ZERO-SUM FLOWS FOR STEINER SYSTEMS

SAIEED AKBARI, HAMID REZA MAIMANI, LEILA PARSAEI MAJD, AND IAN M. WANLESS

Abstract. Given a t-(v, k, λ) design, $\mathcal{D} = (X, \mathcal{B})$, a zero-sum n-flow of $\mathcal{D}$ is a map $f : \mathcal{B} \longrightarrow \{\pm 1, \ldots, \pm(n-1)\}$ such that for any point $x \in X$, the sum of f over all blocks incident with x is zero. For a positive integer k, we find a zero-sum k-flow for an STS(uw) and for an STS$(2v+7)$ for $v \equiv 1 \pmod 4$, if there are STS(u), STS(w) and STS(v) such that the STS(u) and STS(v) both have a zero-sum k-flow. In 2015, it was conjectured that for $v > 7$ every STS(v) admits a zero-sum 3-flow. Here, it is shown that many cyclic STS(v) have a zero-sum 3-flow. Also, we investigate the existence of zero-sum flows for some Steiner quadruple systems.

1. Introduction

For a graph G we use $V(G)$ and $E(G)$ to denote the vertices and edges of G, respectively. A *zero-sum flow* of G is an assignment of non-zero real numbers to the edges of G such that the sum of the values of all edges incident with any given vertex is zero. For a natural number $n \geqslant 2$, a *zero-sum n-flow* is a zero-sum flow with values from the set $\{\pm 1, \ldots, \pm(n-1)\}$. For a subset $S \subseteq E(G)$, the *weight* of S is defined to be the sum of the values of all edges in S.

A t-(v, k, λ) design $\mathcal{D}$ (briefly, *t-design*), is a pair $(X, \mathcal{B})$, where X is a v-set of points and $\mathcal{B}$ is a collection of k-subsets of X, called *blocks*, with the property that every t-subset of X is contained in exactly λ blocks. A t-(v, k, λ) design is also denoted by $S_\lambda(t, k, v)$. If $\lambda = 1$, then $S_\lambda(t, k, v)$ is called a *Steiner system*, and λ is usually omitted. If $t = 2$ and $k = 3$, then a 2-$(v, 3, \lambda)$ design is denoted by TS(v, λ), and it is called a *triple system*. For a triple system if $\lambda = 1$, then the design is called a *Steiner triple system* and is denoted by STS(v).

Given an indexing of the points and blocks of a t-design $\mathcal{D}$ with the block set $\mathcal{B} = \{B_1, \ldots, B_b\}$, the *incidence matrix* of $\mathcal{D}$ is a $v \times b$ $(0, 1)$-matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } x_i \in B_j, \\ 0 & \text{otherwise.} \end{cases}$$

We refer the reader to [3] for notation and further results on designs.

2010 *Mathematics Subject Classification.* 05B05; 05B20; 05C21.

Key words and phrases. Zero-sum flow, Steiner triple system; Steiner quadruple system.

The research of the first author was partly funded by Iranian National Science Foundation (INSF) under the contract No. 96004167. The research of the fourth author was supported by Australian Research Council grant DP150100506.

Given a t-(v,k,λ)-design, $\mathcal{D}=(X,\mathcal{B})$, a zero-sum n-flow of $\mathcal{D}$ is a map $f:\mathcal{B}\longrightarrow\{\pm1,\dots,\pm(n-1)\}$ such that for any point $x\in X$, the sum of f over all blocks incident with x is zero. In other words, the sum of the block weights around any point is zero, i.e.

$$w(x)=\sum_{x\in B}f(B)=0.$$

This is equivalent to finding a vector in the nullspace of the incidence matrix of the design whose entries are all in the set $\{\pm1,\dots,\pm(n-1)\}$. The following theorem and two conjectures appeared in [2].

Theorem 1.1. *Every non-symmetric* 2-(v,k,λ) *design admits a zero-sum k-flow for some positive integer k.*

Conjecture 1.2. *Every non-symmetric design admits a zero-sum* 5*-flow.*

Conjecture 1.3. *Every* $\mathrm{STS}(v)$*, with $v>7$, admits a zero-sum* 3*-flow.*

Motivated by Conjecture 1.3, in Section 3 we prove that every cyclic $\mathrm{STS}(v)$ with $v>7$ admits a zero-sum k-flow for $k=3$ or $k=4$. In particular, we prove Conjecture 1.3 for cyclic $\mathrm{STS}(v)$ of order $v\equiv 1 \pmod 6$ and $v\equiv 9 \pmod{18}$ and for many cyclic $\mathrm{STS}(v)$ of other orders.

For graphs G and H, the *join* of G and H is the graph $G\vee H$ with vertex set $V=V(G)\cup V(H)$ and edge set $E=E(G)\cup E(H)\cup\{uv:u\in V(G),v\in V(H)\}$. The *complete graph* K_n is the graph with n vertices in which every two distinct vertices are adjacent. The *complete bipartite graph* $K_{n,m}$ is $U\vee V$ where U and V are disjoint independent sets with $|U|=n$ and $|V|=m$. The *complete tripartite graph* $K_{\ell,n,m}$ is $U\vee V\vee W$, where U, V and W are disjoint independent sets with $|U|=\ell$, $|V|=n$ and $|W|=m$.

