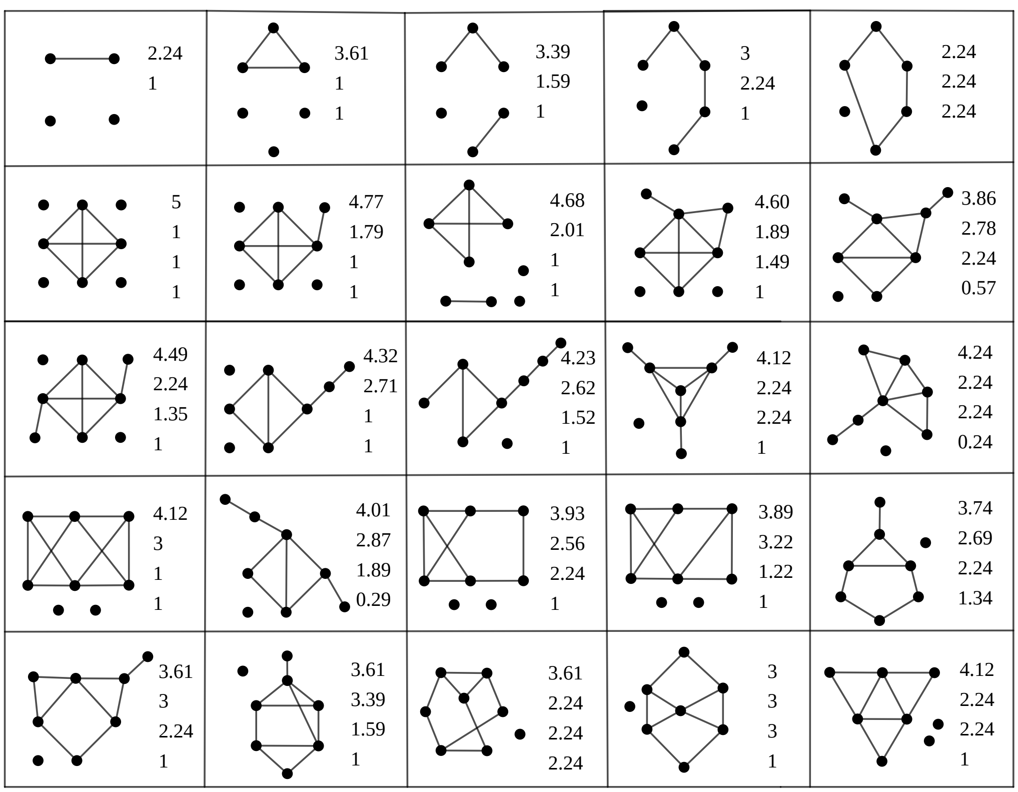

Figure 6. Complete signed graphs (up to switching isomorphism and negation) of order 4, 6, 8 having symmetric spectrum. The numbers next to the graphs are the non-negative eigenvalues. Only the last graph on the right is not sign-symmetric.

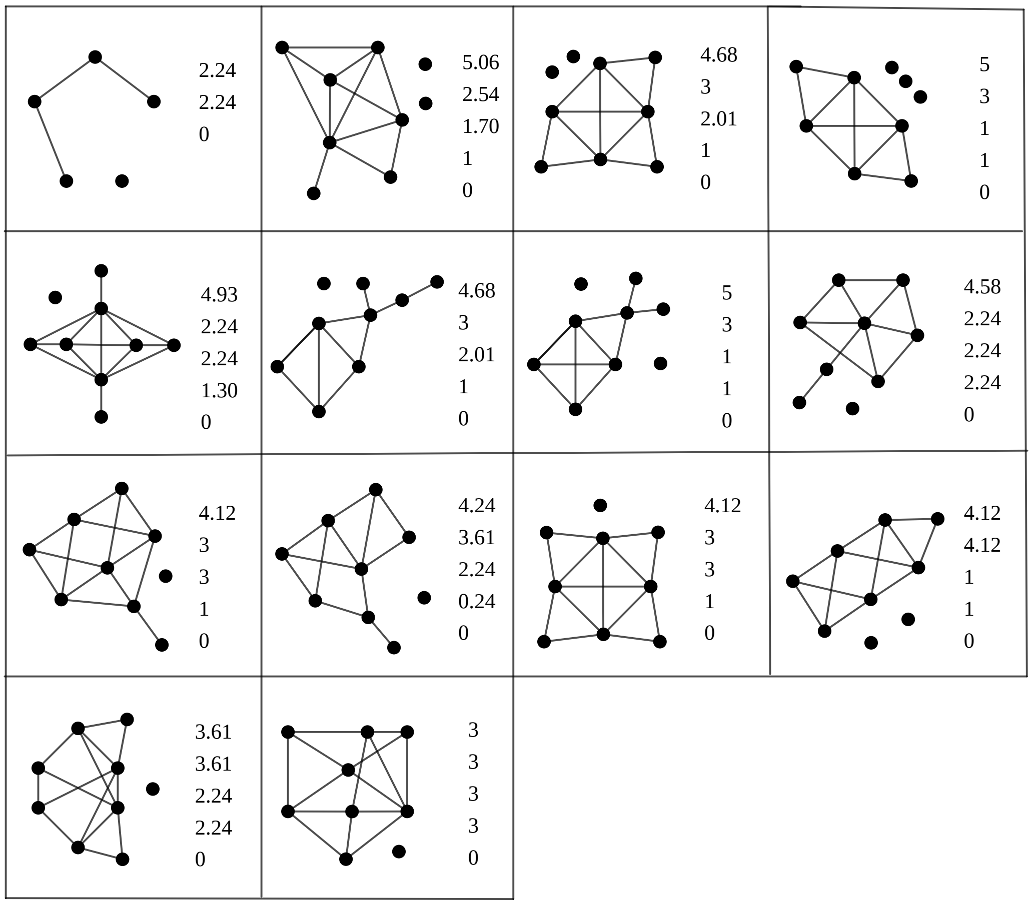

Figure 7. Complete signed graphs (up to switching isomorphism and negation) of order $5, 9$ having symmetric spectrum. The numbers next to the graphs are the non-negative eigenvalues.The first three signed graphs of order 9 are not sign-symmetric.