An Approximate Generalization of the Okamura-Seymour Theorem

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Abstract

We consider the problem of multicommodity flows in planar graphs. Okamura and Seymour [11] showed that if all the demands are incident on one face, then the cut-condition is sufficient for routing demands. We consider the following generalization of this setting and prove an approximate max flow-min cut theorem: for every demand edge, there exists a face containing both its end points. We show that the cut-condition is sufficient for routing $\Omega(1)$-fraction of all the demands. To prove this, we give a L_1-embedding of the planar metric which approximately preserves distance between all pair of points on the same face.

1 Introduction

Given a graph G with edge capacities and multiple source-sink pairs, each with an associated demand, the multicommodity flow problem is to route all demands simultaneously without violating edge capacities. The problem was first formulated in the context of VLSI routing in the 70's and since then it has seen a long and impressive line of work.

The demand graph, H is the graph obtained by including an edge (s_i, t_i) for a demand with source-sink s_i, t_i. A necessary condition for the flow to be routed is that the capacity of every cut exceeds the demand across the cut. This condition is known as the **cut-condition** and is known to be sufficient when G is planar and all the source-sink pairs are on one face [11] or when $G+H$ is planar [14]. However, one can construct small instances where the cut-condition is not sufficient for routing flow. When G is series-parallel, if every cut has capacity at least twice the demand across it, then flow is routable [6, 3]. The flow-cut gap of a certain graph class is the smallest α such that flow is routable when capacity of every cut is at least α times the demand across it. Thus, for series-parallel graphs, the flow-cut gap is 2. For general graphs, the flow-cut gap is $\Theta(\log k)$ [10], where k is the number of demand pairs.

The flow-cut gap for planar graphs (G planar, H arbitrary) is $\mathcal{O}(\sqrt{\log n})$ [12] and is conjectured to be $\mathcal{O}(1)$ [6]. Chekuri et al. [4] showed a flow-cut gap of $2^{\mathcal{O}(k)}$ for k-outerplanar graphs. Lee et al. [8] made progress towards this conjecture by showing an $\mathcal{O}(\log h)$ bound on the flow-cut gap, where h is the number of faces on which source-sink vertices are incident. Filtser [5] further improved his bound by showing a flow-cut gap of $\mathcal{O}(\sqrt{\log h})$, when all the source-sink vertices are incident on h faces. In this paper, we consider instances where the source and sink of every demand lie on the same face, but all source-sink pairs don't necessarily lie on a single face, and show that the flow-cut gap of such instances is $\mathcal{O}(1)$. It is well known that the cut-condition is not sufficient for such instances (see Figure 1). A common approach to establish bounds on the flow-cut gap is to bound the L_1 distortion incurred in embedding an arbitrary metric on the graph G into a normed space. This, for instance, has been the method used to establish flow-cut gaps for general graphs [10], series-parallel graphs [6, 3] and planar graphs [12]. We too build on this technique to prove our results (see Theorem 5 and 6).

