

# Simultaneous PQ-Ordering with Applications to Constrained Embedding Problems\*

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## Abstract

In this paper, we define and study the new problem SIMULTANEOUS PQ-ORDERING. Its input consists of a set of PQ-trees, which represent sets of circular orders of their leaves, together with a set of child-parent relations between these PQ-trees, such that the leaves of the child form a subset of the leaves of the parent. SIMULTANEOUS PQ-ORDERING asks whether orders of the leaves of each of the trees can be chosen *simultaneously*, that is, for every child-parent relation the order chosen for the parent is an extension of the order chosen for the child. We show that SIMULTANEOUS PQ-ORDERING is  $\mathcal{NP}$ -complete in general and we identify a family of instances that can be solved efficiently, the *2-fixed instances*. We show that this result serves as a framework for several other problems that can be formulated as instances of SIMULTANEOUS PQ-ORDERING.

In particular, we give linear-time algorithms for recognizing simultaneous interval graphs and extending partial interval representations. Moreover, we obtain a linear-time algorithm for PARTIALLY PQ-CONSTRAINED PLANARITY for biconnected graphs, which asks for a planar embedding in the presence of PQ-trees that restrict the possible orderings of edges around vertices, and a quadratic-time algorithm for SIMULTANEOUS EMBEDDING WITH FIXED EDGES for biconnected graphs with a connected intersection. Both results can be extended to the case where the input graphs are not necessarily biconnected but have the property that each cutvertex is contained in at most two non-trivial blocks. This includes for example the case where both graphs have maximum degree 5.

## 1 Introduction

Ordering objects in a specific way is a fundamental concept behind many applications. Probably the most basic ordering problem is sorting a totally ordered set. However, there may be less restrictive requirements on

an order of elements in a set than a total order. Examples for such requirements are partially ordered sets or the requirement that subsets of elements have to appear consecutively. Such requirements yield sets of possible (circular or linear) orders and in the two mentioned examples these sets admit compact representations (i.e., polynomial in the number of elements although the set of orderings may be exponentially large). More precisely, the possible orders for a partially ordered set may be represented by a directed acyclic graph (DAG) and all orders in which some specific subsets of elements appear consecutively can be represented by a PQ-tree [7]. A PQ-tree represents orders of its leaves by allowing edges around inner nodes to be either ordered arbitrarily (P-nodes) or by fixing this order up to reversal (Q-nodes); see Fig. 1a. Similarly, a matching on a set of vertices describes a set of possible orders, namely all orders where no pair of matched vertices alternates. In this work we do not consider the case where the order of elements of a single set is restricted in a specific way but we introduce the concept of simultaneous orders for a family of sets. Namely, given sets of orders  $\mathcal{L}_1, \dots, \mathcal{L}_n$  on element sets  $L_1, \dots, L_n$ , we seek orderings  $O_i \in \mathcal{L}_i$  such that the common elements are ordered consistently. Note that this is generally  $\mathcal{NP}$ -hard if the sets of orders  $\mathcal{L}_i$  are given as compact representations, as it contains the  $\mathcal{NP}$ -hard problem CYCLIC ORDERING [14].

Nevertheless, many special cases with interesting applications admit polynomial-time algorithms. For example Klavík et al. [25] essentially find a simultaneous ordering of a partially ordered set and a superset constrained by a PQ-tree to extend partial interval representations of graphs (an *interval representation* assigns an interval to each vertex such that intervals intersect if and only if the corresponding vertices are adjacent). Haeupler et al. [17] solve a special case of the simultaneous embedding problem SEFE, which asks for planar drawings of two graphs such that common parts are drawn the same (see Fig. 1b), by repeatedly finding simultaneous orders for two PQ-trees. Angelini et al. [2] show that a more general case of SEFE is equivalent to finding simultaneous orderings for a PQ-tree and two matchings. To find simultaneous interval

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representations of two graphs (where common vertices are represented by the same intervals) Jampani and Lubiw [22] seek compatible clique orderings represented by a pair of PQ-trees. Angelini et al. [1] and Gutwenger et al. [16] find planar embeddings subject to constraints on orderings of edges around vertices. The problem PARTIALLY PQ-CONSTRAINED PLANARITY combines these problems by restricting the orders of subsets of edges using PQ-trees. Such constrained embedding problems also fall into the domain of simultaneous ordering problems for the following reason. Planar embeddings of graphs are determined by circular orders of edges around vertices, and thus for each vertex there is a set of possible orders. However, to obtain a planar embedding by choosing an ordering for each vertex, extensive compatibility conditions need to be satisfied, yielding a simultaneous ordering problem.

In this paper we make a first step to unify simultaneous ordering problems within a common framework. We consider the case where orders are represented by PQ-trees leading to the problem SIMULTANEOUS PQ-ORDERING that is defined as follows. Let  $D = (N, A)$  be a DAG with nodes  $N = \{T_1, \dots, T_k\}$ , where  $T_i$  is an unrooted PQ-tree representing a set  $\mathcal{L}_i$  of circular orderings of its leaves  $L_i$ . Each arc  $a \in A$  consists of a source  $T_i$ , a target  $T_j$  and an injective map  $\varphi: L_j \rightarrow L_i$ , and it is denoted by  $(T_i, T_j; \varphi)$ . SIMULTANEOUS PQ-ORDERING asks whether there are orders  $O_1, \dots, O_k$  with  $O_i \in \mathcal{L}_i$  such that an arc  $(T_i, T_j; \varphi) \in A$  implies that  $\varphi(O_j)$  is a suborder of  $O_i$ . Note that this strictly generalizes the above simultaneous ordering problem for PQ-trees; consistent orderings of common elements can be enforced by introducing a common child, using  $\varphi = \text{id}$ . On the other hand, the injective maps can express more general relations between elements.

**Contribution and Outline** We show that SIMULTANEOUS PQ-ORDERING is  $\mathcal{NP}$ -hard in general and identify a large class of instances, the so-called *2-fixed instances*, that can be solved efficiently; see Section 3.2 for a formal definition. This result serves as a framework for several applications. In particular, we obtain algorithms for several problems mentioned above. The algorithms obtained in this way either solve more general cases that were not known to be efficiently solvable or significantly improve over the previously best running times.

We first define basic notation and preliminaries in Section 2. In Section 3, which is the main part of this paper, we show how to solve SIMULTANEOUS PQ-ORDERING for 2-fixed instances in quadratic time. We present several applications in Section 4, where we show how to formulate various problems as 2-fixed instances in the framework of SIMULTANEOUS PQ-ORDERING.

In particular, we give a linear-time algorithm for recognizing simultaneous interval graphs, improving upon the previously best algorithm with running time  $O(n^2 \log n)$  [22] and an  $O(n + m)$  algorithm for extending partial interval representations, improving upon the previously best algorithm with running time  $O(n^2)$  [25]. Moreover, we obtain a linear-time algorithm for PARTIALLY PQ-CONSTRAINED PLANARITY of biconnected graphs and a quadratic-time algorithm for SEFE for biconnected graphs with connected intersection. Both algorithms generalize to input graphs having the property that each cutvertex is contained in at most two non-trivial blocks. This significantly extends the results requiring that the common graph is biconnected [17, 2] for the following reason. If the intersection  $G$  of two graphs  $G^\circledast$  and  $G^\circledcirc$  is biconnected, it is completely contained in a single maximal biconnected component of  $G^\circledast$  and  $G^\circledcirc$ , respectively. Thus, testing SEFE for  $G^\circledast$  and  $G^\circledcirc$  is equivalent to testing it for these two biconnected components since all remaining biconnected components can be attached if and only if they are planar.

We emphasize that all applications follow directly from the main results in Section 3. The formulations as instances of SIMULTANEOUS PQ-ORDERING we use are straightforward and can easily be verified to be 2-fixed, at which point the machinery developed in the main part of this paper takes over.

Due to space constraints, we omit several proofs from this extended abstract. For detailed proofs, we refer the reader to the full version of this paper [5].

## 2 Preliminaries

**Orders and Permutations** Let  $L$  be a finite set and  $S \subseteq L$ . A linear order (circular order)  $O$  of  $L$  induces a unique linear order (circular order)  $O'$  on  $S$ . We say that  $O'$  is a *suborder* of  $O$  and that  $O$  is an *extension* of  $O'$ . Sometimes we do not have  $S \subseteq L$  but an injective map  $\varphi: S \rightarrow L$ . We then use the terms suborder and extension if  $\varphi(O')$  is a suborder of  $O$ , where  $\varphi(O')$  is the order obtained from  $O'$  by applying  $\varphi$  to each element.

Given a circular order  $O$  of a set  $L$ , a permutation  $\varphi$  of  $L$  is *order preserving* (*order reversing*) with respect to  $O$ , if  $\varphi(O)$  coincides with  $O$  (is the reverse of  $O$ ). It is *order preserving* (*order reversing*) if it is *order preserving* (*order reversing*) with respect to some order. Note that this definition allows a permutation to be order preserving and order reversing at the same time (with respect to different orders). We will use the following lemma.

LEMMA 2.1. *A permutation  $\varphi$  is a) order preserving and b) order reversing if and only if*

- a) *all its permutation cycles have the same length*

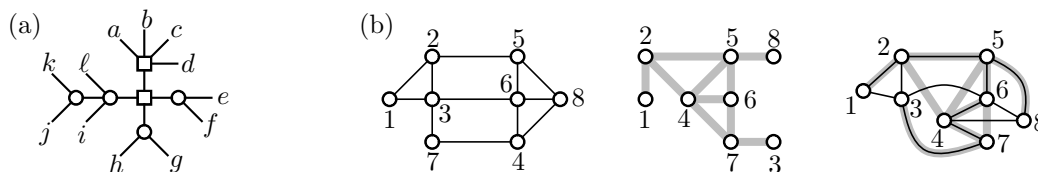


Figure 1: (a) A PQ-tree; P- and Q-nodes are depicted as circles and boxes, respectively. (b) Two graphs on a common node set. The SEFE on the right makes differences and similarities visible.

b) all its permutation cycles have length 2, except for at most two cycles with length 1.