2. Zero-sum flows on $\mathrm{STS}(vw)$ and $\mathrm{STS}(2v+7)$

Let $\mathrm{STS}(v)$ and $\mathrm{STS}(w)$ be two Steiner triple systems such that the $\mathrm{STS}(v)$ has a zero-sum k-flow for $k\geqslant 3$. In this section, we provide a zero-sum k-flow for a Steiner triple system $\mathrm{STS}(vw)$. Moreover, we find a zero-sum k-flow for an $\mathrm{STS}(2v+7)$, where $v\equiv 1 \pmod 4$.

Our constructions will use Latin squares. A *Latin square* of order n with entries from a set X is an $n\times n$ array L such that every row and column of L is a permutation of X. Suppose that L_1 and L_2 are two Latin squares of order n with entries from X and Y, respectively. We say that L_1 and L_2 are *orthogonal* provided that, for every $x\in X$ and $y\in Y$, there is a unique cell (i,j) such that $L_1(i,j)=x$ and $L_2(i,j)=y$. Note that by [3, p.12] for every positive integer $v\notin\{2,6\}$, there are orthogonal Latin squares of order v. A *transversal* of a Latin square is a set of entries which includes exactly one representative from each row and column and one of each symbol.

Remark 2.1. *It is not hard to see that a Latin square has an orthogonal mate if and only if it can be decomposed into disjoint transversals.*

We refer the reader to [12] for a survey of results on transversals in Latin squares.

Next, we recall the following construction for STS(vw), see [4].

Construction A. STS(vw)-Construction

Let $(X, \mathcal{B})$ be an STS(v) on the set $X = \{x_1, \ldots, x_v\}$ and $(Y, \mathcal{B}')$ be an STS(w) on the set $Y = \{y_1, \ldots, y_w\}$. Then define $(Z, \mathcal{C})$ as an STS(vw) on the set $Z = \{z_{ij}, 1 \leqslant i \leqslant v, 1 \leqslant j \leqslant w\}$ with two types of blocks as follows:

For $j = 1, \ldots, w$, consider a copy K_v^j of the complete graph K_v, with vertex set $\{z_{1j}, \ldots, z_{vj}\}$. Using $\mathcal{B}$, one can partition the edges of each K_v^j into triangles, for $j = 1, \ldots, w$. We say that the blocks made by these triangles are of Type A. Now, consider the complete graph K_w with vertex set K_v^j for $1 \leqslant j \leqslant w$. Using $\mathcal{B}'$ one can partition the edges of K_w into triangles. Join every vertex of K_v^i to every vertex of K_v^j, for $1 \leqslant i < j \leqslant w$. Using the partition of K_w, every triangle in K_w corresponds to a complete tripartite graph $K_{v,v,v}$ which has $3v^2$ edges. Now, for each triangle $\{K_v^p, K_v^s, K_v^t\}$ of K_w, where $1 \leqslant p < s < t \leqslant w$, consider a Latin square $L = L(p, s, t)$ of order v on the set $\{z_{1t}, \ldots, z_{vt}\}$ such that the rows and columns are indexed by $\{z_{1p}, \ldots, z_{vp}\}$ and $\{z_{1s}, \ldots, z_{vs}\}$, respectively. For $1 \leqslant i \leqslant v$ and $1 \leqslant j \leqslant v$, we make a block $\{z_{ip}, z_{js}, L(z_{ip}, z_{js})\}$ of Type B. It is not hard to see that all blocks of Type A and Type B together form an STS(vw).

This construction allows us to prove the following lemma.

Lemma 2.2. *Let v and w be two positive integers for which there exist* STS(v) *and* STS(w)*, where at least one of the* STS(v) *and* STS(w) *has a zero-sum k-flow for some $k \geqslant 3$. Then there exists an* STS(vw) *which has a zero-sum k-flow.*

Proof. Suppose that an STS(v) has a zero-sum k-flow for $k \geqslant 3$. In Construction A, we let the blocks of Type A inherit a zero-sum k-flow from the STS(v). According to Remark 2.1, since $v \notin \{2, 6\}$, in Construction A one can choose Latin squares that decompose into transversals $T_1, \ldots, T_v$, each of which corresponds to a collection of blocks in the STS(vw). Now, assign values $+2, -1, -1$ to the blocks from T_1, T_2, T_3, respectively. Then, label the blocks from T_i with $(-1)^i$ for $i = 4, \ldots, v$. In this way, the Type B blocks defined by each Latin square contribute a total of zero to the weight of every vertex. □

We need the following observation to prove our next results. This can be found in [7, p.41].

Remark 2.3. *For odd v, the edges of K_{v+7} can be partitioned into $v+7$ triangles and v 1-factors. Note that each vertex appears in exactly three triangles.*

Construction B. STS($2v + 7$)-Construction

Let $(X, \mathcal{A})$ be a Steiner triple system of order v, with $X = \{x_1, \ldots, x_v\}$, and let Y be a set of size $v + 7$, such that $X \cap Y = \varnothing$. Using Remark 2.3, partition the edges of K_{v+7} with vertex set Y into a set L containing $v+7$ triangles and a set $F = \{F_1, \ldots, F_v\}$ containing v 1-factors. Set $Z = X \cup Y$ and define a collection of triples $\mathcal{B}$ as follows: We can consider a block corresponding to each triangle in L. Put all such blocks in a set N. Now, join x_i to the end vertices of each edge of F_i, for $i = 1, \ldots, v$, to obtain some new triangles. Let T be a set of blocks corresponding to these new triangles. Then, $(Z, \mathcal{B})$ is a Steiner triple system of order $2v+7$, where $\mathcal{B} = \mathcal{A} \cup N \cup T$. See [7, p.41–42].