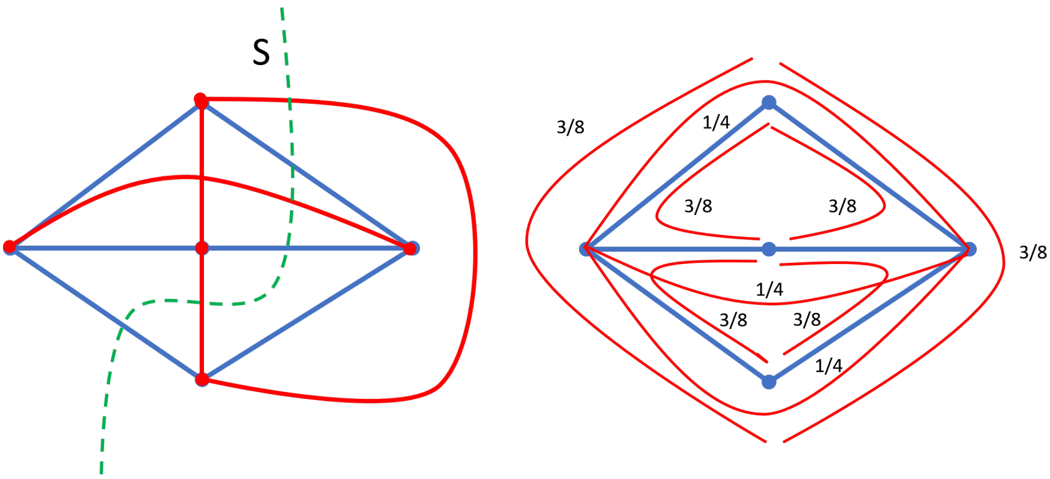

Figure 1: This example first appeared in the work of Okamura and Seymour [11]. All supply (blue) and demand (red) edges have value 1. S (green) is a cut. The total capacity of supply edges across S is three while S separates three units of demand; hence S satisfies the cut-condition. One can check that no cut violates the cut-condition. Since the source-sink of every demand is distance two apart, a total capacity of $4 \cdot 2 = 8$ is required for a feasible routing, but only six are available. Hence, no feasible routing is possible. This also implies that no more than 3/4 of every demand can be routed simultaneously. Figure on the right shows a feasible routing of 3/4 of every demand, which implies a flow-cut gap of 4/3.

2 Definitions and Preliminaries

Let $G = (V, E)$ be a simple graph with edge capacities $c : E \to R_{\geq 0}$. We call this the supply graph. Let $H = (V, D)$ be a simple graph with demands on edges $d : D \to R_{\geq 0}$. We call this the demand graph. The objective of the **multicommodity flow** problem is to find paths between the end points of demand edges in the supply graph such that the following hold: for every demand edge $e \in D$, $d(e)$ paths are picked in the supply graph and every supply edge $e \in E$ is present in at most $c(e)$ paths.

We say that an instance is feasible if paths satisfying the above two conditions can be found. We also call such an instance integrally feasible. If there exists an assignment of positive real numbers to paths such that total flow for every demand edge $e \in F$ is d_e and total value of paths using a supply edge $e \in E$ is at most c_e, then we say that the instance is (fractionally) feasible. A cut $S \subseteq V$ is a partition of the vertex set $(S, V \setminus S)$. The number of edges of G going across the cut is denoted by $\delta_G(S)$. Similarly, $\delta_H(S)$ denotes the number of demand edges going from S to $V \setminus S$. One necessary condition for routing the flow (fractionally or integrally) is as follows: for every $S \subseteq V, \delta_G(S) \geq \delta_H(S)$. In other words, across every cut, total supply should be at least the total demand. This condition is also known as the **cut-condition**. In general, the cut-condition is not sufficient for a feasible routing. We can ask for the following relaxation: given an instance for which the cut-condition is satisfied, what is the maximum value of f, such that f fraction of every demand can be routed? The number f^{-1} is known as the **flow-cut gap** of the instance. There is an equivalent definition of the flow-cut gap: given an instance (G, H) satisfying the cut condition, the smallest number k, such that (kG, H) is feasible, where kG denotes the graph with every edge capacity multiplied by k (see Figure 1 for an illustration). The following two classic results identify settings where the cut-condition is also sufficient for routing demands in planar graphs. We will be invoking these to prove our results.

Theorem 1 (Okamura-Seymour [11]) *If G is a planar graph, all the edges of H are restricted to a face and $G + H$ is eulerian, then the cut-condition is necessary and sufficient for integral routing of all the demands.*

Theorem 2 (Seymour [14]) *If $G + H$ is planar and eulerian, then the cut-condition is necessary and sufficient for integral routing of all the demands.*

Note that if eulerian condition is not satisfied, we get a half integral flow. Both the results can be converted into algorithms which run in polynomial time. In this paper, we will not be concerned with (half) integral flows and focus on proving flow-cut gap for instances that generalize the setting of the above stated results (see Theorem 5).

From now on, we assume a fixed planar embedding of G. Without loss of generality, one can assume that G is 2-vertex connected. If there is a cut vertex v and ab is a demand separated by removal of v, then replacing ab by av, vb maintains the cut-condition. By doing this for every cut vertex and demand separated by them, we get separate smaller instances for each 2-vertex connected component. Hence, every vertex is a part of cycle corresponding to some face. By our assumption, for every demand edge there exists a face such that both its end points lie on that face. Hence, we can associate every demand with a face. We abuse notation and use f to also denote the edges and vertices associated with the cycle of face f. Given a set S, we denote the subgraph induced by vertices in S as $G[S]$. We call a subset $A \subseteq V$ central if both $G[A]$ and $G[V - A]$ are connected. The following is well-known.