Checking these conditions and constructing corresponding orders can be done in linear time.

**PQ-Trees** PQ-Trees were originally introduced by Booth and Lueker [7]. They were designed to decide whether a set  $L$  has the CONSECUTIVE ONES property with respect to a family  $\mathcal{S} = \{S_1, \dots, S_k\}$  of subsets  $S_i \subseteq L$ . The set  $L$  has this property if a linear order of its elements can be found, such that the elements in each subset  $S_i \in \mathcal{S}$  appear consecutively. Booth and Lueker showed how to solve CONSECUTIVE ONES in linear time and that all possible orders can be represented by a PQ-tree having the elements in  $L$  as leaves. PQ-trees are used for testing planarity in linear time and for recognizing interval graphs [7]. Unrooted PQ-trees are also called PC-trees [19, 20, 21]. There is an equivalence between PQ- and PC-trees [18], and we thus do not make this distinction.

Given an unrooted tree  $T$  with leaves  $L$  having a fixed circular order of edges around every vertex, the circular order of the leaves is also fixed. In an *unrooted PQ-tree* for some inner nodes, the *Q-nodes*, the circular order of incident edges is fixed up to reversal, for the other nodes, the *P-nodes*, this order can be chosen arbitrarily. Hence, an unrooted PQ-tree represents a set of circular orders of its leaves. Formally, the empty set, saying that no order is possible, is represented by the *null tree*, whereas the *empty tree* has the empty set as leaves and represents the set containing only the empty order. Analogously, one can define *rooted PQ-trees* representing linear orders. Haeupler and Tarjan [18] show that there is a correspondence between rooted and unrooted PQ-trees. Their construction adds a special leaf on top of the root of a rooted PQ-tree and then unroots it, leading to an unrooted PQ-tree representing essentially the same orders. PQ-trees were originally introduced in the rooted case by Booth and Lueker [7]. Unless stated otherwise, we refer to circular orders and unrooted PQ-trees if we write orders and PQ-trees, respectively.

Let  $T$  be a PQ-tree with leaves  $L$  representing the set of orders  $\mathcal{L}$  and let  $S \subseteq L$ . There exists a PQ-tree  $T|_S$  that represents exactly the orders of  $S$  that are suborders of some  $O \in \mathcal{L}$ , called the *projection* of  $T$

so  $S$ . Let  $T_1$  and  $T_2$  be two PQ-trees with the same leaf set  $L$  representing orders  $\mathcal{L}_1$  and  $\mathcal{L}_2$ , respectively. There exists a PQ-tree  $T_1 \cap T_2$  representing  $\mathcal{L}_1 \cap \mathcal{L}_2$ , called the *intersection* of  $T_1$  and  $T_2$ . Projections and intersections can be computed in linear time [8, 7]. The projection of a PQ-tree  $T$  to a subset  $S$  of the leaves  $L$  can be computed as follows. First, remove all leaves in  $L \setminus S$ , then iteratively remove former inner nodes with degree 1 and replace degree-2 nodes by single edges connecting their neighbors. Clearly, every inner node  $\mu'$  of  $T|_S$  stems from a distinct inner node  $\mu$  of  $T$ , and every edge  $\varepsilon'$  incident to  $\mu'$  stems from a distinct edge  $\varepsilon$  incident to  $\mu$ . Fixing an order for  $T|_S$  thus partially fixes  $T$ . In particular, if  $\mu'$  is a Q-node of  $T|_S$ , then the node  $\mu$  of  $T$  it stems from, which is also a Q-node, is completely determined; we call it *fixed*. A Q-node is *free* if it is not fixed. Similarly, a P-node  $\mu'$  of  $T|_S$  with incident edges  $\varepsilon'_1, \dots, \varepsilon'_k$ , determines the order of the edges  $\varepsilon_1, \dots, \varepsilon_k$  around  $\mu$ . We say that these edges are *fixed*, the remaining edges incident to  $\mu$  are *free*. A P-node is *fixed* if any of its incident edges is fixed, it is *free* otherwise. A *representative* of an edge  $\varepsilon$  with respect to  $\mu$  is a leaf of  $T$  whose path to  $\mu$  contains  $\varepsilon$ . Note that a node is fixed if and only if at least three incident edges have representatives in  $S$ .

Let  $T$  and  $T'$  be two PQ-trees with leaves  $L$  and  $L'$  and let  $\varphi: L' \rightarrow L$  be an injective map. To simplify notation, we consider  $L'$  as a subset of  $L$  via  $\varphi$  in the following. We seek *simultaneous orders* for  $T$  and  $T'$ , i.e., orders  $O$  and  $O'$  represented by  $T$  and  $T'$ , respectively, such that  $O$  extends  $O'$ . Orders of  $T'$  that cannot be extended to  $T$  can be eliminated by replacing  $T'$  by  $T' \cap T|_{L'}$ . Therefore, without loss of generality we assume that all orders represented by  $T'$  can be extended to an order represented by  $T$ . Fixing an order for  $T'$ , completely determines the order of  $T|_{L'}$ , and thus the notions of free and fixed edges/nodes can be extended to this case. Since  $T'$  is more restricted than  $T|_{L'}$ , a Q-node of  $T'$  may fix several nodes of  $T$ . However, the orientation of every fixed Q-node  $\mu$  in  $T$  is determined by exactly one Q-node of  $T'$ , called the *representative* of  $\mu$ , denoted by  $\text{rep}(\mu)$ . Conversely, a P-node of  $T$  may be fixed by several nodes of  $T'$ , but each P-node of  $T'$  fixes exactly one P-node of  $T$ .

### 3 Simultaneous PQ-Ordering

Recall that the problem SIMULTANEOUS PQ-ORDERING has a set  $N$  of PQ-trees  $T_1, \dots, T_k$  with leaf sets  $L_1, \dots, L_k$  as input, together with a DAG  $D = (N, A)$  specifying several child–parent relations between them. Two orders  $O_i$  and  $O_j$  represented by the PQ-trees  $T_i$  and  $T_j$ , respectively, satisfy the arc  $(T_i, T_j; \varphi)$  with the associated injective mapping  $\varphi: L_j \rightarrow L_i$  if  $O_i$  extends  $\varphi(O_j)$ . We seek *simultaneous orders*  $O_1, \dots, O_k$ , such that  $T_i$  represents  $O_i$  and all arcs are satisfied. In most cases it is not important to consider the map  $\varphi$  explicitly, hence we often simply write  $(T_i, T_j)$  instead of  $(T_i, T_j; \varphi)$ . An instance  $D = (N, A)$  of SIMULTANEOUS PQ-ORDERING is *normalized*, if an arc  $(T_i, T_j) \in A$  implies that  $\mathcal{L}_i$  contains an order  $O_i$  extending  $O_j$  for every order  $O_j \in \mathcal{L}_j$ , i.e., the child does not represent orders that cannot be extended at all. It is easy to see that every instance of SIMULTANEOUS PQ-ORDERING can be normalized by traversing  $D$  in top-down fashion and replacing  $T_j$  for each arc  $(T_i, T_j)$  by  $T_i|_{L_j} \cap T_j$ , consuming quadratic time. From now on, all instances of SIMULTANEOUS PQ-ORDERING are assumed to be normalized. The size of a node of  $D = (N, A)$  is the number of vertices of the corresponding PQ-tree, which is linear in its number of leaves since it does not contain degree-2 vertices. For every arc  $(T_i, T_j; \varphi) \in A$  we need to store the injective map  $\varphi$  from the leaves of  $T_j$  to the leaves of  $T_i$ , which takes space  $O(|T_j|)$ . The size of  $D$ , denoted by  $|D|$ , is the total size of all nodes plus the sizes of all arcs.

Not surprisingly, SIMULTANEOUS PQ-ORDERING turns out to be  $\mathcal{NP}$ -complete. The proof is by reduction from CYCLIC ORDERING, which is known to be  $\mathcal{NP}$ -complete [14].

**THEOREM 3.1.** SIMULTANEOUS PQ-ORDERING is  $\mathcal{NP}$ -complete.

In the following we identify a class of instances that can be solved efficiently, the 2-fixed instances.

#### 3.1 Critical Triples and the Expansion Graph

Let  $D = (N, A)$  be an instance of SIMULTANEOUS PQ-ORDERING and let  $(T, T_1) \in A$  be an arc. By choosing an order  $O_1 \in \mathcal{L}_1$  represented by  $T_1$  and extending  $O_1$  to an order  $O \in \mathcal{L}$ , we ensure that the constraint given by  $(T, T_1)$  is satisfied. Hence, our strategy will be to choose orders bottom-up, which can always be done for a single arc since our instances are normalized. However,  $T$  can have several children  $T_1, \dots, T_\ell$ , and orders  $O_i \in \mathcal{L}_i$  for  $i = 1, \dots, \ell$  cannot always be extended simultaneously to an order  $O \in \mathcal{L}$ . We derive necessary and sufficient conditions for the orders  $O_i$  to be simultaneously extendable under the additional

assumption that every P-node in  $T$  is fixed with respect to at most two children. We consider the Q- and P-nodes separately.

