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SIGNED GRAPHS COSPECTRAL WITH THE PATH

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ABSTRACT. A signed graph Γ is said to be determined by its spectrum if every signed graph with the same spectrum as Γ is switching isomorphic with Γ. Here it is proved that the path P_n, interpreted as a signed graph, is determined by its spectrum if and only if $n \equiv 0, 1$, or $2 \pmod 4$, unless $n \in \{8, 13, 14, 17, 29\}$, or $n = 3$.
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1. Introduction

Throughout this paper all graphs are simple, without loops or parallel edges. A *signed graph* $\Gamma = (G, \sigma)$ (with $G = (V, E)$) is a graph with the vertex set V and the edge set E together with a function $\sigma : E \to \{-1, +1\}$, called the *signature function*. So, every edge becomes either positive or negative. The adjacency matrix A of Γ is obtained from the adjacency matrix of the underlying graph G, by replacing 1 by -1 whenever the corresponding edge is negative. The spectrum of A is also called the spectrum of the signed graph Γ. For a vertex subset X of Γ, the operation that changes the sign of all outgoing edges of X, is called switching. In terms of the matrix A, switching multiplies the rows and columns of A corresponding to X by -1. The switching operation gives rise to an equivalence relation, and equivalent signed graphs have the same spectrum (see [7, Proposition 3.2]). If a signed graphs can be switched into an isomorphic copy of another signed graph, the two signed graphs are called *switching isomorphic*. Clearly switching isomorphic graphs are cospectral (that is, they have the same spectrum). A signed graph Γ is determined by spectrum whenever every graph cospectral with Γ is switching isomorphic with Γ. For unsigned graphs it is known that the path P_n is determined by the spectrum of the adjacency matrix. Among the signed graphs this is in general not true anymore. In this paper we determine precisely for which n this is still the case, see Theorems 4.4, 5.1, and Corollary 5.3.

We refer to [7] and [8] for more information about signed graphs. For the relevant background on graphs we refer to [3], [4], or [5]. The initial problem was, possibly, first introduced by Acharya in [1].

2. preliminaries

A *walk* of length k in a signed graph Γ is a sequence $v_1e_1v_2e_2 \dots v_ke_kv_{k+1}$ of vertices $v_1, v_2, \dots, v_{k+1}$ and edges $e_1, e_2, \dots, e_k$ such that $v_i \neq v_{i+1}$ and $e_i = \{v_i, v_{i+1}\}$ for each

$i = 1, 2, \ldots, k$. A walk is said to be *positive* if it contains an even number of positive edges, otherwise it is called *negative.* Let $w_{ij}^{+}(k)$ (resp. $w_{ij}^{-}(k)$) denote the number of positive (resp., negative) walks of length k from the vertex v_i to the vertex v_j. A closed walk is a walk that starts and ends at the same vertex.

In the unsigned case, the (i,j)-entry of A^k represents the number of walks of length k from v_i to v_j. But in the signed case, powers of A count walks in a signed way. The (i,j)-entry of A^k is $w_{ij}^{+}(k) - w_{ij}^{-}(k)$ ([2, Lemma 3.2], [8, Theorem II.1]). For simplicity, we set $W_k(\Gamma) = \sum_{i=1}^{n} (w_{ii}^{+}(k) - w_{ii}^{-}(k))$. It is easy to see that if Γ and Γ' are two cospectral signed graphs, then $W_k(\Gamma) = W_k(\Gamma')$ for each $k \geqslant 1$. Moreover, if Γ and Γ' are two cospectral signed graphs since the sum of the squares of the eigenvalues is twice of the number of edges, we obtain that the order and the size of Γ and Γ' are the same. Note that in the following figures thick lines indicate negative edges.

The following lemma can be easily proved by induction.

Lemma 2.1.
$$W_4(P_n) = 14 + 6(n-4), \ \textit{for } n \geqslant 2,$$
$$W_6(P_n) = 76 + 20(n-6), \ \textit{for } n \geqslant 3, \ \textit{and } W_6(P_2) = 2$$

A cycle in a signed graph is called *balanced* if it contains an even number of negative edges, otherwise it is called *unbalanced.* A signed graph is balanced if all its circuits are balanced. It is easily seen that a signed path and a balanced cycle is switching isomorphic with the underlying unsigned path and cycle, respectively. An unbalanced cycle is switching isomorphic with the underlying cycle with precisely one negative edge.