Remark 2.4. *Let $n \geqslant 8$ be an even positive integer, and let $Y = \{y_1, \ldots, y_n\}$. It is clear that $n = v+7$, for some odd $v \geqslant 1$. We know that the edges of K_n, with vertex set Y, can be partitioned into n triangles and v 1-factors, $\{F_1, \ldots, F_v\}$. If we assign the value 1 to each of the n triangles, then the sum of the values of the three triangles containing y_i is 3, for $i = 1, \ldots, n$.*

Now, if $v = 1$, then we have just one 1-factor, F_1. Assign -3 to each edge of F_1. Otherwise, $v \geqslant 3$. Assign -1 to the edges of F_1, F_2 and F_3. Then assign $(-1)^j$ to F_j for $j = 4, \ldots, v$. Since v is odd, in all cases the sum of the values of the edges in $\cup_{j=1}^{v} F_j$ incident with y_i is -3, for $i = 1, \ldots, n$. Hence the total weight allocated to the edges and triangles incident with any vertex in Y is 0.

Next, from a zero-sum k-flow for STS(v), we show how to obtain a zero-sum k-flow for an STS($2v+7$), if $v \equiv 1 \pmod 4$. We say that a graph G has a *k-null 1-factorisation* if G has a zero-sum k-flow and there is a 1-factorisation in which the weight of each 1-factor is zero. We call each 1-factor in a k-null 1-factorisation of G a *k-null 1-factor*. We use the following lemma, see the proof of Lemma 4.2 in [1].

Lemma 2.5. *There exists a 3-null 1-factorization of $K_{n,n}$ for every $n \geqslant 3$. If n is even and $n \neq 6$, then $K_{n,n}$ has a 2-null 1-factorization.*

Theorem 2.6. *Let $v > 9$ be a positive integer and $v \equiv 1 \pmod 4$. If there exists an* STS(v) *with a zero-sum k-flow for some positive integer $k \geqslant 2$, then there exists an* STS($2v+7$) *with a zero-sum k-flow.*

Proof. Let $(X, \mathcal{A})$ be an STS(v), with $X = \{x_1, \ldots, x_v\}$, which has a zero-sum k-flow, and let Y be a set of size $v+7$ such that $X \cap Y = \varnothing$. Keep the values of the blocks in $\mathcal{A}$. Consider the Steiner triple system on $X \cup Y$ given in Construction B. Since $v \equiv 1 \pmod 4$ and $v > 9$, we know that $v + 7 = 4s$ for some integer $s \geqslant 5$. Let $2s = t + 7$, for some odd $t \geqslant 3$. We have $K_{v+7} = \mathcal{K} \vee \mathcal{K}'$, where $\mathcal{K}$ and $\mathcal{K}'$ are both copies of K_{t+7}. By Remark 2.3 we can decompose the edges of $\mathcal{K}$ into 1-factors $M_1, \ldots, M_t$ and $t+7$ triangles. We give each of these triangles a weight of 1. For $1 \leqslant i \leqslant t$ and for each edge e in M_i we then make a new block containing x_i and the end vertices of e. We assign this block a weight equal to the value that e was assigned in Remark 2.4. We then decompose $\mathcal{K}'$ in a similar way into $t+7$ triangles and 1-factors $M'_1, \ldots, M'_t$. We allocate a weight of -1 to the $t+7$ triangles and we give each edge in M'_i the negative of the weight that the edges in M_i were given. In this way, when we join x_i to M'_i in the same way that we joined x_i to M_i, the total weight of the blocks incident with x_i will be zero for $1 \leqslant i \leqslant t$. Similarly, Remark

2.4 shows that for any vertex in Y, there is zero total weight for the blocks so far constructed that are incident with that vertex.

The edges between $\mathcal{K}$ and $\mathcal{K}'$ form a $K_{t+7,t+7}$, which has a 2-null 1-factorization $F_1, \ldots, F_{t+7}$, by Lemma 2.5. For $i = 1, \ldots, v-t$ and for each edge e' in F_i, make a new block containing x_{t+i} and the end vertices of e'. Assign this block a weight equal to the value that e' received in the 2-null 1-factorization. By this process we obtain a zero-sum k-flow for the STS$(2v+7)$ formed by Construction B. □

Remark 2.7. *If $v = 9$ and there exists an* STS(9) *with a zero-sum* 3*-flow, then we are not able to find a zero-sum* 3*-flow for the* STS(25) *obtained by Construction B. This is because, in Remark* 2.4 *we utilised a weight of* -3 *in the case when* $t = 1$*. Note that in this case, we can find a zero-sum* 4*-flow for the constructed* STS(25)*. However, in* [1] *it was proved that for every pair* (v, λ) *such that a* TS(v, λ) *exists, there is one with a zero-sum* 3*-flow, except when* $(v, \lambda) \in \{(3, 1), (4, 2), (6, 2), (7, 1)\}$*.*

It would be interesting to know if the restriction to $v \equiv 1 \pmod 4$ is really needed in Theorem 2.6.