Let  $\mu$  be a Q-node in  $T$ . If  $\mu$  is fixed with respect to  $T_i$ , there is a unique Q-node  $\text{rep}(\mu)$  in  $T_i$  determining its orientation. By introducing a boolean variable  $x_\eta$  for every Q-node  $\eta$ , which is TRUE if  $\eta$  is oriented the same as a fixed reference orientation and FALSE otherwise, we can express the condition that  $\mu$  is oriented as determined by its representative by  $x_\mu = x_{\text{rep}(\mu)}$  or  $x_\mu \neq x_{\text{rep}(\mu)}$ . Haeupler et al. [17] use a similar technique to enforce consistent orientations of Q-nodes over several PQ-trees. For every Q-node in  $T$  that is fixed with respect to a child  $T_i$  we obtain such an (in)equality and we call the resulting set of (in)equalities the *Q-constraints*. It is obvious that the Q-constraints are necessary. On the other hand, if the Q-constraints are satisfied, all children of  $T$  fixing the orientation of  $\mu$  fix it in the same way. Note that the Q-constraints form an instance of 2-SAT that has linear size in the number of Q-nodes, which can be solved in polynomial [26] and even linear [11, 3] time. Hence, we only need to deal with the P-nodes, which is not as simple.

Let  $\mu$  be a P-node in  $T$ . If  $\mu$  is fixed with respect to only one child  $T_i$ , we can simply choose the order given by  $O_i$ . If  $\mu$  is additionally fixed with respect to  $T_j$ , it is of course necessary that the orders  $O_i$  and  $O_j$  induce the same order for the edges incident to  $\mu$  that are fixed with respect to both,  $T_i$  and  $T_j$ . We call such a triple  $(\mu, T_i, T_j)$ , where  $\mu$  is a P-node in  $T$  fixed with respect to the children  $T_i$  and  $T_j$  a *critical triple*. We say that the critical triple  $(\mu, T_i, T_j)$  is *satisfied* if the orders  $O_i$  and  $O_j$  induce the same order for the edges incident to  $\mu$  commonly fixed with respect to  $T_i$  and  $T_j$ . If we allow multiple arcs, we can also have a critical triple  $(\mu, T', T')$  for two parallel arcs  $(T, T'; \varphi_1)$  and  $(T, T'; \varphi_2)$ . Clearly, all critical triples need to be satisfied by the orders chosen for the children to be able to extend them simultaneously. The following lemma shows that satisfying all critical triples is not only necessary but also sufficient if every P-node is contained in at most one critical triple, that is, it is fixed with respect to at most two children of  $T$ . Note that the lemma does not hold in general if a P-node in  $T$  is fixed by three or more children.

**LEMMA 3.1.** Let  $T$  be a PQ-tree with children  $T_1, \dots, T_\ell$ , such that every P-node in  $T$  is contained in at most one critical triple, and let  $O_1, \dots, O_\ell$  be orders represented by  $T_1, \dots, T_\ell$ . An order  $O$  that is represented by  $T$  and simultaneously extends the orders  $O_1, \dots, O_\ell$  exists if and only if the Q-constraints and all critical triples are satisfied.

*Proof.* The only if part is clear. Conversely, assume that

we have orders  $O_1, \dots, O_\ell$  satisfying the Q-constraints and every critical triple. We show how to construct an order  $O$  represented by  $T$ , extending all orders  $O_1, \dots, O_\ell$  simultaneously. We find  $O$  by choosing orders for the internal nodes of  $T$  separately. For the Q-nodes the variable assignments of the Q-constraints imply consistent orientations. For a P-node  $\mu$  in  $T$  that is fixed with respect to at most one child, we choose the order of fixed edges incident to  $\mu$  as determined by the child and add the free edges arbitrarily. Otherwise,  $\mu$  is contained in exactly one critical triple  $(\mu, T_i, T_j)$ . We first choose the order of edges incident to  $\mu$  that are fixed with respect to  $T_i$  as determined by  $O_i$ . From the point of view of  $T_j$ , some of the fixed edges incident to  $\mu$  are already ordered, but this order is consistent with the order induced by  $O_j$ , since  $(\mu, T_i, T_j)$  is satisfied. The remaining edges incident to  $\mu$  that are fixed with respect to  $T_j$  can be added as determined by  $O_j$ , and the free edges can be added arbitrarily.  $\square$

Since testing whether the Q-constraints are satisfiable is easy, we concentrate on satisfying the critical triples. Let  $\mu$  be a P-node in a PQ-tree  $T$  such that  $\mu$  is fixed with respect to two children  $T_1$  and  $T_2$ , that is,  $(\mu, T_1, T_2)$  is a critical triple. By projecting  $T_1$  and  $T_2$  to representatives of the common fixed edges incident to  $\mu$  and intersecting the results, we obtain a new PQ-tree  $T(\mu, T_1, T_2)$ . There are natural injective maps from the leaves of  $T(\mu, T_1, T_2)$  to the leaves of  $T_1$  and  $T_2$ , hence we can add  $T(\mu, T_1, T_2)$  together with incoming arcs from  $T_1$  and  $T_2$  to our instance  $D$  of SIMULTANEOUS PQ-ORDERING. This procedure of creating  $T(\mu, T_1, T_2)$  is called *expansion step* with respect to the critical triple  $(\mu, T_1, T_2)$ , and the resulting new PQ-tree  $T(\mu, T_1, T_2)$  is called the *expansion tree* with respect to that triple; see Fig. 2 for an example of the expansion step.

We introduce the expansion tree for the following reason. If we find orders  $O_1$  and  $O_2$  represented by  $T_1$  and  $T_2$  that both extend the same order represented by the expansion tree  $T(\mu, T_1, T_2)$ , we ensure that the edges incident to  $\mu$  fixed with respect to both,  $T_1$  and  $T_2$ , are ordered the same in  $O_1$  and  $O_2$ , or in other words, we ensure that  $O_1$  and  $O_2$  satisfy the critical triple  $(\mu, T_1, T_2)$ . By Lemma 3.1, we know that satisfying the critical triples is necessary, thus we do not lose solutions by adding expansion trees to an instance of SIMULTANEOUS PQ-ORDERING. Furthermore, it is also sufficient, if every P-node is contained in at most one critical triple (if we forget about the Q-nodes for a moment). Hence, given an instance  $D$  of SIMULTANEOUS PQ-ORDERING, we want to expand  $D$  iteratively until no unprocessed critical triples are left and find simultaneous orders bottom-up. Unfortunately, it can happen that the expansion does not terminate and thus

yields an infinite graph. Thus, we define a special case where we do not expand further. Let  $\mu$  be a P-node of  $T$  with outgoing arcs  $(T, T_1; \varphi_1)$  and  $(T, T_2; \varphi_2)$  such that  $(\mu, T_1, T_2)$  is a critical triple. If  $T_i$  (for  $i = 1, 2$ ) consists only of a single P-node, the image of  $\varphi_i$  is a set of representatives of the edges incident to  $\mu$  that are fixed with respect to  $T_i$ . Hence  $\varphi_i$  is a bijection between  $L_i$  and the fixed edges incident to  $\mu$ . If additionally the fixed edges with respect to both,  $T_1$  and  $T_2$ , are the same, we obtain a bijection  $\varphi: L_2 \rightarrow L_1$  as  $\varphi = \varphi_1^{-1} \circ \varphi_2$ . Assume without loss of generality that there is no directed path from  $T_2$  to  $T_1$  in the current DAG. If there is neither a directed path from  $T_1$  to  $T_2$  nor from  $T_2$  to  $T_1$ , we achieve uniqueness by assuming that  $T_1$  comes before  $T_2$  with respect to some fixed order of the nodes in  $D$ . Instead of an expansion step we apply a *finalizing step* by simply creating the arc  $(T_1, T_2; \varphi)$ . This new arc ensures that the critical triple  $(\mu, T_1, T_2)$  is satisfied if we have orders for the leaves  $L_1$  and  $L_2$  respecting  $(T_1, T_2; \varphi)$ . It can be shown that infinite expansion results from repeated creation of trees consisting of a single P-node; the finalizing step prevents this.

For the case that  $(\mu, T', T')$  is a critical triple resulting from two parallel arcs  $(T, T'; \varphi_1)$  and  $(T, T'; \varphi_2)$ , we can apply the expansion step as described above. However, the finalizing step would introduce a loop. In this case, we omit the loop and mark  $(T, T'; \varphi_1)$  and  $(T, T'; \varphi_2)$  as a *critical double arc*. When choosing orders bottom-up in the DAG, we have to explicitly ensure that all critical triples stemming from critical double arcs are satisfied. To simplify this, we ensure that all targets of critical double arcs are sinks in the expansion graph. This follows from the construction, except for the case when the critical double arc is already contained in the input instance. In this case, we apply one additional expansion step, which essentially clones the double arc.

To sum up, we start with an instance  $D$  of SIMULTANEOUS PQ-ORDERING. As long as  $D$  contains unprocessed critical triples  $(\mu, T_1, T_2)$  we apply expansion and finalizing steps. The resulting graph is called the *expansion graph* of  $D$  and is denoted by  $D_{\text{exp}}$ . Note that  $D_{\text{exp}}$  is also an instance of SIMULTANEOUS PQ-ORDERING. The finalizing step ensures finiteness of  $D_{\text{exp}}$ , and moreover, it is obtained by adding necessary conditions, and hence is equivalent to the original instance  $D$ .

**LEMMA 3.2.** *The expansion graph  $D_{\text{exp}}$  of  $D$  is unique and finite.*

**LEMMA 3.3.** *The instances  $D$  and  $D_{\text{exp}}$  of SIMULTANEOUS PQ-ORDERING are equivalent.*

For now, we know that we can consider the expansion graph instead of the original instance to solve

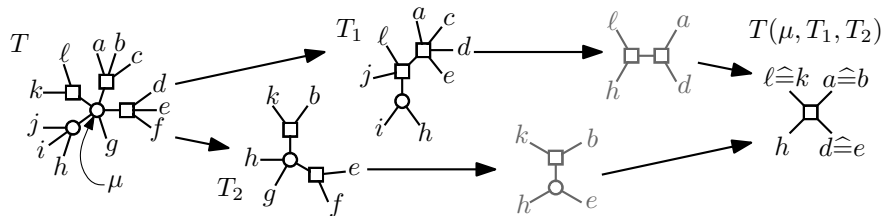


Figure 2: The P-node  $\mu$  in the PQ-tree  $T$  is fixed with respect to the children  $T_1$  and  $T_2$ . We first project  $T_1$  and  $T_2$  to representatives of the common fixed edges incident to  $\mu$  and intersect the result to obtain  $T(\mu, T_1, T_2)$ ; the gray shaded projections only illustrate an intermediate step.