Lemma 2.2. [2, Lemma 4.4]. *Let P_n and C_n (resp. C_n^-) be the path and the balanced cycle (resp. unbalanced cycle) on n vertices, respectively. Then the following hold:*

$$\mathrm{Spec}(C_n) = \left\{2\mathrm{cos}\frac{2i\pi}{n} : \ i = 0, 1, \ldots, n-1\right\},$$
$$\mathrm{Spec}(C_n^-) = \left\{2\mathrm{cos}\frac{(2i+1)\pi}{n} : \ i = 0, 1, \ldots, n-1\right\},$$
$$\mathrm{Spec}(P_n) = \left\{2\mathrm{cos}\frac{i\pi}{n+1} : \ i = 1, \ldots, n\right\}.$$

Observe that C_n has largest eigenvalue 2, and that C_n^- has smallest eigenvalue -2 when n is odd, while all eigenvalues of the path are strictly between -2 and 2. Moreover, all eigenvalues of the path are simple (have multiplicity 1), while C_n and C_n^- have (many) eigenvalues of multiplicity 2.

Suppose Γ is a signed graph of order n with adjacency matrix A. Then we write $\det(\Gamma)$ instead of $\det(A)$. So $\det(\Gamma)$ equals the product of the eigenvalues of Γ, and if $p(x) = a_0 + a_1 x + \ldots a_{n-1} x^{n-1}$ is the characteristic polynomial of Γ, then clearly $\det(\Gamma) = a_0 = p(0)$. We define $\det'(\Gamma) = a_1 = p'(0)$. If Γ has an eigenvalue 0, then $\det(\Gamma) = 0$, and $\det'(\Gamma)$ is the product of the $n-1$ remaining eigenvalues.

Lemma 2.3. (a) *If n is even, then* $\det(P_n) = \pm 1$.

(b) *If n is odd then* $\det'(P_n) = (n+1)/2$.

Proof. (a) Clearly $\det(P_2) = -1$, and expanding $\det(P_{n+2})$ with respect to an end vertex of P_{n+2} gives $\det(P_{n+2}) = -\det(P_n)$.
(b) Let B_n be the adjacency matrix of P_n. When n is odd, we can write

$$B_n = \begin{bmatrix} O & N \\ N^\top & O \end{bmatrix}, \text{ where } N = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

The eigenvalues of B_n^2 are the eigenvalues of $NN^\top$ together with the eigenvalues of $N^\top N$. Since $NN^\top$ and $N^\top N$ have the same nonzero eigenvalues it follows that $\det'(B_n) = \det(NN^\top)$. We easily have that $NN^\top = 2I + B_m$, where $m = (n-1)/2$. Write $d_m = \det(2I + B_m)$, then $d_1 = 2$, $d_2 = 3$ and $d_{m+2} = 2d_{m+1} - d_m$, so $d_m = m + 1 = (n+1)/2$. □

Lemma 2.4. *Let B be a symmetric matrix of order n with two equal rows (and columns), and let B' be the matrix of order $n-1$ obtained from B by deleting one repeated row and column. Then* $\det(B) = 0$, *and* $\det'(B) = 2\det(B')$.

Proof. Clearly B is singular, so $\det(B) = 0$. Without loss of generality we assume that the first two rows and columns of B are equal. Consider the following orthogonal matrices $Q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, and $Q = \begin{bmatrix} Q_2 & O \\ O & I_{n-2} \end{bmatrix}$. Then $Q^\top BQ = \begin{bmatrix} 0 & \underline{0}^\top \\ \underline{0} & B'' \end{bmatrix}$, where B'' is obtained from B' by multiplying the first row and column by $\sqrt{2}$. On the other hand, B and $Q^\top BQ$ are cospectral, therefore $\mathrm{Spec}(B'') = \mathrm{Spec}(B) \setminus \{0\}$. So $\det'(B) = \det(B'') = 2\det(B')$. □

3. Signed graphs cospectral with the path

In the remaining of the paper we assume that Γ is a signed graph cospectral but not switching isomorphic with the path P_n. We know that Γ has n vertices and $n-1$ edges. Since Γ is not a signed path, Γ has at least two components. In this section we obtain conditions for the components of Γ.