Lemma 1 ([13]) *(G, H) satisfies the cut-condition if and only if all the central sets satisfy the cut-condition.*

The set of all faces of G will be denoted by F. The **dual** of a planar graph $G^D = (V^D, E^D)$ is defined as follows: $V^D = F$ and if $f_i, f_j \in F$ share an edge in G, then $(f_i, f_j) \in E^D$. It is a well known fact that edges of a central cut in G correspond to a simple circuit in G^D and vice versa. Given a graph $G = (V, E)$ with edge length $l : E \to \mathbb{R}_{\geq 0}$, we use $d_G(u, v)$ to denote the shortest path distance between u and v in G w.r.t l. We now describe the connection between flow-cut gap and embedding vertices into normed space.

2.1 Embedding Metrics into L_1

Given an edge weighted graph $G = (V, E)$ with edge length $l : E \to \mathbb{R}_{\geq 0}$, associated shortest path metric d_G, a graph $H = (V, F)$ and an embedding f of vertices V into L_1, the contraction and expansion of f are the smallest α, β respectively, such that $||f(u) - f(v)||_1 \geq d_G(u, v)/\alpha$ for all $(u, v) \in F$ and $||f(u) - f(v)||_1 \leq \beta \cdot d_G(u, v)$ for all $(u, v) \in E$. The distortion of the embedding, $\texttt{dist}(G, H, f)$, is $\alpha \cdot \beta$. Given G, H, we are generally interested in finding an embedding with low distortion. If H is a clique, we refer to $\texttt{dist}(G, H, f)$ simply as $\texttt{dist}(G, f)$ and call it the distortion of G with respect to f. Linial et al. [10] built on the result of Bourgain [1], and gave a polynomial time algorithm that embeds any graph on n vertices into L_1 with distortion $\mathcal{O}(\log n)$. Furthermore, this result is asymptotically the best possible, as there exist instances for which any embedding into L_1 has distortion $\Omega(\log n)$. There is a rich literature on finding low distortion embeddings for special graph classes. It is well known that a tree can be embedded into L_1 with distortion 1, outerplanar graphs with distortion 1 [11], k-outerplanar graphs with distortion $2^{\mathcal{O}(k)}$ [4], series-parallel graphs with distortion 2 [3]. Rao [12] showed that any planar metric can be embedded into L_1 with distortion $\mathcal{O}(\sqrt{\log n})$ and there has essentially been no improvement upon this in the last two decades. It is conjectured that any planar graph can be embedded into L_1 with distortion $\mathcal{O}(1)$ [6] (also known as the `GNRS-Conjecture`). In this paper, we make progress towards this conjecture. Given a drawing of a planar graph in the plane, let H be the set of all pairs of vertices (u, v) such that u and v lie on the same face. We show the existence of a polynomial time computable $f : V \to L_1$ such that $\texttt{dist}(G, H, f) = \mathcal{O}(1)$. In this paper, we will work exclusively with the 1-norm, so we drop the subscript and denote $||f(u) - f(v)||_1$ simply as $||f(u) - f(v)||$. Also, if $\beta = 1$, we say that the embedding is non-expansive. By scaling, we may convert any L_1 embedding into a non-expansive one.

2.2 Flow-Cut Gap and Embedding into L_1

Flow-cut gaps and embedding graphs into L_1 are intimately related. This connection was first observed by Linial et al. [10], who used it to prove flow-cut gap results for arbitrary graphs. We describe this connection formally now. Let $G = (V, E), H = (V, F)$ be fixed graphs and $c : E \to \mathbf{R}_{\geq 0}$, $d : F \to \mathbf{R}_{\geq 0}$ and $l : E \to \mathbf{R}_{\geq 0}$ be the capacity, demand and length functions on respective edge sets. Let $\mathcal{I}$ be the set of all multicommodity flow instances $G = (V, E, c), H = (V, F, d)$ for which the cut-condition is satisfied. Let $\texttt{cong}(G, H)$ denote the maximum congestion required for routing in any multicommodity flow instance in $\mathcal{I}$. Let $\texttt{dist}(G, H)$ be the minimum number such that for any length function l, there exists a f such that $\texttt{dist}(G, H, f) \leq \texttt{dist}(G, H)$, where $G = (V, E)$ with edge-length l. The **congestion-distortion theorem** states that $\texttt{cong}(G, H) = \texttt{dist}(G, H)$. See Section 3 of [2] for a simple proof of this fact using LP duality. This connection has been exploited extensively to prove flow-cut gap results for general graphs [10], series-parallel graphs [3, 6] and planar graphs [12]. All these results proceed by showing the existence of a low distortion embedding of the corresponding metric into L_1. Using the congestion-distortion theorem, we now restate the theorem of Okamura and Seymour [11] and Seymour [14] in terms of metric embedding.