**SIMULTANEOUS PQ-ORDERING.** Lemma 3.1 motivates that we can solve the instance given by the expansion graph by simply choosing orders bottom-up, if the Q-constraints and the critical double arcs are satisfied. However, this only works for “simple” instances where P-nodes in the expansion graph are fixed with respect to at most two children; we call such instances *1-critical*. To make this approach useful, we need to somehow bound the size of the expansion graph.

**3.2 1-Critical and 2-Fixed Instances** In this section we consider the expansion graph  $D_{\text{exp}}$  for 1-critical instances  $D$  and show that in this case, the size of  $D_{\text{exp}}$  is polynomial in the size of  $D$ , which is not true for general instances. We make use of the following lemma. Afterwards, we show how to solve 1-critical instances efficiently.

**LEMMA 3.4.** *For every 1-critical instance  $D$ , targets of critical double arcs in  $D_{\text{exp}}$  are sinks.*

**LEMMA 3.5.** *A 1-critical instance  $D$  can be solved in time polynomial in  $|D_{\text{exp}}|$ .*

*Proof.* Due to Lemma 3.3, we can solve the instance  $D_{\text{exp}}$  of SIMULTANEOUS PQ-ORDERING instead of  $D$  itself. Of course we cannot find simultaneous PQ-orders for the PQ-trees in  $D_{\text{exp}}$  if any of these PQ-trees is the null tree. Additionally, Lemma 3.1 states that  $D$  admits a solution if and only if we can satisfy the Q-constraints and all critical triples. The Q-constraints can be checked in linear time [11, 3]. The critical triples, except for the ones defined by critical double arcs, are enforced by the expansion graph  $D_{\text{exp}}$  when choosing orders bottom-up. It remains to deal with the critical double arcs, whose targets are sinks by Lemma 3.4.

Let  $(T, T'; \varphi_1)$  together with  $(T, T'; \varphi_2)$  be a critical double arc. By construction,  $T'$  consists of a single P-node fixing the same edges incident to a single P-node  $\mu$  in  $T$  with respect to both arcs. To satisfy the critical triple  $(\mu, T', T')$ , we need to find an order  $O'$  of  $L'$  such that  $\varphi_1(O') = \varphi_2(O')$ . This is equivalent to finding an

order  $O'$  for which  $\varphi = \varphi_2^{-1} \circ \varphi_1$  is order-preserving, which can be tested with Lemma 2.1a. If such orders can be found for all critical double arcs, we simply choose the remaining orders bottom-up, which is always possible if the Q-constraints are satisfied and  $D_{\text{exp}}$  does not contain the null-tree.  $\square$

Our next step is to bound the size of the expansion graph. It can be exponential in the size of the original instance. However, for 1-critical instances, it turns out to be polynomially bounded.

**LEMMA 3.6.** *Let  $D$  be a 1-critical instance of SIMULTANEOUS PQ-ORDERING. The size of its expansion graph  $D_{\text{exp}}$  is quadratic in  $|D|$ .*

Lemma 3.5 and 3.6 together imply a polynomial-time algorithm for SIMULTANEOUS PQ-ORDERING for 1-critical instances. A detailed analysis of the running time gives the following.

**THEOREM 3.2.** *1-critical instances of SIMULTANEOUS PQ-ORDERING can be solved in quadratic time.*

Note that 1-criticality is not only important to ensure that the expansion graph has polynomial size (Lemma 3.6) but also to be able to solve the instance at all (Lemma 3.5). This comes from the fact that satisfying critical triples is not sufficient to be able to extend several orders simultaneously if a P-node is contained in more than one critical triple; see Lemma 3.1.

Actually, Theorem 3.2 tells us how to solve 1-critical instances, which was the main goal of this section. However, the characterization of the 1-critical instances is not really satisfying, since we need to know the expansion graph, which may be exponentially large for instances that are not 1-critical, to check whether an instance is 1-critical or not. In the following we present a simple sufficient criterion for 1-criticality, which does not involve the expansion graph.

Let  $D$  be an instance of SIMULTANEOUS PQ-ORDERING. We define the *fixedness*  $\text{fixed}(\mu)$  of a P-node  $\mu$  in a PQ-tree of  $D$  as follows. For a P-node  $\mu$

belonging to a tree  $T$  that is a source in  $D$ , we define  $\text{fixed}(\mu)$  to be the number of children fixing it. Now let  $\mu$  be a P-node of some internal PQ-tree  $T$  of  $D$  with parents  $T_1, \dots, T_\ell$ . Each of the trees  $T_i$  contains exactly one P-node  $\mu_i$  that is fixed by  $\mu$ . Additionally, let  $k'$  be the number of children fixing  $\mu$ . We set  $\text{fixed}(\mu) = k' + \sum (\text{fixed}(\mu_i) - 1)$ . We say that an instance  $D$  is  $k$ -fixed for some integer  $k$ , if  $\text{fixed}(\mu) \leq k$  holds for all P-nodes  $\mu$  of all PQ-trees in  $D$ . The motivation for this definition is that a P-node with fixedness  $k$  in  $D$  is fixed with respect to at most  $k$  children in the expansion graph  $D_{\text{exp}}$ . We obtain the following theorem providing sufficient conditions for  $D$  to be a 1-critical instance.

**THEOREM 3.3.** *Every 2-fixed instance of SIMULTANEOUS PQ-ORDERING is 1-critical.*

Theorem 3.2 and 3.3 together provide a framework for solving problems that can be formulated as instances of SIMULTANEOUS PQ-ORDERING. For some applications it will be useful to allow so-called *reversing arcs*, not enforcing an order to be an extension of the order provided by the child, but requiring that it is an extension of the *reversal* of this order.

In the following we briefly sketch why all results still hold when we also allow reversing arcs. First of all, for the case that every P-node of a parent is fixed by at most two children, it is readily seen that orders for children can be simultaneously extended if and only if the Q-constraints and the critical triples are satisfied, i.e., Lemma 3.1 still holds in the presence of reversing arcs. Moreover, the construction of the expansion graph can be naturally extended to instances containing reversing arcs with the only exception that arcs created due to reversing arcs in an expansion step are also reversing, and similarly for the finalizing step. Hence, the expansion graph of an instance with reversing arcs can be obtained by taking the usual expansion graph, ignoring that some arcs are reversing, and then suitably replacing some of the normal arcs by reversing arcs. This shows that all structural results on the expansion graph still hold. The equivalence of an instance with its expansion graph can be proved literally. The only difference is when choosing orders for the targets of critical double arcs containing exactly one reversing arc. In this case we need to ensure that the corresponding permutation (see Lemma 3.5) is order-reversing, instead of order-preserving, and we use case b) of Lemma 2.1. Hence, Theorems 3.2 and 3.3 also hold in the presence of reversing arcs, and in the following we allow reversing arcs without further notice.

## 4 Applications

In this section we present several applications. In particular, we give an optimal linear-time algorithm for recognizing simultaneous interval graphs and apply it to the problem of extending partial interval representations. Moreover, we present a novel representations of all planar embeddings of biconnected graphs in terms of simultaneous PQ-orders, leading to efficient algorithms for the problems PARTIALLY PQ-CONSTRAINED PLANARITY for biconnected graphs and SEFE for biconnected graphs with a connected intersection.

### 4.1 Simultaneous and Partial Interval Graph Representations

An *interval representation* of a graph  $G$  represents every vertex as an interval such that two vertices are adjacent if and only if their intervals intersect;  $G$  is then called an *interval graph*. Fulkerson and Gross [13] characterize interval graphs in terms of orderings of maximal cliques. A linear order of sets is *v-consecutive* if the sets containing  $v$  appear consecutively. A *clique ordering* is a linear order of the maximal cliques of a graph. It is *valid* if it is *v-consecutive* for every vertex  $v$ . The first algorithm recognizing interval graphs in linear time was given by Booth and Lueker [7] and was based on the characterization by Fulkerson and Gross [13], stating that a graph is an interval graph if and only if it admits a valid clique ordering. Rooted PQ-trees were originally designed to model exactly such constraints.

**THEOREM 4.1.** (FULKERSON AND GROSS [13]) *A graph  $G$  is an interval graph if and only if there is a linear order of all maximal cliques of  $G$  that is v-consecutive with respect to every vertex  $v$ .*

Jampani and Lubiw [22] introduce *simultaneous interval graphs*, which are graphs  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$  sharing a common subgraph admitting interval representations such that the common vertices are represented by the same intervals. They give an  $\mathcal{O}(n^2 \log n)$ -time algorithm for the corresponding recognition problem SIMULTANEOUS INTERVAL REPRESENTATION. Our algorithm is based on a novel characterization of simultaneous interval graphs in terms of the orderings of the union of the maximal cliques of  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$ .