Observation 3.1. (1) By the interlacing theorem and Lemma 2.2, Γ contains no odd cycle, no balanced even cycle, and no star $K_{1,4}$ as an induced subgraph. Hence, all cycles in Γ are unbalanced of even order, and the maximum degree of Γ is at most 3.

(2) We checked (by computer) that a signed graph for which the underlying unsigned graph is one of the graphs given in Fig. 1 has largest eigenvalue at least 2. Therefore, no graph in Fig. 1 has an induced subgraph of Γ. Also each graph of Fig. 2 has at least one eigenvalue of multiplicity at least 2. Therefore none of these can be a component of Γ. Note that Graph (g) in Fig. 2, has an eigenvalue of multiplicity 3, so by the interlacing theorem, each graph on 8

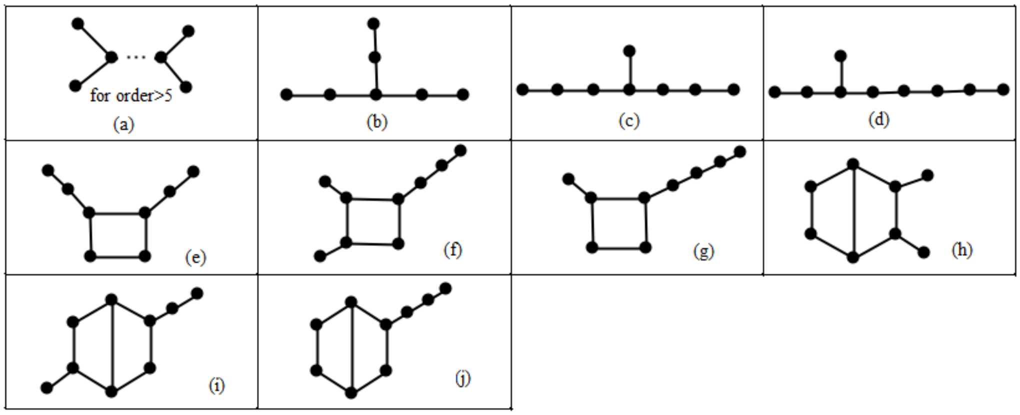

FIGURE 1. Graphs with largest eigenvalue at least 2

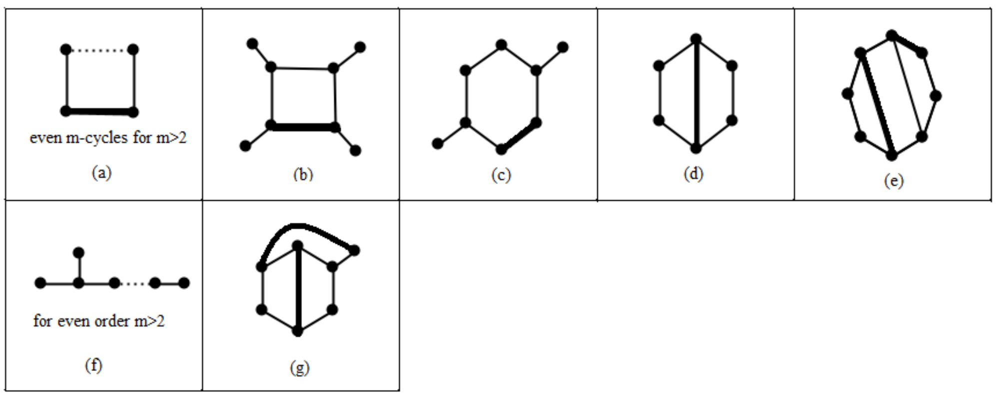

FIGURE 2. Graphs with some non-simple eigenvalues

vertices having Graph (g) as an induced subgraph has at least one non-simple eigenvalue, and therefore cannot be a component of Γ.

(3) Let M be Graph (e) of Fig. 2. Then M is not an induced subgraph of Γ. Indeed, M is not a component of Γ, and every graph on 9 vertices with maximum degree 3 that contains M as an induced subgraph contains an odd cycle, or Graph (a) from Fig. 1.