Theorem 3 (Okamura-Seymour[11]) *Let $G = (V, E)$ be a planar graph with edge length $l : E \to \mathbb{R}_{\geq 0}$ and $t \in F$ be one of its faces. Then there exists an embedding of V into L_1 such that for all $u, v \in t, ||f(u) - f(v)|| = d_G(u, v)$ and for all $(u, v) \in E, ||f(u) - f(v)|| \leq d_G(u, v) = l(u, v)$.*

Theorem 4 (Seymour[14]) *Let $G = (V, E)$ be a planar graph with edge length $l : E \to \mathbb{R}_{\geq 0}$ and $H = (V, T)$ be a demand graph such that $G + H$ is planar. Then there exists an embedding of V into L_1 such that for all $(u, v) \in T, ||f(u) - f(v)|| = d_G(u, v)$ and for all $(u, v) \in E, ||f(u) - f(v)|| \leq d_G(u, v) = l(u, v)$.*

2.3 Cut Metrics and L_1 Embedding

Suppose we have a set of cuts with non-negative weights $\mathcal{C} = \{(C_1, w_1), \ldots, (C_k, w_k)\}$. Define $\delta_{C_i}(u, v)$ to be w_i if exactly one of u, v is contained in C_i and 0 otherwise. Let $\delta_{\mathcal{C}}(u, v) = \sum_{i=1}^{k} \delta_{C_i}(u, v)$. It is easy to verify that δ_C induces a metric on V. We refer to δ_C as the distance induced by the cuts in $\mathcal{C}$ or a cut-metric $\mathcal{C}$. One can construct $f : V \to \mathbb{R}^k$ such that $\forall u, v$ we have $||f(u) - f(v)||_1 = \delta_{\mathcal{C}}(u, v)$ as follows: for any vertex u, define $f(u) = (u_1, u_2, \ldots, u_k)$ where $u_i = w_i$ if $u \in C_i$, 0 otherwise. In fact, the converse of above is also true: given any embedding of vertices into L_1, there exists a set of weighted cuts $\mathcal{C}$ such that the distance metric induced by $\mathcal{C}$ is equal to the distance metric induced by the L_1 embedding (see Lemma 15.2 of [15] for a proof). Hence, to show a low distortion L_1 embedding of a metric, it is equivalent to show a collection of cuts which preserve distances with low distortion. Using the aforementioned equivalence of the cut-metric and L_1 embedding, we us them interchangeably from now on. Given a scalar α and a collection of weighted cuts $\mathcal{C}$, $\alpha \cdot \mathcal{C}$ denotes the the same collection of cuts with the weight of all cuts scaled by a (multiplicative) factor of α.

3 Our Contribution

We generalize the result of Okamura and Seymour [11] and prove the following approximate max flow-min cut theorem:

Theorem 5 *Let G be an edge-capacitated planar graph and H be a set of demand edges such that for each $(u, v) \in H$, there exists a face f containing both u and v. If the cut-condition is satisfied, then there exists a feasible routing of $\Omega(1)$-fraction of all the demands.*

Using the congestion-distortion theorem, Theorem 5 can be stated in terms of metric embedding as follows:

Theorem 6 *Let $G = (V, E)$ be a planar graph with edge length $l \to \mathbb{R}_{\geq 0}$ and T be pairs of vertices (u, v) such that both u and v lie on the same face. Then there exists a constant $c > 1$ and an embedding $g : V \to L_1$ such that $||g(u) - g(v)|| \geq d_G(u, v)/c$ for $(u, v) \in T$ and $||g(u) - g(v)|| \leq l(u, v)$ for $(u, v) \in E$.*

We now give a brief overview of our approach. As mentioned before, we work with a fixed embedding of the given planar graph in the plane. We call a face $f \in F$ **geodesic** if for all $u, v \in f$, $d_G(u, v)$ is equal to the shortest path distance between u, v using only the edges of the cycle corresponding to f. A face which is not geodesic is called **non-geodesic**. Let F_G and F_N denote the set of all the geodesic and non-geodesic faces. Given a face f, let G_f be the subgraph enclosing minimal area in the plane that supports the metric on the vertices of face f. Let S_f be the minimal area cycle bounding G_f in the plane. Given a cycle S, let $R(S) \subseteq \mathbb{R}^2$ be the open region contained inside S.