Question 2.8. Let v, k be positive integers such that $v \equiv 3 \pmod 4$ and $k \geqslant 2$. Suppose that in Construction B we use an STS(v) that has a zero-sum k-flow. Is there necessarily a zero-sum k-flow for the resulting STS$(2v+7)$?

3. Flows in cyclic STS

In this section we are going to verify that for $v > 7$ each cyclic STS(v) has a zero-sum 4-flow and that many such systems have a zero-sum 3-flow. First we need some definitions.

An *automorphism* of a t-(v, k, λ) design, $(X, \mathcal{B})$, is a bijection $\alpha : X \longrightarrow X$ such that $B = \{x_1, \ldots, x_k\} \in \mathcal{B}$ if and only if $B\alpha = \{x_1\alpha, x_2\alpha, \ldots, x_k\alpha\} \in \mathcal{B}$. A t-(v, k, λ) design is called *cyclic* if it has an automorphism that is a permutation consisting of a single cycle of length v; this automorphism is called a *cyclic automorphism.* Throughout, we will assume for our cyclic t-(v, k, λ) design that $X = \mathbb{Z}_v$, and $\alpha : i \longrightarrow i+1 \pmod v$ is its cyclic automorphism. The blocks of a cyclic t-(v, k, λ) design are partitioned into orbits under the action of the cyclic group generated by α. Each orbit of blocks is completely determined by any of its blocks, and $\mathcal{B}$ is determined by a collection of blocks called *base blocks* (sometimes also called *starter blocks* or *initial blocks*) containing one block from each orbit. For an example, $X = \{1, 2, 3, 4, 5, 6, 7\}$ and

$$\mathcal{B} = \big\{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}\big\},$$

form an STS(7) which is cyclic, since the permutation $\alpha = (1234567)$ is an automorphism.

In 1939, Rose Peltesohn solved both of Heffter's Difference Problems, see [10]. This solution provides the following theorem, see [7, Section 1.7].

Theorem 3.1. *For all* $v \equiv 1$ *or* $3 \pmod 6$ *with* $v \neq 9$*, there exists a cyclic* STS(v).

Remark 3.2. *If* $v \equiv 1 \pmod 6$*, every cyclic* STS(v) *has* $\frac{v-1}{6}$ *full orbits. Also, if* $v \equiv 3 \pmod 6$*, every cyclic* STS(v) *has* $\frac{v-3}{6}$ *full orbits and one short orbit which contains the block* $\{0, \frac{v}{3}, \frac{2v}{3}\}$*. Moreover, note that every full orbit contains each point* 3 *times, and each point appears once in the short orbit, see* [4].

For $v \equiv 3 \pmod 6$, we will classify orbits of a cyclic STS(v) into three types. For $i = 1, 2, 3$ an orbit is of Type i if every block in the orbit contains representatives of precisely i different congruence classes modulo 3. As v is divisible by 3, every orbit will be of Type 1, Type 2 or Type 3 and its type can be established by examining any single block in the orbit.

Since the incidence matrix of STS(7) has full rank, STS(7) has no zero-sum k-flow. Also, by [7, Section 1.7], there is no cyclic STS(9). In the following we are going to show that every cyclic STS(v) for $v > 7$ admits a zero-sum k-flow for $k = 3$ or $k = 4$.

We will split the $v \equiv 3 \pmod 6$ case into three subcases: $v \equiv 3, 9$ or $15 \pmod{18}$. In the following we prove that if $v \equiv 1 \pmod 6$ or $v \equiv 9 \pmod{18}$ and $v \neq 7$, then each cyclic STS(v) admits a zero-sum 3-flow. In other words, Conjecture 1.3 is true for these families of Steiner triple systems. Also, we show that for $v \equiv 3$ or $15 \pmod{18}$, each cyclic STS(v) has a zero-sum 4-flow. We need the following lemmas to prove our main results.

Lemma 3.3. *For* $v \equiv 9 \pmod{18}$*, every cyclic* STS(v) *has a full orbit of Type* 3.

Proof. Suppose that there exists a cyclic STS(v), S, with no full orbit of Type 3. Let S have t full orbits of Type 2 and s full orbits of Type 1. Note that t and s are two non-negative integers and $t + s = (v - 3)/6$. Now, count the number of pairs $\{a, b\}$ where $a \not\equiv b \pmod 3$, among all blocks of S. Since the short orbit has Type 1, and every full orbit has v blocks, we obtain the following equality:

$$2vt = 3\frac{v}{3} \times \frac{v}{3}.$$

Hence $t = v/6$, a contradiction. □

Lemma 3.4. *Let* $v \equiv 3$ *or* $15 \pmod{18}$ *and* S *be a cyclic* STS(v) *with no full orbit of Type* 3*. Then* S *has no full orbit of Type* 1.