**THEOREM 4.2.** *Two graphs  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$  are simultaneous interval graphs if and only if there exists a linear order of the union of their maximal cliques that is v-consecutive with respect to every common vertex  $v$ , and whose restrictions to the maximal cliques of  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$  yield valid clique orderings.*

*Proof.* Assume  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$  are simultaneous interval graphs and let for every vertex  $v$  be  $I(v)$  the interval

representing  $v$ . Assume  $\mathcal{C}^{\textcircled{1}} = \{C_1^{\textcircled{1}}, \dots, C_k^{\textcircled{1}}\}$  and  $\mathcal{C}^{\textcircled{2}} = \{C_1^{\textcircled{2}}, \dots, C_\ell^{\textcircled{2}}\}$  are the maximal cliques in  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$  respectively. When considering  $G^{\textcircled{1}}$  for itself, we again obtain for every maximal clique  $C^{\textcircled{1}} = \{v_1, \dots, v_r\}$  a position  $x$  such that  $x$  is contained in  $I(v_i)$  for every  $v_i \in C^{\textcircled{1}}$  but in no other interval representing a vertex in  $G^{\textcircled{1}}$ . The same can be done for the maximal cliques of  $G^{\textcircled{2}}$ , yielding a linear order  $O$  of all maximal cliques  $\mathcal{C} = \mathcal{C}^{\textcircled{1}} \cup \mathcal{C}^{\textcircled{2}}$ . It is clear that the projection of this order to the cliques in  $G^{\textcircled{1}}$  is  $v$ -consecutive for every vertex  $v$  in  $G^{\textcircled{1}}$  due to Theorem 4.1 and the same holds for  $G^{\textcircled{2}}$ . It remains to show that  $O$  is  $v$ -consecutive for each common vertex  $v$ . Assume  $O$  is not  $v$ -consecutive for some common vertex  $v$ . Then there need to be three cliques  $C_i, C_j$  and  $C_k$ , no matter if they are maximal cliques in  $G^{\textcircled{1}}$  or in  $G^{\textcircled{2}}$ , with positions  $x_i, x_j$  and  $x_k$  such that  $x_i < x_j < x_k$  and  $v \in C_i, C_k$  but  $v \notin C_j$ . However, since the interval  $I(v)$  contains  $x_i$  and  $x_k$  it also contains  $x_j$ , which is a contradiction to the construction of the position  $x_j$  for the clique  $C_j$  since  $v$  is a common vertex. Note that this is the same argument as used in the proof of Theorem 4.1.

Conversely, we need to show how to construct an interval representation from a given linear order of all maximal cliques. Assume we have a linear order  $O$  of all maximal cliques satisfying the conditions of the theorem. Rename the cliques such that  $C_1 \dots C_{k+\ell}$  is this order, neglecting for a moment from which graph the cliques stem. Let  $v$  be a vertex in  $G^{\textcircled{1}}$  or  $G^{\textcircled{2}}$  and let  $C_i$  and  $C_j$  be the leftmost and rightmost clique in  $O$  containing  $v$ . Then we define the interval  $I(v)$  to be  $[i, j]$ , as in the case of a single graph. Our claim is that this yields a simultaneous interval representation of  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$ . Again, it is easy to see that a non-integer position  $x$  is only contained in intervals also containing  $\lfloor x \rfloor$  and  $\lceil x \rceil$ . Thus we only need to consider the positions  $1, \dots, k + \ell$ , let  $i$  be such an integral position. Assume without loss of generality that  $C_i = \{v_1, \dots, v_r\}$  is a clique of  $G^{\textcircled{1}}$ . Then  $i$  is contained in all the intervals  $I(v_1), \dots, I(v_r)$  by definition. The position  $i$  may be additionally contained in the interval  $I(u)$  for a vertex that is exclusively contained in  $G^{\textcircled{2}}$  but this does not create an edge between vertices in  $G^{\textcircled{1}}$ . However, there is no vertex  $u \notin C_i$  contained in  $G^{\textcircled{1}}$  such that  $i$  is contained in  $I(u)$  since this would violate the  $u$ -consecutiveness either of the whole order or of the projection to the cliques in  $G^{\textcircled{1}}$ . Since the same argument works for cliques in  $G^{\textcircled{2}}$ , all edges in maximal cliques of  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$  are represented by the defined interval representation and at the integer positions no edges not contained are represented. Hence, this definition yields a simultaneous interval representation of  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$ .  $\square$

With this characterization it is straightforward to formulate the problem of recognizing simultaneous interval graphs as an instance of SIMULTANEOUS PQ-ORDERING. Let  $\mathcal{C}^{\textcircled{1}}$  and  $\mathcal{C}^{\textcircled{2}}$  denote the sets of maximal cliques of  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$ , respectively. We create a rooted PQ-tree  $T$  representing all orders of  $\mathcal{C}^{\textcircled{1}} \cup \mathcal{C}^{\textcircled{2}}$  that are  $v$ -consecutive for all common vertices of  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$ . As children of  $T$  we add the rooted PQ-trees representing the valid clique orderings of  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$ , respectively. Note that by adding a special leaf to each of the three trees, we obtain an unrooted instance representing the same orderings. Again, this instance obviously models exactly the necessary conditions. Moreover, it is 2-fixed and a special analysis shows that the running time is actually linear.

**THEOREM 4.3.** SIMULTANEOUS INTERVAL REPRESENTATION can be solved in linear time.

*Proof.* Let  $\mathcal{C}^{\textcircled{1}} = \{C_1^{\textcircled{1}}, \dots, C_k^{\textcircled{1}}\}$  and  $\mathcal{C}^{\textcircled{2}} = \{C_1^{\textcircled{2}}, \dots, C_\ell^{\textcircled{2}}\}$  be the maximal cliques of  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$  respectively and let  $\mathcal{C} = \mathcal{C}^{\textcircled{1}} \cup \mathcal{C}^{\textcircled{2}}$  be the set of all maximal cliques. We define three PQ-trees  $T, T^{\textcircled{1}}$  and  $T^{\textcircled{2}}$  having  $\mathcal{C}, \mathcal{C}^{\textcircled{1}}$  and  $\mathcal{C}^{\textcircled{2}}$  as leaves, respectively. The tree  $T$  is defined such that it represents all linear orders of  $\mathcal{C}$  that are  $v$ -consecutive with respect to all common vertices  $v$ . The trees  $T^{\textcircled{1}}$  and  $T^{\textcircled{2}}$  are defined to represent all linear orders of  $\mathcal{C}^{\textcircled{1}}$  and  $\mathcal{C}^{\textcircled{2}}$  that are  $v$ -consecutive with respect to all vertices  $v$  in  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$ , respectively. Note that  $T^{\textcircled{1}}$  and  $T^{\textcircled{2}}$  are the PQ-trees that would be used to test whether  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$  themselves are interval graphs. By the characterization in Theorem 4.2 it is clear that  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$  are simultaneous interval graphs if and only if we can find an order represented by  $T$  extending orders represented by  $T^{\textcircled{1}}$  and  $T^{\textcircled{2}}$ . Hence  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$  are simultaneous interval graphs if and only if the instance  $D$  of SIMULTANEOUS PQ-ORDERING consisting of the nodes  $T, T^{\textcircled{1}}$  and  $T^{\textcircled{2}}$  and the arcs  $(T, T^{\textcircled{1}})$  and  $(T, T^{\textcircled{2}})$  has a solution. This can be checked in quadratic time using Theorem 3.2 since  $D$  is obviously 1-critical. Furthermore, normalization can of course be done in linear time and the expansion tree of linear size can be computed in linear time since expansion stops after a single expansion step. Hence the instance  $D$  of SIMULTANEOUS PQ-ORDERING can be solved in linear time, which concludes the proof.  $\square$

Klavík et al. [25] consider the problem PARTIAL INTERVAL GRAPH EXTENSION asking, whether there exists an interval representation of the input graph  $G$  such that certain vertices are represented by prescribed intervals. More precisely, let  $G$  be a graph,  $H = (V, E)$  be a subgraph of  $G$  and let  $I$  be an interval representation of  $H$ . The problem PARTIAL INTERVAL GRAPH EXTENSION asks, whether there exists an interval graph



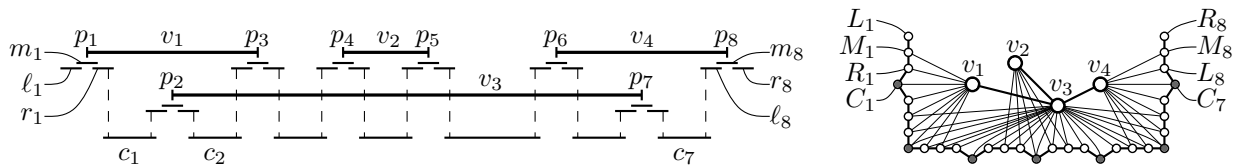


Figure 3: Prescribed intervals corresponding to vertices  $v_1, \dots, v_4$  and additional intervals fixing this representation. The resulting graph  $G'$  with the new vertices is shown on the right.

representation  $I'$  of  $G$  such that for all  $v \in V$  we have that  $I'(v) = I(v)$ . We call an instance  $(G, H, I)$  of PARTIAL INTERVAL GRAPH EXTENSION a *partial interval graph*. Klavík et al. [25] show that PARTIAL INTERVAL GRAPH EXTENSION can be solved in time  $O(n^2)$ . We show that PARTIAL INTERVAL GRAPH EXTENSION can be reduced in  $O(n + m)$  time to an instance of SIMULTANEOUS INTERVAL REPRESENTATION. It then follows from Theorem 4.3 that the partial interval graph extension problem can be solved in  $O(n + m)$  time.

Without loss of generality, we assume that the endpoints of all intervals  $I(v), v \in V(H)$  are distinct. For  $v \in V(H)$  let  $\ell(v)$  and  $r(v)$  denote the left and right endpoint of  $I(v)$ , respectively. Further let  $S(I)$  denote the sequence of these endpoints in increasing order of coordinate. We call this order the *signature* of  $I$ . We say that two interval representations  $I$  and  $I'$  of the same graph  $H$  are *equivalent* if they have the same signature. Klavík et al. [25] show that PARTIAL INTERVAL GRAPH EXTENSION for a partial interval graph  $(G, H, I)$  is equivalent to deciding whether there exists an interval representation  $I'$  of  $G$  whose restriction to  $H$  is equivalent to  $I$ . In the following we construct an interval graph  $G'$  containing  $H$  as an induced subgraph such that every interval representation of  $G'$  induces an interval representation of  $H$  that is equivalent to  $I$ .