If f is a geodesic face, then G_f is exactly the cycle bounding the face, i.e. $G_f = S_f$. In Section 4, we first show that the set $\{R(S_f)|f \in F\}$ forms a laminar or non-crossing set system. This implies that there is a face f with minimal $R(S_f)$, i.e. for any $f' \neq f$ either $R(S_f) \subseteq R(S_{f'})$ or $R(S_f) \cap R(S_{f'}) = \emptyset$. We then go on to show that removing the edges of $(G_f \setminus S_f)$ doesn't modify the metric on any of the non-geodesic faces other than f. We then go on to find a suitable embedding of the graph $(G \setminus G_f) \cup S_f$ inductively and show how to extend the embedding to include all the faces contained in $R(S_f)$. This extension argument turns out to be non-trivial and forms the core of our proof. To do this extension, we need to develop several new tools. In Section 4, we give an algorithm to modify the original length function so as to allow a nice inductive decomposition. We call such length functions α-good and believe that this could be a useful tool in proving flow-cut gaps for other planar instances as well. In Section 5, we come up with an embedding for all geodesic pairs of vertices,i.e. pairs of vertices for which there exists a shortest path using only the edges on the corresponding face. In Section 6, we develop a low distortion embedding for all pairs of vertices whose shortest path uses a fixed vertex of the graph. We believe that this problem is interesting in its own right and could prove to be useful in other settings. In Sections 7 and 8, we combine the tools developed in previous sections to complete the inductive step.

4 Laminar Structure of Face Supports

We partition the set of faces of a planar graph G into two sets: **geodesic** and **non-geodesic**. A face f is called geodesic if for all $u, v \in f$, there exists a shortest path between u and v using only the vertices of the cycle associated with f. A face which is not geodesic is called a non-geodesic face. Let F_G, F_N denote the set of geodesic and non-geodesic faces. Observe that $F = F_G \cup F_N$.

Let f be a face of G. A set of edges $E' \subseteq E$ is called a **support** of f if for all $u, v \in f$, $d_{G'}(u, v) = d_G(u, v)$, where $G' = (V, E')$. In other words, restricting to E' doesn't change the shortest path metric on f. A support of f is called minimal if deleting any edge from it changes the shortest path metric on vertices of f. Given a cycle C, let $R(C)$ (resp. $\overline{R(C)}$) denote the open (resp. closed) region contained inside the cycle C in the planar embedding of G. We choose $R(C)$ and $\overline{R(C)}$ such that it does not contain the infinite face. Let S_f be a cycle such that $\overline{R(S_f)}$ is the inclusion wise minimal region containing a support of f (i.e. there exists a support E_f of f such that $u, v \in \overline{R(S_f)}$ for all $(u, v) \in E_f$). Note that if $f \in F_G$, then $S_f = f$. Also, for any $f \in F$, $\overline{R(f)} \subseteq \overline{R(S_f)}$. We stress that the S_f is a function of the edge-lengths.

Lemma 2 *Let $f_1, f_2 \in F_N$. Then one of the following must hold: $R(S_{f_1}) \cap R(S_{f_2}) = \emptyset$ or $R(S_{f_1}) \subseteq R(S_{f_2})$ or $R(S_{f_2}) \subseteq R(S_{f_1})$.*