(4) A Θ-graph is a union of three internally disjoint paths P_p, P_q, P_r with common end vertices, where $p, q, r \geqslant 2$ and at most one of them equals 2. If $p, q, r \geqslant 3$ we call the Θ-graph *proper*. A proper signed Θ-graph has at least one balanced cycle. Then using the interlacing theorem for this induced balanced cycle, we conclude that a Γ has no proper Θ-graph as an induced subgraph.

(5) A unbalanced even cycle has eigenvalues of multiplicity 2, and therefore cannot occur as a component of Γ. Furthermore, we claim that Γ contains no induced even cycle of order more than 6. Indeed, let C_r^- be an unbalanced induced

cycle for $r \geqslant 8$. Since C_r^- is not a component, there exist a vertex v out of C_r^-, which is adjacent to one, two or three vertices of C_r^-. If v is adjacent to just one vertex of C_r^-, then we have Graph (c) given in Fig. 1 as an induced subgraph. If v is adjacent to two vertices of C_r^-, then graph $\langle V(C_r) \cup \{v\} \rangle$ is a proper Θ-graph (recall that Γ has no odd cycle). If v is adjacent to three vertices of C_r^-, then it is easy to check that the graph $\langle V(C_r) \cup \{v\} \rangle$ has one of the graphs (a) or (b) given in Fig. 1 as an induced subgraph. In all three case we have a contradiction, and since no vertex of Γ has degree more than three, the claim is proved.

Now, based on the previous observations we prove the following result.

Lemma 3.2. *If H is a component of Γ containing an induced 6-cycle, then H is one of the graphs presented in Fig. 3*

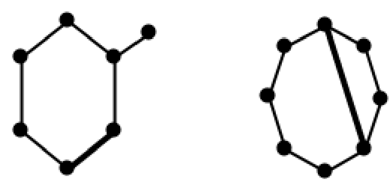

FIGURE 3.

Proof. We know that the 6-cycle is unbalanced. Also, the unbalanced 6-cycle C_6^- has eigenvalues of multiplicity 2, so $H \neq C_6^-$. So, there is at least one vertex v in H not contained in C_6^-. We claim that each vertex out of C_6 is adjacent with at least one vertex of C_6. Indeed, suppose there is a vertex v at distance 2 from C_6^-. Let u be a vertex adjacent to v and C_6^-. If u is adjacent to one vertex of C_6^-, then H has graph (b) given in Fig. 1 as an induced subgraph. If u is adjacent to two vertices of C_6^-, then H contains a proper Θ-graph. Since the degree of u is at most three, u cannot be adjacent to more than two vertices of C_6^-, from which the claim follows. Each vertex of C_6^- has at most one outgoing edge. Therefore H has at most 12 vertices, and by straightforward checking it follows that only the two graphs of Fig. 3 survive all conditions. □

Theorem 3.3. *The only graphs that can occur as a component of Γ are listed in Fig. 4.*

Proof. Let H be a component of Γ. If H is a tree, then since all eigenvalues are strictly less than 2, H is one of the trees in Fig. 4 (see [3, Theorem 3.1.3]). Now, suppose that H has a cycle. By Observation 3.1, H has no induced unbalanced cycle of order more than 6. Hence, every induced cycle of H has order 4 or 6. If H has an induced 6-cycle, then H is Graph (f) or (j) in Fig. 4, by Lemma 3.2.

Now, assume that H has an induced unbalanced 4-cycle but no induced 6-cycle. If $H = C_4^-$, then Γ has non-simple eigenvalues. Let X be the set of vertices outside C_4^- which are adjacent to C_4^-. Then $1 \leq |X| \leq 4$. The graph induced by X has no two intersecting edges, because otherwise H contains an induced 5-cycle or 6-cycle. So X contains two disjoint edges, or one edge, or no edge at all. In the first case, H contains M (Graph (e) of Fig. 2) as an induced subgraph, which is impossible by Observation 3.1. Next suppose that X contains no edges. Then, since H contains no induced t-cycles with $t \geq 5$, the only path between two vertices of X goes via an edge of C_4^-, and moreover, all vertices of H outside C_4^- have degree at most 2, because otherwise Graph (a) of Fig. 1 or a proper signed Θ-graph is an induced subgraph of H. Therefore H consists of C_4^- with a path attached to some of its vertices, and only Graph (a), (b), (c), (d), (e), (k) and (l) of Fig. 4 survive. Finally we assume that X contains exactly one edge e, say. Then the vertices of e and C_4^- induce a graph F consisting of a hexagon with one more edge connecting opposite vertices. Similar as in the previous case, it follows that H consists of F with a path attached to some of the vertices of degree 2, and only (g), (h), and (i) survive, see Figs. 1 and 2. □