Proof. Suppose S has t full orbits of Type 2 and s full orbits of Type 1. We have $t + s = (v - 3)/6$. Since $v/3$ is not divisible by 3, the short orbit has Type 3. Now, count the number of pairs $\{a, b\}$ in all blocks of S, where $a \not\equiv b \pmod 3$. We have

$$2tv + 3\frac{v}{3} = 3\frac{v}{3} \times \frac{v}{3}.$$

Hence, $t = (v - 3)/6$ and $s = 0$. □

Remark 3.5. *Let* $v \equiv 9 \pmod{18}$*, and suppose that a cyclic* STS(v) *has a full orbit of Type* 3 *generated from a base block* $\{a, b, c\}$*. Then the blocks* $\{a + 3i, b + 3i, c + 3i\}$ *for* $0 \leqslant i \leqslant \frac{v}{3} - 1$*, contain exactly one occurrence of each point in* $\mathbb{Z}_v$*. This is because*

$\{a+3i : 0 \leqslant i \leqslant \frac{v}{3}-1\}$ *contains the* $v/3$ *points that are congruent to* $a \pmod 3$. *Similar statements holds for* $\{b+3i\}$ *and* $\{c+3i\}$, *and these sets are disjoint because the orbit is of Type* 3.

Using Lemmas 3.3 and 3.4, and Remark 3.5, we have the following theorems about the existence of a zero-sum k-flow with $k=3$ or $k=4$, for every cyclic STS(v).

Theorem 3.6. *Every cyclic* STS(v) *for* $v \equiv 1 \pmod 6$ *or* $v \equiv 9 \pmod{18}$ *with* $v \neq 7$ *admits a zero-sum* 3*-flow.*

Proof. There is no cyclic STS(9), so $v > 9$ and we have at least two full orbits. The case when $v \equiv 1 \pmod 6$ is handled by [1, Theorem 1.7], so we assume that $v \equiv 9 \pmod{18}$. In this case, by Lemma 3.3, there exists a full orbit with a block $\{a,b,c\}$ congruent to $\{0,1,2\} \pmod 3$. So, assign the weight of all blocks within a full orbit of Type 3 as follows:

$$-1, +1, +1, -1, +1, +1, -1, +1, +1, \ldots.$$

Note that by Remark 3.5, each point gets weight $+1$ along this orbit. Now, if $O_2, O_3, \ldots, O_{\frac{v-3}{6}}$ are the other full orbits, assign weight $(-1)^{i+1}$ to every block O_i, for $2 \leqslant i \leqslant \frac{v-3}{6}$. If $\frac{v-3}{6}$ is odd, assign weight -1 to the blocks in the short orbit. Otherwise, assign value 2 to the blocks in the short orbit. □

For the cases not covered by Theorem 3.6, we have the following result.

Theorem 3.7. *Suppose that* S *is a cyclic* STS(v), *where* $v \equiv 3$ *or* $15 \pmod{18}$ *and* $v > 3$. *Then* S *has a zero-sum* 4*-flow. If* S *has any full orbit of Type* 1 *or Type* 3, *then* S *has a zero-sum* 3*-flow.*

Proof. We first show that S admits a zero-sum 4-flow. Assign value -3 to the blocks in the short orbit. For the first full orbit, assign a value of 2 if there are an even number of full orbits, and a value of 1 otherwise. For the other full orbits, alternate between assigning -1 and 1 to the orbit. This produces a zero-sum 4-flow for S. If S has a full orbit of Type 3, then similar to the proof of Theorem 3.6, there exists a zero-sum 3-flow for S. By Lemma 3.4, we know that if some full orbit has Type 1 then there will be a full orbit of Type 3, so we are also done in that case. □

Corollary 3.8. *Every cyclic* STS(v) *with* $v > 7$ *admits a zero-sum* 4*-flow.*

We stress that Theorem 3.7 does not rule out the existence of a zero-sum 3-flow for a cyclic STS(v) that has no full orbits of Type 1 or 3. Such triple systems do exist. For example, any triple system built using three identical cyclic quasigroups in the Bose Construction ([7, Section 1.2]), will have only full orbits of Type 2. We next show that such STS may still have a zero-sum 3-flow. There are two cyclic STS(15). The cyclic STS(15) with the base blocks $\{0,1,4\}$, $\{0,2,8\}$ and $\{0,5,10\}$ is not obtained from the Bose construction, but the other one constructed by the base blocks $\{0,1,4\}$, $\{0,2,9\}$ and $\{0,5,10\}$ arises from the Bose construction. However, both of them admit a zero-sum 3-flow and the full orbits of these cyclic STS(15) are all of Type 2.

In the following one can find a zero-sum 3-flow for the cyclic STS(15) with the base blocks $\{0,1,4\}$, $\{0,2,8\}$ and $\{0,5,10\}$. The fourth number (after each block) is the flow value assigned to that block. We omit the { } symbols in each block.

0	1	4	-1	0	2	8	1	0	5	10	2
1	2	5	-1	1	3	9	1	1	6	11	2
2	3	6	1	2	4	10	-1	2	7	12	2
3	4	7	-1	3	5	11	-1	3	8	13	2
4	5	8	1	4	6	12	-1	4	9	14	2
5	6	9	-1	5	7	13	-1				
6	7	10	1	6	8	14	-1				
7	8	11	-1	7	9	0	1				
8	9	12	-1	8	10	1	-1				
9	10	13	-1	9	11	2	-1				
10	11	14	1	10	12	3	-1				
11	12	0	-1	11	13	4	1				
12	13	1	1	12	14	5	1				
13	14	2	-1	13	0	6	-1				
14	0	3	-1	14	1	7	-1				

Also, a cyclic STS(15) with the base blocks $\{0,1,4\}$, $\{0,2,9\}$ and $\{0,5,10\}$ has a zero-sum 3-flow as follows:

0	1	4	1	0	2	9	-1	0	5	10	1
1	2	5	-2	1	3	10	-2	1	6	11	1
2	3	6	1	2	4	11	2	2	7	12	1
3	4	7	-2	3	5	12	1	3	8	13	1
4	5	8	-1	4	6	13	2	4	9	14	-1
5	6	9	-2	5	7	14	2				
6	7	10	1	6	8	0	-2				
7	8	11	-2	7	9	1	2				
8	9	12	1	8	10	2	1				
9	10	13	2	9	11	3	-1				
10	11	14	-2	10	12	4	-1				
11	12	0	1	11	13	5	1				
12	13	1	-2	12	14	6	-1				
13	14	2	-2	13	0	7	-2				
14	0	3	2	14	1	8	2				

4. Steiner Quadruple Systems

In this section we study zero-sum k-flows in Steiner quadruple systems (SQS). For $k \geqslant 3$ we show the following results. If we have a zero-sum k-flow for two SQS(v), then we can find a zero-sum k-flow for an SQS$(2v)$. Also, if there are an SQS(u) and an SQS(v) both with a zero-sum k-flow, then we can find a zero-sum k-flow for an SQS(uv).

First we recall some definitions and background about Steiner quadruple systems from [8] and [11]. A *Steiner quadruple system* (or simply a quadruple system) is a pair $(X, \mathcal{B})$ which is a 3-design with parameters $(v, 4, 1)$ such that any 3-subset of X belongs to exactly one block of $\mathcal{B}$. A Steiner quadruple system of order v is denoted by SQS(v). One obtains immediately that $v \equiv 2$ or 4 (mod 6) is a necessary condition for the existence of an SQS(v). The total number of quadruples is $\frac{1}{24}v(v-1)(v-2)$, the number of quadruples containing a given element is $\frac{1}{6}(v-1)(v-2)$, and the number of quadruples containing a given pair of elements is $\frac{1}{2}(v-2)$. In 1960, Hanani [5] proved that the set of possible orders for quadruple systems consists of all positive integers $v \equiv 2$ or 4 (mod 6). If $(X, \mathcal{B})$ is a quadruple system and x is any element in X, put $X_x = X \setminus \{x\}$ and $\mathcal{B}(x) = \{B \setminus \{x\} : B \in \mathcal{B}, x \in B\}$. It can be easily checked that $(X_x, \mathcal{B}(x))$ is a Steiner triple system which is called a *derived triple system* of the quadruple system $(X, \mathcal{B})$.

We now recall two recursive constructions of SQS($2v$) and SQS(uv) from [8].

Construction C. SQS($2v$)-Construction

Let $v \equiv 2$ or 4 (mod 6). Consider two disjoint copies of K_v, with vertex sets X and Y such that $|X| = |Y| = v$. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be any two SQS(v). Let $F = \{F_1, \ldots, F_{v-1}\}$ and $G = \{G_1, \ldots, G_{v-1}\}$, be two 1-factorizations of K_v on X and Y, respectively. Assume that $\mathcal{C} = \mathcal{A} \cup \mathcal{B} \cup T$ on the point set $Z = X \cup Y$, where the elements of T are defined as follows:

If $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, then $\{x_1, x_2, y_1, y_2\} \in T$ if and only if there exists i, with $1 \leqslant i \leqslant v-1$ such that x_1x_2 and y_1y_2 are edges in F_i and G_i, respectively. It is shown in [8] that $(Z, \mathcal{C})$ is an SQS($2v$).

In the following lemma, we assume that there are two SQS(v) with a zero-sum k-flow. Then, we find a zero-sum k-flow for an SQS($2v$).

Lemma 4.1. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two* SQS(v) *with $X \cap Y = \varnothing$, where both* SQS(v) *have a zero-sum k-flow for $k \geqslant 3$. Then there is an* SQS($2v$) *with a zero-sum k-flow.*

Proof. In Construction C, we keep the values of all blocks in $\mathcal{A} \cup \mathcal{B}$. Hence, it only remains to define weights for the blocks in T. First, we assign 2, -1 and -1, to the elements of F_1, F_2, and F_3, respectively, and assign $(-1)^i$ to F_i, for $4 \leqslant i \leqslant v-1$. Note that $v-1$ is odd. Now, each block of T contains exactly one element of one of the F_i, so we may assign the value of that element to the block. In this way, we obtain a zero-sum 3-flow for an SQS($2v$). □

Construction D. SQS(uv)-Construction

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be an SQS(u) and an SQS(v), respectively, and consider the following properties: Define a ternary operation $\langle\ ,\ ,\ \rangle$ on X by $\langle a, b, c\rangle = d$

whenever $\{a, b, c, d\} \in \mathcal{A}$, and $\langle a, a, b\rangle = b$. Now, denote $X_y = X \times \{y\}$, and for every $y \in Y$, let $\mathcal{A}_y$ be a collection of quadruples on X_y such that $(X_y, \mathcal{A}_y)$ is an SQS(u). Let $Y = \{y_1, \ldots, y_v\}$, and $F^{(y_i)} = \{F_1^{(y_i)}, F_2^{(y_i)}, \ldots, F_{u-1}^{(y_i)}\}$ for $i \in \{1, \ldots, v\}$, be a 1-factorization of K_u on X_{y_i}. For the set $X \times Y$ define the following collection $\mathcal{C}$ of quadruples:

(1) $\mathcal{C}$ contains every quadruple belonging to $\mathcal{A}_{y_i}$ for any $y_i \in Y$.
(2) If $(a, y_i), (b, y_i) \in X_{y_i}$ and $(c, y_j), (d, y_j) \in X_{y_j}$ for $i < j$, then

$$\{(a, y_i), (b, y_i), (c, y_j), (d, y_j)\} \in \mathcal{C}$$

if and only if $(a, y_i)(b, y_i)$ and $(c, y_j)(d, y_j)$ are edges in $F_k^{(y_i)}$ and $F_k^{(y_j)}$, respectively, for some $1 \leqslant k \leqslant u - 1$.
(3) For every quadruple $\{y_i, y_j, y_t, y_s\} \in \mathcal{B}$ and for every three (not necessarily distinct) elements $a, b, c \in X$, $\mathcal{C}$ contains $\{(a, y_i), (b, y_j), (c, y_t), (\langle a, b, c\rangle, y_s)\}$ where $i < j < t < s$.

It is shown in [8] that $(X \times Y, \mathcal{C})$ is an SQS(uv).

In the following lemma we present a zero-sum k-flow for an SQS(uv) using Construction D.

Lemma 4.2. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be an* SQS(u) *and an* SQS(v)*, respectively, both having a zero-sum k-flow for some $k \geqslant 3$. Then there is an* SQS(uv) *which admits a zero-sum k-flow.*

Proof. In Construction D, one can ignore the blocks from (1) because they inherit their value from the zero-sum flow of the SQS(u). It is not hard to see that there exists a zero-sum 3-flow on the blocks from (2), by treating them as a complete bipartite graph similar to the proof of Lemma 4.1. That leaves the blocks from (3), where for each given block of $\mathcal{B}$ we have u^3 quadruples in SQS(uv) because we have u choices for each of a, b and c. There are exactly u^2 blocks obtained from a given block $\{y_i, y_j, y_t, y_s\} \in \mathcal{B}$ that contain an element (a, y_i) for any fixed $a \in X$. Now, assign to all blocks obtained from $\{y_i, y_j, y_t, y_s\}$, the weight of the block $\{y_i, y_j, y_t, y_s\}$ in the zero-sum k-flow for the SQS(v). In this way we obtain an SQS(uv) with a zero-sum k-flow. □

A t-design $(X, \mathcal{B})$ is said to be α-*resolvable* if there exists a partition of the collection $\mathcal{B}$ into parts called α-*parallel classes* (or α-*resolution classes*) such that each point of X occurs in exactly α blocks in each class. When $\alpha = 1$, α is omitted. We denote the number of α-parallel classes by $\rho = r/\alpha$, where r is the number of appearances of each point $x \in X$ among the blocks of the design. A t-(v, k, λ) design is called an *even design* when it is α-resolvable with even ρ. Moreover, a t-$(v, k, 1)$ design, $S(t, k, v)$, is called *i-partitionable* (some literature uses the alternative term *i-resolvable*, but to avoid confusion we will not) if the block set can be partitioned into $S(i, k, v)$ designs for $0 < i < t$. Note that by [8, Section 11], if $\alpha = i = 2$, then 2-resolvability and 2-partitionability are the same for $SQS(v)$. We refer the reader to [9] for more information about these concepts.

Lemma 4.3. *A t-(v, k, λ) design has a zero-sum 2-flow if and only if it is even.*

Proof. Let $(X, \mathcal{B})$ be a t-(v, k, λ) design. If $(X, \mathcal{B})$ is even, it is sufficient to assign $+1$ to each block in half, namely $\frac{\rho}{2}$, of the α-parallel classes and assign -1 to each block in the other half of the α-parallel classes. Note that $\alpha = r/\rho$, where r is the number of appearances of each point $x \in X$ among the blocks of the design. For the converse, suppose $(X, \mathcal{B})$ has a zero-sum 2-flow. Since for each arbitrary element $x \in X$, there exist r blocks containing x, exactly half of these blocks have the value $+1$ and the rest have the value -1. If we take all blocks with the same value in a set, we have two sets such that in each of them every element appears in $\frac{r}{2}$ blocks. Therefore, $\alpha = \frac{r}{2}$ and $\rho = 2$. Hence, $(X, \mathcal{B})$ is an even design. □

Remark 4.4. *By* [11, Theorem 10.1], *a resolvable $S(2, 4, v)$ exists if and only if $v \equiv 4 \pmod{12}$. Moreover, a 2-partitionable* SQS(v) *is one that can be decomposed into $S(2, 4, v)$ designs. According to* [6], *a Steiner system $S(2, 4, v)$ exists if and only if $v \equiv 1$ or $4 \pmod{12}$. So, a necessary condition for the existence of a 2-partitionable* SQS(v) *is $v \equiv 4 \pmod{12}$. For any positive integer n, there exists a 2-partitionable* SQS$(4n)$ *as well as a 2-partitionable* SQS$(2pn + 2)$, *for $p \in \{7, 31, 127\}$, see* [9].