Let  $p_1, \dots, p_{2n}$  denote the interval endpoints of  $I$  in increasing order. We now add several intervals to the representation. Namely, for each point  $p_i$  we put three intervals of length  $\varepsilon$ . The interval  $\ell_i$  is to the left of  $p_i$ , interval  $r_i$  is to the right of  $p_i$  and  $m_i$  contains  $p_i$  and intersects both  $\ell_i$  and  $r_i$ . We choose  $\varepsilon$  small enough so that no two intervals of distinct points  $p_i$  and  $p_j$  intersect. We call these intervals *markers*. Finally, we add  $2n - 1$  *connectors*, where the connector  $c_i$ , for  $i = 1, \dots, 2n - 1$  lies strictly between  $p_i$  and  $p_{i+1}$ , and intersects  $r_i$  and  $\ell_{i+1}$ ; see Figure 3 for an example. Now consider the graph  $G'$  given by this interval representation containing  $H$  as induced subgraph and the new vertices  $L_i, M_i, R_i$  and  $C_i$  corresponding to the intervals  $\ell_i, m_i, r_i$  and  $c_i$ . Then  $(G, G')$  defines an instance of SIMULTANEOUS INTERVAL REPRESENTATION corresponding to the instance  $(G, H, I)$  of PARTIAL IN-

TERVAL GRAPH EXTENSION and we obtain the following theorem by showing their equivalence.

**THEOREM 4.4.** PARTIAL INTERVAL GRAPH EXTENSION can be solved  $O(n + m)$  time.

*Proof.* Let  $(G, H, I)$  be an instance of PARTIAL INTERVAL GRAPH EXTENSION and let  $(G, G')$  be the corresponding instance of SIMULTANEOUS INTERVAL REPRESENTATION as defined above. We need to show that these two instances are equivalent and that  $(G, G')$  has size linear in the size of  $(G, H, I)$ .

Obviously  $G'$  contains  $H$  as an induced subgraph. We claim that in any interval representation  $I'$  of  $G'$  the subrepresentation  $I'|_H$  is equivalent to  $I$ . First, note that the sequence  $L_1, M_1, R_1, C_1, \dots, C_{2n-1}, L_{2n}, M_{2n}, R_{2n}$  is an induced path in  $G'$ . Hence, in every representation of  $G'$  the starting points of their intervals occur either in this or in the reverse order. In particular, the marker intervals  $I'(M_i)$  are pairwise disjoint and sorted. Let  $v_i$  denote the vertex whose interval has  $p_i$  as an endpoint. Since  $M_i$  is adjacent to  $L_i$  and  $R_i$ , exactly one of which is adjacent  $v_i$ , it follows that  $I'(M_i)$  contains an endpoint of  $I'(v_i)$ . Since this holds for each marker  $M_i$ , the claim follows.

With this result the equivalence of the instance  $(G, H, I)$  and  $(G, G')$  is easy to see. If  $(G, H, I)$  admits an interval representation of  $G$ , then the above construction shows how to construct a corresponding simultaneous representation of  $(G, G')$ . On the other hand, if  $G$  and  $G'$  admit a simultaneous interval representation, then the endpoints of the intervals corresponding to vertices of  $H$  must occur in the same order as in  $I$ , and hence the interval representation of  $G$  extends  $I$ .

It remains to show that  $G'$  has size linear in the size of  $H$ . To this end, we revisit the construction of  $G'$  from  $H$ . Let  $H'$  be the subgraph of  $G'$  obtained by removing the vertices corresponding to connectors. We first show that the size of  $H'$  is linear in the size of  $H$ .

Clearly,  $H'$  contains exactly six additional vertices for each vertex of  $H$  (three for each endpoint of an interval representing a vertex of  $H$ ), and thus  $|V(H')| = 7n$ . Now consider the edges of  $H'$ . We denote by  $I(p)$

the set of vertices whose intervals contain  $p$  in the interior. Let again  $p_1, \dots, p_{2n}$  denote the endpoints of the intervals in the interval representation  $I$  of  $H$ . Recall that for each such endpoint we add three vertices, which are represented by the intervals  $\ell_i, m_i$  and  $r_i$ , respectively. Note that the endpoints  $p_{i-1}$  and  $p_{i+1}$  (if they exist) lie to the left of  $\ell_i$  and to the right of  $r_i$ , respectively, and hence do not intersect with these intervals. The neighbors of  $L_i, M_i$  and  $R_i$  belonging to  $H$  are contained in  $I(p_i) \cup \{v_i\}$ . This implies that the degree of  $L_i, M_i$  and  $R_i$  is linear in the degree of  $v_i$  in  $H$ , and hence the total number of edges in  $H'$  is linear in  $|E(H)|$ .

For the step from  $H'$  to  $G'$ , we add the connectors. Consider the  $i$ th connector  $C_i$ , which is adjacent to  $R_i$  and  $L_{i+1}$ . Since no other intervals start or end in between, the vertex corresponding to the connector  $C_i$  is adjacent to the same vertices as  $R_i$  and  $L_{i+1}$ . Thus, the size of  $G'$  is linear in the size of  $H'$  and the claim follows. Moreover, it is clear that, assuming the intervals of  $I$  are given in sorted order,  $G'$  can be constructed from  $G$  in  $O(n+m)$  time.  $\square$

**4.2 Constrained Embedding Problems** Constrained embedding problems ask for a planar graph whether it can be drawn without crossings in the plane, satisfying some additional constraints. Angelini et al. [1] and Jelínek et al. [23] study the problem of finding embeddings of graphs for which the embedding of a subgraph is fixed. The case that the order of every edge around every vertex is constrained by a PQ-tree has been studied by Gutwenger et al. [16]. None of their approaches can be applied to PARTIALLY PQ-CONSTRAINED PLANARITY; either a subgraph is completely fixed, or the whole graph is constrained by PQ-trees, whereas we also allow a subgraph to be PQ-constrained.

Concerning the simultaneous embedding problem SEFE, the most prevalent open question is the computational complexity for two graphs. It is known that the problem is  $\mathcal{NP}$ -complete for three graphs [15] and for two graphs the problem is linear-time solvable if one of the graphs has at most one cycle [12], if the intersection graph is biconnected [2, 17], and if the common graph consists of a set of cycles [6]. Schaefer [27] gives an efficient testing algorithm for the cases that (1) the common graph consists of disjoint biconnected components and isolated vertices, (2) the common graph has maximum degree 3, and (3) the first graph is a disjoint union of triconnected graphs. Bläsius et al. [4] give an extensive survey. Jünger and Schulz [24] show that two graphs admit a SEFE if and only if they have planar embeddings that coincide on the intersection graph.

To address these problems we introduce the *PQ-embedding representation*, a 1-fixed instance of SIMULTANEOUS PQ-ORDERING representing all planar embeddings of a biconnected graph. We then show that the additional constraints of the two problems can be added to obtain a 2-fixed instance, which can then be solved efficiently as shown by our main result.

**A New Embedding Representation** Recall that a graph is *biconnected* if it is connected and does not contain a vertex whose removal disconnects the graph. A *separation pair* in a biconnected graph is a pair of vertices whose removal disconnects the graph. A biconnected graph without separation pairs is *triconnected*. A *split pair* is either a separation pair or an edge. The SPQR-tree  $\mathcal{T}$  of a biconnected planar graph  $G$  is a tree representing a recursive decomposition along its split pairs [9, 10]. The SPQR-tree provides a succinct description of all planar embeddings of  $G$ . Its internal nodes are either S-, P- or R-nodes, each of them is associated with a *skeleton*, which is a multigraph on a subset of the vertices of  $G$ . The embedding choices are 1) flips for the R-nodes (whose skeleton is triconnected and thus has a unique embedding up to a flip) and 2) circular orders for P-nodes (whose skeleton consists of parallel edges between two nodes). These choices are in bijection with the embeddings of  $G$ .

It follows from the SPQR-tree that the possible circular orders around every vertex  $v$  in a planar embedding can be represented by a PQ-tree  $T(v)$ , the *embedding tree*. In particular,  $T(v)$  has a P-node for every P-node in the SPQR-tree whose skeleton contains  $v$  and a Q-node (which is fixed up to a flip) for every R-node whose skeleton contains  $v$ . We say that the nodes of the PQ-tree *stem* from the corresponding node of the SPQR-tree. Choosing orders for all embedding trees independently does generally not give a planar embedding. To obtain an embedding, we need to ensure that all nodes of the embedding trees stemming from the same node in the SPQR-tree are ordered consistently, i.e., 1) they fix the same flip for R-nodes and 2) they fix opposite orders for P-nodes.