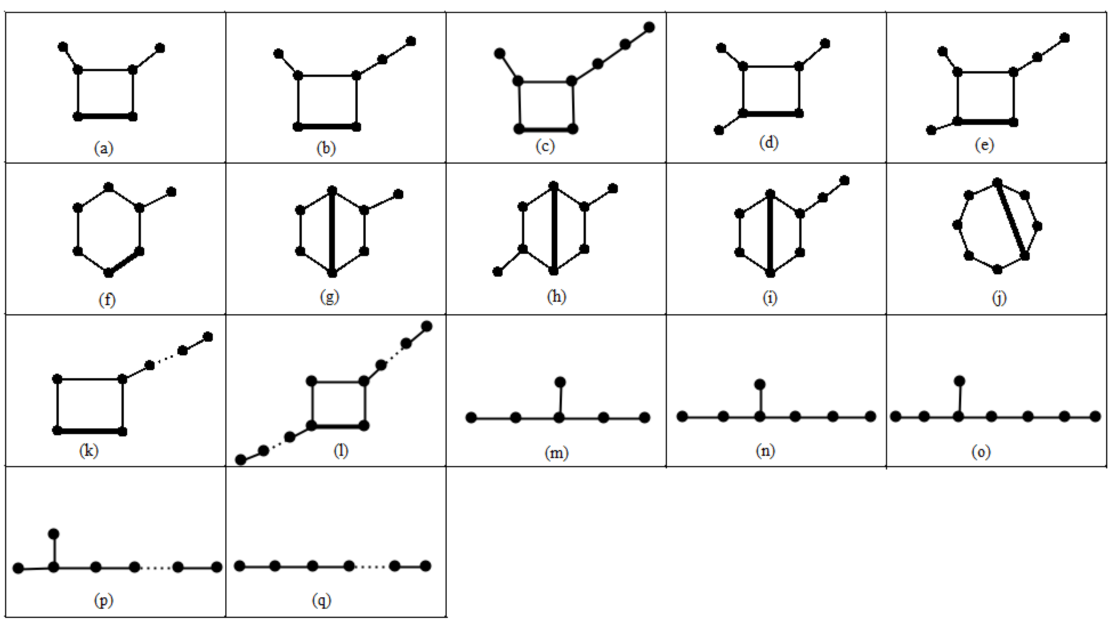

FIGURE 4. All possible components for Γ

Theorem 3.4. *If Γ is cospectral, but not switching isomorphic with P_n, then Γ contains an unbalanced 4-cycle as induced subgraph.*

Proof. Suppose Γ does not contain an C_4^-. Then, since Γ contains an induced unbalanced cycle C_r^- with $r \geq 6$, Theorem 3.3 implies that Graph (f) of Fig. 4 is a component of Γ. Also there cannot be two or more components equal to graph (f), since then Γ would have eigenvalues of multiplicity at least 2. So we can conclude that Γ has just one component with a cycle, which equals Graph (f), and that there is just one more component equal to (m), (n), (o), (p), or (q) of Fig. 4. By verification it follows that none of these possibilities has the spectrum of P_n. $\square$

Note that only four cases in Fig. 4 represent an infinite family. Graph (q) of order m is the path P_m, and Graph (p) of order m is known as D_m. Graph (k) and (l) will be denoted by H_t and H_t^{t+m}, respectively. More precisely, H_t is the union of C_4^- and P_t, where an end vertex of P_t is joined to a vertex of C_4^-, and H_t^{t+m} is the union of C_4^-, P_t and P_{m+t} where an end vertex of P_t is joined to one vertex of C_4^-, and an end vertex of P_{m+t} is joined to the opposite vertex of C_4^-.