Proof. For the sake of contradiction, assume that there exist $f_1, f_2 \in F_N$ such that $R(S_{f_1}) \setminus R(S_{f_2}) \neq \emptyset$ and $R(S_{f_2}) \setminus R(S_{f_1}) \neq \emptyset$. Then either $R(f_1) \in R(S_{f_2})$ or $R(f_1) \cap R(S_{f_2}) = \emptyset$. We present the argument for the case when $R(f_1) \in R(S_{f_2})$, an analogous argument works for the second case as well. Suppose $R(f_1) \in R(S_{f_2})$. Then there must exist $u_1, v_1 \in f_1$ such that any shortest path from u_1 to v_1 exits and enters the region $R(S_{f_2})$ at least once. Let $P(u_1, v_1)$ be one such shortest path and x_1, y_1 be the first point of entry/last point of exit of $P(u_1, v_1)$ from/into $R(S_{f_2})$ respectively. Let $P(x_1, y_1)$ be the portion of the path between x_1 and y_1 on the cycle S_{f_2}. Note that $P(x_1, y_1)$ is a path on the boundary of $\overline{R(S_{f_2})}$. Since $\overline{R(S_{f_2})}$ defines the minimal region containing the support of the metric on f_2, there must exist $u_2, v_2 \in f_2$ such that some shortest path between u_2 and v_2 uses an edge on the path $P(x_1, y_1)$. Let $P(u_2, v_2)$ be such a shortest path and let x_2, y_2 be the first and the last point of intersection of $P(u_2, v_2)$ with $P(u_1, v_1)$. Then replacing the portion of the path $P(u_1, v_1)$ between x_2 and y_2 by portion of the path $P(u_2, v_2)$ between x_2 and y_2, we obtain a shortest u_1, v_1 path contained completely inside S_{f_2}. This contradicts our assumption on the minimality of $\overline{R(S_{f_1})}$. We can use an analogous argument to arrive at a contradiction in case $R(f_1) \cap R(S_{f_2}) = \emptyset$. Hence such f_1, f_2 can't exist and it must be true that $R(S_{f_1}) \cap R(S_{f_2}) = \emptyset$ or $R(S_{f_1}) \subseteq R(S_{f_2})$ or $R(S_{f_2}) \subseteq R(S_{f_1})$ for any $f_1, f_2 \in F_N$. ∎

Given a cycle C, let $I(C)$ be the set of vertices contained in $R(C)$. Note that $I(C)$ doesn't contain vertices of C. Let $\alpha > 1$ be a given constant. Given a face $f \in F_N$ and its support cycle S_f, we say that S_f is **α-loose** if the following holds: for any $u, v \in f$, the length of any path between u and v using only the vertices in $I(S_f) \cup \{u, v\}$ is at least $\alpha \cdot d_G(u, v)$.

To make the induction argument work, only the laminar property of face supports is not sufficient and we need a stronger structure. We now describe this property in more detail now. By Lemma 2, the set of regions $\{R(S_f) | f \in F_N\}$ forms a laminar structure. This implies that there exists a face $f \in F_N$ such that for any $f' \in F_N$, either $R(S_f) \subseteq R(S_{f'})$ or $R(f) \cap R(S_{f'}) = \emptyset$. We call such faces **innermost**. Note that there could be multiple innermost faces. An **α-good** length function $l : E \to \mathbb{R}^+$ is defined inductively as follows: if $f \in F_N$ is an innermost face, then the graph obtained by removing all the edges contained completely inside $R(S_f)$ is α-good and S_f is α-loose. We now show than any length function can be converted into a α-good one by modifying the edge lengths by a factor of at most α.

Theorem 7 *Let $G = (V, E)$ be a planar graph with f_I as the infinite face, edge length $l : E \to \mathbb{R}_{\geq 0}$ and $\alpha > 1$ be a constant. Then there exists a new edge length $l' : E \to \mathbb{R}^+$ such that G is α-good with respect to l', $l'(e) = l(e)$ for $e \in E \cap f_I$ and $l(e)/\alpha \leq l'(e) \leq l(e)$ for $e \in E$.*

Proof. We prove the theorem by induction on the number of faces in the graph. In case $|F| = 1$ or there are no non-geodesic faces, the statement of the theorem follows trivially by setting $l' = l$. Suppose that G has at least one non-geodesic face w.r.t the length function l. By Lemma 2, regions in the set $\{R(S_f) | f \in F_N\}$ form a laminar structure. A face $f \in F_N$ is called maximal if for any other face f', either $R(S_{f'}) \subseteq R(S_f)$ or $R(S_{f'}) \cap R(S_f) = \emptyset$. Let $f_1, f_2, \ldots, f_k$ be the maximal faces of G w.r.t length function l. Let G_i be the graph consisting of vertices and edges contained completely inside $\overline{R(S_{f_i})}$. If $k = 1$ and $R(f_I) \subset R(S_{f_1})$ or $k \geq 2$, then each G_i has strictly less number of faces than G. By induction, for each G_i we have length functions l_i satisfying the conditions of the theorem. We construct the new length function l' as follows: for an edge (u, v) not contained in any of the G_i, we set $l'(u, v) = l(u, v)$. If an edge (u, v) is contained in G_i, we set $l'(u, v) = l_i(u, v)$. Note that if an edge (u, v) is contained in G_i and G_j, then it must be present on the infinite face of G_i, G_j and $l_i(u, v) = l_j(u, v) = l(u, v)$ (by induction). Hence l' is a valid length function and by construction G is α-good with w.r.t l'. Suppose that $k = 1$ and $R(S_{f_1}) = R(f_I)$. We consider two cases depending on whether face f_I is α-loose or not.