We formulate these conditions as an instance  $D(G)$  of SIMULTANEOUS PQ-ORDERING by introducing common children to the embedding trees. To ensure 1), we create for each R-node  $\eta$  in the SPQR-tree  $\mathcal{T}$  a tree  $Q(\eta)$  consisting of a single Q-node and add it as a common child of all embedding trees that contain a Q-node stemming from  $\eta$  with suitable injective maps. For 2) we add for each P-node  $\mu$  in the SPQR-tree whose skeleton contains  $u$  and  $v$  and  $k$  parallel edges between them, a common child  $P(\mu)$  of  $T(u)$  and  $T(v)$  consisting of a single P-node with  $k$  leaves. Exactly one of the arcs is reversing to ensure that the orders are chosen oppo-

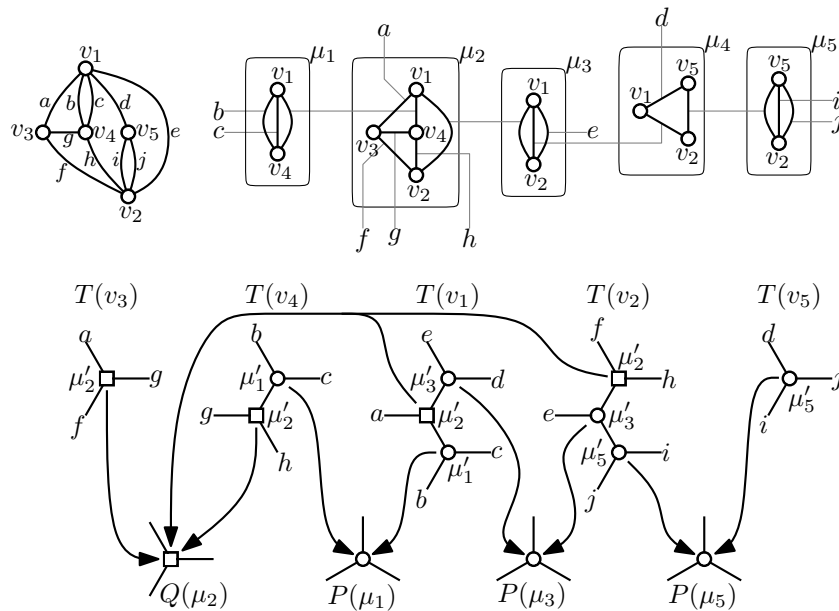


Figure 4: A biconnected planar graph and its SPQR-tree on the top and the corresponding PQ-embedding representation on the bottom. The injective maps on the edges are not explicitly depicted, but the starting points of the arcs suggests which maps are suitable.

sitely. It is not hard to see that the planar embeddings of  $G$  are in bijection with the simultaneous orderings of  $D(G)$ , and that the size of  $D(G)$  is linear in the size of  $G$ . Moreover,  $D(G)$  is 1-fixed by construction. We call this instance the PQ-embedding representation of  $G$ ; see Figure 4 for an example. As  $D(G)$  contains the embedding trees of all vertices, it has the advantage that we can now easily add additional constraints concerning the orders of edges.

**Partially PQ-Constrained Embedding** With the PQ-embedding representation it is easy to solve PARTIALLY PQ-CONSTRAINED PLANARITY for a biconnected graph  $G = (V, E)$  with *constraint trees*  $T'(v)$  for  $v \in V$ , each describing a set of circular orders of a subset of the edges incident to  $v$ . We simply take the PQ-embedding representation of  $G$  and for each vertex  $v \in V$  we add the tree  $T'(v)$  and an arc  $(T(v), T'(v))$ . Clearly, we have added exactly the necessary conditions for PARTIALLY PQ-CONSTRAINED PLANARITY. It is not hard to verify that the instance is 2-fixed, and can therefore be solved in polynomial time. A separate analysis shows that the running time is actually linear for this special case.

**THEOREM 4.5.** PARTIALLY PQ-CONSTRAINED PLANARITY can be solved in  $O(n^2)$  time for biconnected graphs.

*Proof.* Consider  $(G, C)$  to be an instance of PARTIALLY PQ-CONSTRAINED PLANARITY where  $G$  is a bicon-

nected planar graph and  $C$  the set of constraint trees. Let further  $D(G, C)$  be the corresponding instance of SIMULTANEOUS PQ-ORDERING. Since  $D(G, C)$  contains the PQ-embedding representation  $D(G)$ , every solution of  $D(G, C)$  yields a planar embedding of  $G$ . Additionally, this planar embedding respects the constraint trees since the order of edges around every vertex is an extension of an order of the leaves in the corresponding constraint tree. On the other hand, it is clear that a planar embedding of  $G$  respecting the constraint trees yields simultaneous orders for all trees in  $D(G, C)$ . Since the size of  $D(G, C)$  is linear in the size of  $(G, C)$ , we can solve  $(G, C)$  in quadratic time using Theorem 3.2, if  $D(G, C)$  is 1-critical. We will show that the instance  $D(G, C)$  is at most 2-fixed, and hence, due to Theorem 3.3 also 1-critical.

To compute the fixedness of every P-node in every PQ-tree in  $D(G, C)$ , we distinguish between three kinds of trees, the embedding trees, the consistency trees and the constraint trees. If we consider a P-node  $\mu$  in an embedding tree  $T(v)$ , this P-node is fixed with respect to exactly one consistency tree, namely the tree that represents the P-node in the SPQR-tree  $\mu$  stems from. In addition to the consistency trees,  $T(v)$  has the constraint tree  $T'(v)$  as child, thus  $\mu$  can be fixed with respect to  $T'(v)$ . Since  $T(v)$  has no parents and no other children,  $\mu$  is at most 2-fixed, that is  $\text{fixed}(\mu) \leq 2$ . Consider a P-node  $\mu'$  in a constraint tree  $T'(v)$ . Since  $T'(v)$  has no children and its only parent

is  $T(v)$  containing the P-node  $\mu$  that is fixed by  $\mu'$ , we have by the definition of fixedness that  $\text{fixed}(\mu') = \text{fixed}(\mu) - 1$ . Since  $\mu$  is a P-node in an embedding tree we, obtain  $\text{fixed}(\mu') \leq 1$ . We have two kinds of consistency trees, some stem from P- and some from R-nodes in the SPQR-tree. We need to consider only trees  $P(\mu)$  stemming from P-nodes since the consistency trees stemming from R-nodes only contain a single Q-node. Denote the single P-node in  $P(\mu)$  also by  $\mu$  and let  $\mu_1$  and  $\mu_2$  be the two P-nodes in the embedding trees  $T(v_1)$  and  $T(v_2)$  that are fixed with respect to  $P(\mu)$ . Since  $P(\mu)$  has no child and only these two parents, we obtain  $\text{fixed}(\mu) = (\text{fixed}(\mu_1) - 1) + (\text{fixed}(\mu_2) - 1)$ . Since  $\mu_1$  and  $\mu_2$  are P-nodes in embedding trees this yields  $\text{fixed}(\mu) \leq 2$ . Hence, all P-nodes in all PQ-trees in  $D(G, C)$  are at most 2-fixed, thus  $D(G, C)$  itself is 2-fixed. Finally, we can apply Theorem 3.3 yielding that  $D(G, C)$  is 1-critical and thus can be solved in  $O(n^2)$  time, due to Theorem 3.2.  $\square$

A more specific analysis of the instance and its expansion graph shows that the computation actually can be done in linear time.

**THEOREM 4.6.** PARTIALLY PQ-CONSTRAINED PLANARITY can be solved in linear time for biconnected graphs.

**Simultaneous Embedding** Similarly, we can solve SEFE for two biconnected input graphs  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$  whose intersection graph  $G$  is connected. Jünger and Schulz show that this is equivalent to the question whether embeddings of  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$  exist that induce the same embedding for  $G$  [24, Theorem 4]. We start with the embedding representations  $D(G^{\textcircled{1}})$  and  $D(G^{\textcircled{2}})$ . For each common vertex  $v$  we denote the embedding trees of  $v$  in  $D(G^{\textcircled{1}})$  and  $D(G^{\textcircled{2}})$  by  $T(v^{\textcircled{1}})$  and  $T(v^{\textcircled{2}})$ , respectively. To force the orders to coincide on  $G$ , we add a P-node whose leaves are the common edges incident to  $v$  as a child of  $T(v^{\textcircled{1}})$  and  $T(v^{\textcircled{2}})$ . Again this adds exactly the necessary conditions and it can be verified that the resulting instance is 2-fixed.

**THEOREM 4.7.** SEFE can be solved in quadratic time, if both graphs are biconnected and the common graph is connected.

*Proof.* Let  $(G^{\textcircled{1}}, G^{\textcircled{2}})$  be an instance of SEFE with the common graph  $G$  such that  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$  are biconnected and  $G$  is connected. Let further  $D(G^{\textcircled{1}}, G^{\textcircled{2}})$  be the corresponding instance of SIMULTANEOUS PQ-ORDERING as defined above. Since  $D(G^{\textcircled{1}}, G^{\textcircled{2}})$  contains the PQ-embedding representations  $D(G^{\textcircled{1}})$  and  $D(G^{\textcircled{2}})$ , every solution of  $D(G^{\textcircled{1}}, G^{\textcircled{2}})$  yields planar embeddings of  $G^{\textcircled{1}}$

and  $G^{\textcircled{2}}$ . Furthermore, the common edges incident to a common vertex  $v \in V$  are ordered the same in the two embedding trees  $T(v^{\textcircled{1}})$  and  $T(v^{\textcircled{2}})$  since both orders extend the same order of common edges represented by the common embedding tree  $T(v)$ . Thus, the embeddings for  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$  induced by a solution of  $D(G^{\textcircled{1}}, G^{\textcircled{2}})$  induce the same embedding on the common graph and hence are a solution of  $(G^{\textcircled{1}}, G^{\textcircled{2}})$ . On the other hand, if we have a SEFE of  $G^{\textcircled{1}}$  and  $G^{\textcircled{2}}$ , these embeddings induce orders for the leaves of all PQ-trees in  $D(G^{\textcircled{1}}, G^{\textcircled{2}})$  and since the common edges around every common vertex are ordered the same in both embeddings, all constraints given by arcs in  $D(G^{\textcircled{1}}, G^{\textcircled{2}})$  are satisfied.