Lemma 3.5. *For integers $t \geqslant 1$, $k \geqslant 0$ and $m \geq 1$, $\det(H_t)$, $\det(H_t^{t+m})$, $\det(D_m)$, $\det'(H_t)$, $\det'(H_t^{t+m})$, and $\det'(D_m)$ are even.*

Proof. Let B be the adjacency matrix of the underlying graphs H_t or H_t^{t+m}. Then B contains two repeated rows (and columns), so by Lemma 2.4 $\det(B) = 0$ and $\det'(B) = 2\det(B')$, so $\det'(B)$ is even. On the other hand, the signed and the unsigned graph have equal adjacency matrices modulo 2. $\square$

Theorem 3.6. *The eigenvalues of H_t^{t+m} $(t \geq 1,\ m \geq 0)$ are as follows: The first type of eigenvalues are*

$$2\cos\frac{(2i-1)\pi}{2k},\ \textit{for } k = t+m+2,\ i = 1,\dots,k.$$

The second type of eigenvalues are

$$2\cos\frac{(2i-1)\pi}{2t+4},\ \textit{for } i = 1,\dots,t+2.$$

Proof. It is easy to see that we can write the adjacency matrix A of H_t^{t+m} as follows:

$$A = \begin{bmatrix} O & N \\ N^T & O \end{bmatrix}.$$

Then it is seen that

$$A^2 = \begin{bmatrix} NN^T & O \\ O & N^TN \end{bmatrix}.$$

We can write NN^T as the following matrix

$$NN^T = \begin{bmatrix} K & O \\ O & L \end{bmatrix},$$

where K and L are tridiagonal matrices with all-ones on the upper and lower diagonal, and $[3,2,2,\dots,2,1]$ or $[3,2,2,\dots,2]$ on the diagonal. Assume that K and L are square matrices of size s and r, respectively. If t is even, then $s = \frac{t}{2}+1$ and $r = [\frac{t+m}{2}]+1$. Otherwise $s = [\frac{t+m}{2}]+1$ and $r = \lceil\frac{t+1}{2}\rceil$. Moreover, by [6, Theorems 2, 3], we can obtain the eigenvalues of K and L using the following equalities, respectively.

$$\lambda_j = 2 + 2\cos\frac{(2j-1)\pi}{2s},\ j = 1,2,\dots,s,$$

$$\lambda_i = 2 + 2\cos\frac{(2i-1)\pi}{2r+1},\ i = 1, 2, \ldots, r.$$

Now, using a simple trigonometric relation the assertion is proved. □

4. Paths of even order

Suppose n is even. By Lemma 2.3 $\det(\Gamma) = \det(P_n) = \pm 1$. Therefore each component of Γ has determinant ± 1, hence Graphs (a), (c), (e), (h), (i), (j), (m), (o), and (q) (P_k with k even) given in Fig. 4 are the only possible components of Γ. We easily check that Graphs (c), (e), (i) and (m) have eigenvalues which are not a subset of eigenvalues of P_n for some n. Therefore the only graphs which can occur as a component of Γ for even n are the graphs presented in Fig. 5.

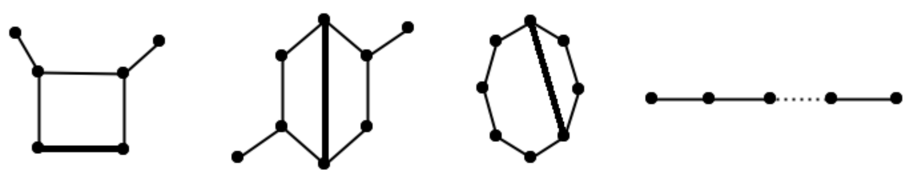

Figure 5. All possible components of Γ with n even

We note that the second and the third graph in Fig. 5 are cospectral. Therefore, at most one of them can be a component of Γ.