Suppose the face f_I is not α-loose. By the definition of a α-loose face, there must exist $u, v \in f_I$ such that the shortest u, v path using no edges on f_I has length $\beta \cdot d_G(u, v)$, for some

$\beta \leq \alpha$. Let $P = \{u, u_1, \dots, u_l, v\}$ be such a path. For every edge $e \in P$, we set $l'(e) = l(e)/\beta$. Since P contains no edge of the infinite face f_I, the length of edges on f_I remain unchanged due to this operation. The path P divides $\overline{R(f_I)}$ into two closed regions, say R_1 and R_2. Let $G_1 \subset G$ and $G_2 \subset G$ be the graphs contained inside R_1 and R_2 respectively. Since P is a shortest u, v path w.r.t l', it follows that for any $f \in G_i$, $R(S_f) \subseteq R_i$ for $i = 1, 2$. Hence, if $u, v \in G_i$, then $d_{G_i}(u, v) = d_G(u, v)$ for $i = 1, 2$. Therefore we can compute length functions for G_1, G_2 separately by using induction and combine them as before. Using induction, we find length functions l_1, l_2 satisfying the conditions of the theorem and set $l'(e) = l_i(e)$ depending on whether $e \in G_1$ or $e \in G_2$. If e belongs to both to G_1 and G_2, then e is on the infinite face of G_1 and G_2 and by the statement of the theorem, $l_1(e) = l_2(e)$. Hence l' is a valid length function satisfying the conditions of the theorem.

Suppose that face the f_I is α-loose. Then the statement of the theorem holds for non-geodesic face f_1. Let $g_1, g_2, \dots, g_k$ be the maximal non-geodesic faces contained strictly inside $R(f_I)$. By the laminar structure of regions in $\{R(S_f) | f \in F_N\}$, we have $R(S_{g_i}) \cap R(S_{g_j}) = \emptyset$ for $i \neq j$. Let G_i be the graph consisting of vertices and edges contained completely inside $\overline{R(S_{g_i})}$. By induction hypothesis, we have length functions $l_1, \dots, l_k$ such that each one of them satisfies the statement of the theorem. We set $l'(e) = l(e)$ for any edge not contained inside any of the $G_i's$ and set $l'(e) = l_i(e)$ if $e \in G_i$. Since for any edge $e \in S_{g_i}$, $l'(e) = l(e)$, we do not create any new non-geodesic face in $R(f_I) \setminus \cup_{i=1}^{k} R(S_{g_i})$. We complete the proof of the theorem by showing that $\overline{R(S_{f_1})} = \overline{R(f_I)}$ and f_I is α-loose w.r.t l'. To show this, we prove that the metric on f_1 w.r.t to l and l' are the same.

Lemma 3 *Let $u, v \in f \in F_N$ and $P = \{u, u_1, u_2, \dots, u_l, v\}$ be a shortest u, v path, then $P \cap I(S_f) = \{u, v\}$.*

Proof. Suppose there exists u, v and a shortest path $P = \{u, u_1, u_2, \dots, u_l, v\}$ between them such that $\{u_1, u_2, \dots, u_l\} \cap I(S_f) \neq \emptyset$. Let S'_f be the cycle created by replacing the u, v path on the cycle S_f by P such that $\overline{R(f)} \subseteq \overline{R(S'_f)}$. Since $\{u_1, u_2, \dots, u_l\} \cap I(S_f) \neq \emptyset, \overline{R(S'_f)} \subset \overline{R(S_f)}$ and for any $u_1, v_1 \in f$, there exists a u_1, v_1 shortest path contained completely inside $\overline{R(S'_f)}$. Hence, $\overline{R(S'_f)}$ contains a support of face f and this contradicts the minimality of S_f. Hence, $P \cap I(S_f) = \{u, v\}$ and this completes proof of the lemma. ■