To compute the fixedness of every P-node in every PQ-tree in  $D(G^{\textcircled{1}}, G^{\textcircled{2}})$  we distinguish between three kinds of trees, the embedding trees, the consistency trees and the common embedding trees. The proof that  $\text{fixed}(\mu) \leq 2$  for every P-node  $\mu$  in every embedding and consistency tree works as in the proof of Theorem 4.5. For a P-node  $\mu$  in a common embedding tree  $T(v)$  we have two P-nodes  $\mu^{\textcircled{1}}$  and  $\mu^{\textcircled{2}}$  in the parents  $T(v^{\textcircled{1}})$  and  $T(v^{\textcircled{2}})$  of  $T(v)$  it stems from. Since  $T(v)$  has no other parents and no children, we obtain  $\text{fixed}(\mu) = (\text{fixed}(\mu^{\textcircled{1}}) - 1) + (\text{fixed}(\mu^{\textcircled{2}}) - 1)$  by the definition of fixedness. Since  $\mu^{\textcircled{1}}$  and  $\mu^{\textcircled{2}}$  are P-nodes in embedding trees, we know that their fixedness is at most 2. Thus, we have  $\text{fixed}(\mu) \leq 2$ . Hence, all P-nodes in all PQ-trees in  $D(G^{\textcircled{1}}, G^{\textcircled{2}})$  are at most 2-fixed, thus  $D(G^{\textcircled{1}}, G^{\textcircled{2}})$  itself is 2-fixed.  $\square$

We note that both results extend to not necessarily biconnected graphs where each cutvertex belongs to at most two non-trivial blocks (i.e., blocks not consisting of a single edge).

## 5 Conclusion

In this work we introduced a new problem called SIMULTANEOUS PQ-ORDERING. Its input consists of a set of PQ-trees with a child-parent relation (a DAG with PQ-trees as nodes) and the question is whether for every PQ-tree a circular order can be chosen such that it is an extension of the orders of all its children. This was motivated by the possibility to represent the possible circular orders of edges around every vertex of a biconnected planar graph by a PQ-tree. Unfortunately, SIMULTANEOUS PQ-ORDERING turned out to be  $\mathcal{NP}$ -complete in general. However, we were able to find an algorithm solving SIMULTANEOUS PQ-ORDERING in polynomial time for “simple” instances, the 1-critical instances. To achieve this result we showed that satisfying the Q-constraints and the critical triples is sufficient to extend orders of several children simultaneously to a parent, if each P-node is contained in at most one

critical triple. We were able to ensure that a critical triples are satisfied automatically when choosing orders bottom-up by inserting new PQ-trees, the expansion trees. Creating the expansion trees iteratively for every critical triple led to the expansion graph that turned out to have polynomial size for 1-critical instances. Hence, we are able to solve a 1-critical instance of SIMULTANEOUS PQ-ORDERING in polynomial time, essentially by choosing orders bottom-up in the expansion graph. We have shown how this framework can be applied to solve PARTIALLY PQ-CONSTRAINED PLANARITY for biconnected graphs and SIMULTANEOUS EMBEDDING WITH FIXED EDGES for biconnected graphs with a connected intersection in polynomial time (linear and quadratic, respectively), which were both not known to be efficiently solvable before. Furthermore, we have shown how to solve SIMULTANEOUS INTERVAL REPRESENTATION and PARTIAL INTERVAL GRAPH EXTENSION in linear time, which improves over the best known algorithms with running times  $\mathcal{O}(n^2 \log n)$  and  $\mathcal{O}(n^2)$  algorithm, respectively. We stress that all these results can be obtained in a straightforward way from the main result of this work, the algorithm for SIMULTANEOUS PQ-ORDERING for 2-fixed instances.

## References

- [1] Patrizio Angelini, Giuseppe Di Battista, Fabrizio Frati, Vít Jelínek, Jan Kratochvíl, Maurizio Patrignani, and Ignaz Rutter. Testing planarity of partially embedded graphs. In *Proceedings of the 21st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'10)*, pages 202–221. SIAM, 2010.
- [2] Patrizio Angelini, Giuseppe Di Battista, Fabrizio Frati, Maurizio Patrignani, and Ignaz Rutter. Testing the Simultaneous Embeddability of two Graphs whose Intersection is a Biconnected or a Connected Graph. *J. Discr. Alg.*, 2011. <http://dx.doi.org/10.1016/j.jda.2011.12.015>.
- [3] Bengt Aspvall, Michael F. Plass, and Robert Endre Tarjan. A linear-time algorithm for testing the truth of certain quantified boolean formulas. *Information Processing Letters*, 8(3):121–123, 1979.
- [4] Thomas Bläsius, Stephen G. Kobourov, and Ignaz Rutter. Simultaneous embedding of planar graphs. *CoRR*, abs/1204.5853, 2012.
- [5] Thomas Bläsius and Ignaz Rutter. Simultaneous pq-ordering with applications to constrained embedding problems. *CoRR*, abs/1112.0245, 2011.
- [6] Thomas Bläsius and Ignaz Rutter. Disconnectivity and relative positions in simultaneous embeddings. In *Graph Drawing*, Lecture Notes in Computer Science. Springer Berlin / Heidelberg, 2013. To appear.
- [7] Kellogg S. Booth and George S. Lueker. Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms. *Journal of Computer and System Sciences*, 13:335–379, 1976.
- [8] K.S. Booth. *PQ-Tree Algorithms*. PhD thesis, University of California, Berkeley, 1975.
- [9] Giuseppe Di Battista and Roberto Tamassia. On-line maintenance of triconnected components with SPQR-trees. *Algorithmica*, 15(4):302–318, 1996.
- [10] Giuseppe Di Battista and Roberto Tamassia. On-line planarity testing. *SIAM Journal on Computing*, 25(5):956–997, 1996.
- [11] S. Even, A. Itai, and A. Shamir. On the complexity of timetable and multicommodity flow problems. *SIAM Journal on Computing*, 5(4):691–703, 1976.
- [12] J. Joseph Fowler, Carsten Gutwenger, Michael Jünger, Petra Mutzel, and Michael Schulz. An SPQR-tree approach to decide special cases of simultaneous embedding with fixed edges. In Ioannis Tollis and Maurizio Patrignani, editors, *Proceedings of the 16th International Symposium on Graph Drawing (GD'08)*, volume 5417 of *Lecture Notes in Computer Science*, pages 157–168. Springer Berlin/Heidelberg, 2009.
- [13] D. R. Fulkerson and O. A. Gross. Incidence matrices and interval graphs. *Pacific Journal of Mathematics*, 15(3):835–855, 1965.
- [14] Zvi Galil and Nimrod Megiddo. Cyclic ordering is NP-complete. *Theoretical Computer Science*, 5(2):179–182, 1977.
- [15] Elisabeth Gassner, Michael Jünger, Merijam Percan, Marcus Schaefer, and Michael Schulz. Simultaneous graph embeddings with fixed edges. In Fedor Fomin, editor, *Proceedings of the 32nd Workshop on Graph-Theoretic Concepts in Computer Science (WG'06)*, volume 4271 of *Lecture Notes in Computer Science*, pages 325–335. Springer Berlin/Heidelberg, 2006.
- [16] Carsten Gutwenger, Karsten Klein, and Petra Mutzel. Planarity testing and optimal edge insertion with embedding constraints. *Journal of Graph Algorithms and Applications*, 12(1):73–95, 2008.
- [17] Bernhard Haeupler, Krishnam Jampani, and Anna Lubiw. Testing simultaneous planarity when the common graph is 2-connected. In Otfried Cheong, Kyung-Yong Chwa, and Kunsoo Park, editors, *Proceedings of the 21st International Symposium on Algorithms and Computation (ISAAC'10)*, volume 6507 of *Lecture Notes in Computer Science*, pages 410–421. Springer Berlin/Heidelberg, 2010.
- [18] Bernhard Haeupler and Robert E. Tarjan. Planarity algorithms via PQ-trees (extended abstract). *Electronic Notes in Discrete Mathematics*, 31:143–149, 2008. The International Conference on Topological and Geometric Graph Theory.
- [19] Wen-Lian Hsu. PC-trees vs. PQ-trees. In *Proceedings of the 7th Annual International Conference on Computing and Combinatorics, COCOON '01*, pages 207–217, 2001.
- [20] Wen-Lian Hsu and Ross M. McConnell. PQ trees, PC trees, and planar graphs, 2001. manuscript, available at [www.cs.colostate.edu/~rmm/pc2.pdf](http://www.cs.colostate.edu/~rmm/pc2.pdf).

- [21] Wen-Lian Hsu and Ross M. McConnell. PC trees and circular-ones arrangements. *Theoretical Computer Science*, 296:99–116, March 2003.
- [22] Krishnam Raju Jampani and Anna Lubiw. Simultaneous interval graphs. In Otfried Cheong, Kyung-Yong Chwa, and Kunsoo Park, editors, *Proceedings of the 21st International Symposium on Algorithms and Computation (ISAAC'10)*, volume 6506 of *Lecture Notes in Computer Science*, pages 206–217. Springer Berlin/Heidelberg, 2010.
- [23] Vít Jelínek, Jan Kratochvíl, and Ignaz Rutter. A Kuratowski-type theorem for planarity of partially embedded graphs. In *Proceedings of the 27th Annual ACM Symposium on Computational Geometry (SoCG'11)*, pages 107–116. ACM, 2011.
- [24] Michael Jünger and Michael Schulz. Intersection graphs in simultaneous embedding with fixed edges. *Journal of Graph Algorithms and Applications*, 13(2):205–218, 2009.
- [25] Pavel Klavík, Jan Kratochvíl, and Tomáš Vyskočil. Extending partial representations of interval graphs. In Mitsunori Ogihara and Jun Tarui, editors, *Proceedings of the 8th Annual Conference on Theory and Applications of Models of Computation (TAMC'11)*, volume 6648 of *Lecture Notes in Computer Science*, pages 276–285. Springer Berlin/Heidelberg, 2011.
- [26] M. R. Krom. The decision problem for a class of first-order formulas in which all disjunctions are binary. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 13(1-2):15–20, 1967.
- [27] Marcus Schaefer. Toward a theory of planarity: Hanani-tutte and planarity variants. In *Graph Drawing*, Lecture Notes in Computer Science. Springer Berlin / Heidelberg, 2013. To appear.