Lemma 4.1. *If n is even and Γ has two connected components then $n = 8$. Moreover, Γ is switching isomorphic with the disjoint union of P_2 and Graph (a) from Fig. 4.*

Proof. Based on the possible components for Γ in Fig. 5, we have only one type of Γ with two components, being Graph (a) and P_{n-6}. By considering the values of $W_4(\Gamma)$ and $W_6(\Gamma)$ and using Lemma 2.1, we have $W_4(\Gamma) = W_4(P_n)$ for each n, but $W_6(\Gamma) \neq W_6(P_n)$ for $n \neq 8$. If $n = 8$ it is easily verified that Γ and P_8 are cospectral. □

Lemma 4.2. *If n is even and Γ has three connected components then $n = 14$. Moreover, if $n = 14$, then Γ is switching isomorphic with the disjoint union of either P_2, P_4 and Graph (h), or P_2, P_4 and Graph (j) in Fig. 4.*

Proof. After considering all cases of the components of Γ in Fig. 5, we obtain two types of Γ with three components given in Fig. 6. These two possible types of Γ are similar because the spectrum of the first components are the same. Hence, it is sufficient to verify one of these two cases for Γ. We note that when Γ contains two paths, then the orders of the paths are different because otherwise the multiplicity of some of the eigenvalues will be at least two. We have $W_4(\Gamma) = W_4(P_n)$, but $W_6(\Gamma) \neq W_6(P_n)$, unless $n = 14$. By an easy inspection, we conclude that if $n = 14$ and the path components have orders 2 and 4, then $\mathrm{Spec}(\Gamma) = \mathrm{Spec}(P_{14})$. □

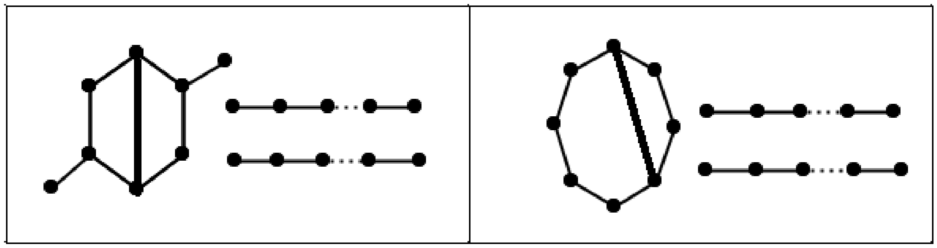

FIGURE 6. All types of Γ with three components and n even

Lemma 4.3. *If n is even, then Γ has at most three components.*

Proof. Assume that Γ has more than three components. Using Fig. 5 we see that there are only the two types for Γ shown in Fig. 7. Similar to the proof of Lemmas 4.1 and 4.2, it is sufficient to determine W_6 for Γ and P_n. In each case we achieve a contradiction. $\square$

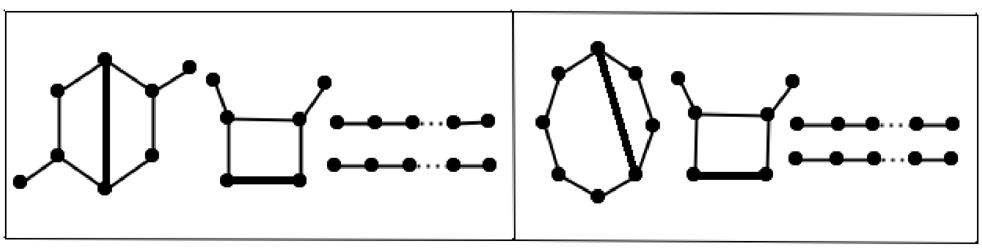

FIGURE 7. All types of Γ with four components and n even

Theorem 4.4. *Suppose n is even. Then P_n is determined by the spectrum if and only if $n \neq 8, 14$.*

Proof. It follows from Lemmas 4.1, 4.2 and 4.3. $\square$

5. PATHS OF THE ODD ORDER

Theorem 5.1. *Suppose $n \equiv 1 \pmod 4$. Then P_n is determined by the spectrum if and only if $n \notin \{13, 17, 29\}$.*

Proof. Since n is odd, $\det(\Gamma) = 0$, and exactly one component H of Γ has an eigenvalue 0. The product of all other eigenvalues of Γ equals $\det'(\Gamma) = (n+1)/2$ by Lemma 2.3. Since $(n+1)/2$ is odd, $\det'(H)$ is odd, and every component different from H has an odd determinant. Hence, by Lemma 3.5, the possible candidates do not include Graphs (k), (l) and (p) in given Fig. 4. So, there is only a small list of possible components of Γ. Clearly $\lambda_1(P_n)$ is equal to $\lambda_1(H)$ for one of the components of Γ.