Lemma 4.5. *Let $(X, \mathcal{B})$ be a 2-resolvable* SQS(v). *Then $(X, \mathcal{B})$ has a zero-sum 3-flow. Moreover, the derived triple system $(X_x, \mathcal{B}(x))$ for any $x \in X$, also has a zero-sum 3-flow.*

Proof. We can decompose $(X, \mathcal{B})$ into $\frac{v-2}{2}$ $S(2, 4, v)$ designs. We know that in this case $v \equiv 4 \pmod{12}$, so $\frac{v-2}{2}$ is an odd number. Using this decomposition, it is not hard to construct a zero-sum 3-flow for $(X, \mathcal{B})$. For the second part, let $x \in X$ and consider all blocks of $(X, \mathcal{B})$ containing x to construct the derived STS$(v - 1)$. Let $y \in X \setminus \{x\}$. As we know each pair of elements of X appears in any obtained $S(2, 4, v)$ exactly once; y appears in all of these $S(2, 4, v)$. By an appropriate assignment (using the values $2, \pm 1$), one can obtain a zero-sum 3-flow on the derived STS$(v - 1)$. □

Remark 4.6. *By* [8], *the constructions of* SQS(8) *and* SQS(10) *are unique. We show that* SQS(8) *and* SQS(10) *admit a zero-sum 3-flow. The following blocks form* SQS(8), *and the value from $\{\pm 1, 2\}$ given on the right hand side of each block is the flow assigned to that block.*

1 2 4 8	1	3 5 6 7	1
2 3 5 8	1	1 4 6 7	1
3 4 6 8	2	1 2 5 7	2
4 5 7 8	−1	1 2 3 6	−1
1 5 6 8	−1	2 3 4 7	−1
2 6 7 8	−1	1 3 4 5	−1
1 3 7 8	−1	2 4 5 6	−1

Moreover, the blocks below form SQS(10), *with the assigned flows of a zero-sum 2-flow specified next to the corresponding blocks. Note that its derived* STS(9) *also* *has a zero-sum 2-flow.*

1 2 4 5	1	1 2 3 7	−1	1 3 5 8	1
2 3 5 6	−1	2 3 4 8	1	2 4 6 9	−1
3 4 6 7	1	3 4 5 9	−1	3 5 7 0	1
4 5 7 8	−1	4 5 6 0	1	1 4 6 8	−1
5 6 8 9	1	1 5 6 7	−1	2 5 7 9	1
6 7 9 0	−1	2 6 7 8	1	3 6 8 0	−1
1 7 8 0	1	3 7 8 9	−1	1 4 7 9	1
1 2 8 9	−1	4 8 9 0	1	2 5 8 0	−1
2 3 9 0	1	1 5 9 0	−1	1 3 6 9	1
1 3 4 0	−1	1 2 6 0	1	2 4 7 0	−1

Corollary 4.7. *Every* SQS(v) *admits a zero-sum k-flow for some positive integer k.*

Proof. Since every 3-design is also a 2-design, by Theorem 1.1, the assertion is proved. □

References

[1] S. Akbari, A.C. Burgess, P. Danziger, E. Mendelsohn, *Zero-sum flows for Steiner triple systems*, Discrete Math. 340 (2017), no. 3, 416–425.

[2] S. Akbari, G.B. Khosrovshahi, A. Mofidi, *Zero-sum flows in designs*, J. Combin. Des. 19 (2011), no. 5, 355–364.

[3] C.J. Colbourn, J.H. Dinitz, editors, *The CRC Handbook of Combinatorial Designs*, Second edition. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2007. xxii+984 pp. ISBN: 978-1-58488-506-1; 1-58488-506-8.

[4] C.J. Colbourn, A. Rosa, *Triple systems*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1999. xvi+560 pp. ISBN: 0-19-853576-7.

[5] H. Hanani, *On quadruple systems*, Canad. J. Math. 12 1960 145–157.

[6] H. Hanani, *The existence and construction of balanced incomplete block designs*, Ann. Math. Statist. 32 1961 361–386.

[7] C. C. Lindner, C. A. Rodger, *Design Theory*, Second edition. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2009. xiv+264 pp. ISBN: 978-1-4200-8296-8.

[8] C. C. Lindner, A. Rosa, *Steiner quadruple systems - a survey*, Discrete Math. 22 (1978), no. 2, 147–181.

[9] F. Montecalvo, *Some constructions of general covering designs*, Electron. J. Combin. 19 (2012), no. 3, Paper 28, 16 pp.

[10] R. Peltesohn, *Eine Lösung der beiden Heffterschen Differenzenprobleme*, (German) Compositio Math. 6 (1939), 251–257.

[11] C. Reid, A. Rosa, *Steiner systems $S(2,4,v)$ - a survey*, Electron. J. Combin., (2010), #DS18.

[12] I. M. Wanless, *Transversals in Latin squares: a survey*, Surveys in combinatorics 2011, 403–437, London Math. Soc. Lecture Note Ser., 392, Cambridge Univ. Press, Cambridge, 2011.

S. Akbari, Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746.
Email address: s_akbari@sharif.ir

H. R. Maimani, Mathematics Section, Department of Basic Sciences, Shahid Rajaee Teacher Training University, P.O. Box 16785-163, Tehran, Iran.
Email address: maimani@ipm.ir

L. Parsaei Majd, Mathematics Section, Department of Basic Sciences, Shahid Rajaee Teacher Training University, P.O. Box 16785-163, Tehran, Iran.
Email address: leila.parsaei84@yahoo.com

I. M. Wanless, School of Mathematics, Monash University, Clayton Vic 3800, Australia.
Email address: ian.wanless@monash.edu