Since $\lambda_1(P_k) < \lambda_1(P_n)$ when $k < n$, $H \neq P_k$, and the largest eigenvalue of each of the other possible components is at most $\lambda_1(P_{29})$. Therefore P_n is determined by the spectrum when $n \geq 33$.

For $n = 5, 9$, it is easy to check that P_n is determined by the spectrum. If $n = 21, 25$, then $\det'(P_n) = 11, 13$ respectively. But none of the components H in Fig. 4 (except P_{21} and P_{25}) has $\det(H)$, or $\det'(H)$ equal to 11 or 13. Hence, P_{21} and P_{25} are determined by their spectrums. Furthermore, we give graphs cospectral with P_{13}, P_{17} and P_{29} in Fig. 8. □

Theorem 5.2. *Let $n = 4k + 3$ for some integer $k \geq 1$. Then there exists a graph Γ which is cospectral but not switching isomorphic with P_n.*

Proof. Consider graph Γ with two components H_2 and P_1. It is easy to check that Γ is cospectral with P_7. For other cases, we show that a signed graph with two components H_{k-1}^{2k} and P_k is a cospectral mate of P_{4k+3}.

$$\begin{aligned}
&\mathrm{Spec}(P_{4k+3}) = \{2\cos\frac{i\pi}{4k+4},\ i = 1, 2, \ldots, 4k+3\} \\
&= \{2\cos\frac{i\pi}{4(k+1)},\ i = 1, 3, \ldots, 4k+3\} \cup \{2\cos\frac{j\pi}{4(k+1)},\ j = 2, 4, \ldots, 4k+2\} \\
&= \{2\cos\frac{i\pi}{4(k+1)},\ i = 1, 3, \ldots, 4k+3\} \cup \{2\cos\frac{j\pi}{2(k+1)},\ j = 1, 2, \ldots, 2k+1\} \\
&= \mathrm{Spec}(H_{k-1}^{2k}) \cup \mathrm{Spec}(P_k).
\end{aligned}$$
□

In Fig. 8 (E_6 is Graph(m) of Fig. 4, and E_8 is Graph (o) of Fig. 4), we give signed graphs cospectral with P_{11}, P_{15} and P_{23}. It shows that the presented graphs in Theorem 5.2 are in general not unique.

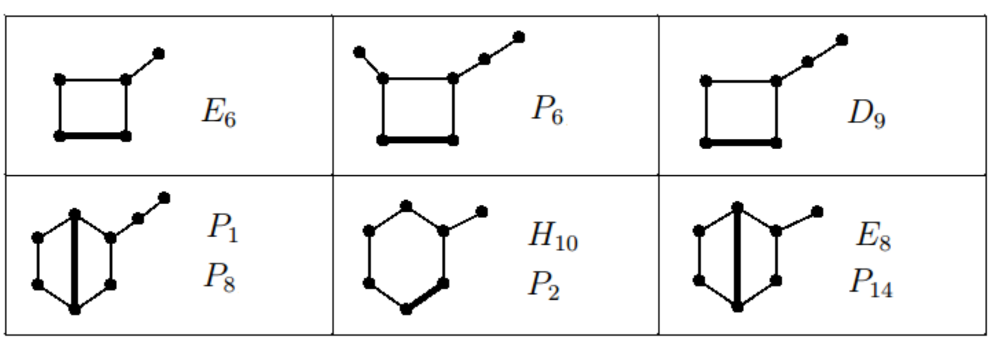

FIGURE 8. Cospectral mates of $P_{11}, P_{13}, P_{15}, P_{17}, P_{23}, P_{29}$

Obviously P_3 is determined by its spectrum, so we have the following conclusion.

Corollary 5.3. *Suppose $n \equiv 3 \mod 4$. Then P_n is determined by its spectrum if and only if $n = 3$.*

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