By Lemma 3, deleting the edges contained completely inside a $\overline{R(S_{g_i})}$ doesn't affect the metric on f_1. Doing this for each $i = 1, 2, \dots, k$ in a sequential order and noting that for any $e \in \cup_{i=1}^{l} S_{g_i}$, $l'(e) = l(e)$, we conclude that the metric on f_1 remains unchanged under l', and this completes the proof of the theorem. ■

5 Embedding For The Geodesic Pairs

Let $G = (V, E)$ be a planar graph and F be the set of its faces and $u, v \in V$ be a pair of vertices on the same face (i.e. there exists a face $f \in F$ such that $u, v \in f$). Furthermore, suppose that there exists a shortest path between u and v using only the edges of f. We call (u, v) a **geodesic pair**. Let T be the set of all the geodesic pairs in G. In Theorem 9, we show that there exists an embedding of V into L_1 which preserves the distances between all the geodesic pairs within a constant factor. Using the congestion-distortion theorem, the following result follows directly from Theorem 12 of Kumar [9].

Theorem 8 *Let $G = (V, E)$ be a planar graph with edge length $l : E \to \mathbb{R}_{\geq 0}$ and T be pairs of vertices (u, v) such that both u and v lie on the same face. Let $T_f \subseteq T$ denote the set of pairs of vertices incident on face f. Suppose for each $f \in F$, there exists disjoint set of vertices $X_1, Y_1, X_2, Y_2, \dots, X_k, Y_k \subseteq f$ such that X_i, Y_i appear contiguously on f in clockwise order and* *for each* $(u,v) \in T_f, u \in X_j, v \in Y_j$ *for some* $j \in \{1,2,\ldots,k\}$. *Then there exists an embedding* $h : V \to L_1$ *such that* $||h(u)-h(v)|| \leq d_G(u,v)$ *for all* $(u,v) \in E$ *and* $||h(u)-h(v)|| \geq d_G(u,v)/3$ *for all* $(u,v) \in T$.

Theorem 9 *Let* G *be a planar graph with edge length* $l : E \to \mathbb{R}^+$. *Then there exists an embedding* $g : V \to L_1$ *such that* $||g(u)-g(v)|| \leq d_G(u,v)$ *for all* $(u,v) \in E$ *and* $||g(u)-g(v)|| \geq d_G(u,v)/21$ *for all* $(u,v) \in T$.

Proof. We first construct a set of geodesic pairs $T_p \subseteq T$ as follows. Initially $T_p = \emptyset$. Let $f \in F$ and $V(f) = \{v_1, v_2, \ldots, v_n, v_{n+1} = v_1\}$ be the vertices of f in clockwise order. We partition $V(f)$ into contiguous sets of vertices as follows: let v_k be the largest k such that (v_1, v_k) is a geodesic pair. We set $T_p = T_p \cup (v_1, v_k)$ and $S_f^1 = \{v_1, v_2, \ldots, v_k\}$. We then start from v_{k+1} and find the largest index l such that (v_{k+1}, v_l) is a geodesic pair, set $T_p = T_p \cup (v_{k+1}, v_l)$ and $S_f^2 = \{v_{k+1}, v_{k+2}, \ldots, v_l\}$, and we continue with this process until all the vertices in $C(f)$ have been exhausted, i.e. $C(f) = \cup_{i\geq 0} S_f^i$. We do the above for all the faces of G. By construction, the graph $G \cup T_p$ is planar. Therefore by Theorem 4, we have an embedding, say $h : V \to L_1$, that preserves the distance between the end points of vertices in T_p exactly. In other words, $||h(u)-h(v)|| = d_G(u,v)$ for all $(u,v) \in T_p$ and $||h(u)-h(v)|| \leq d_G(u,v)$ for all $(u,v) \in E$. Let $T' = \{(u,v) | u, v \in S_f^i, f \in F, i \geq 0\}$. By construction, the first and the last vertex (w.r.t the numbering above) of each of the set S_f^i form a geodesic pair, hence for all $u, v \in T'$, we have that $|||h(u)-h(v)|| = d_G(u,v)$.

In Claim 1, we show that the geodesic pairs in $T \backslash T'$ can be partitioned into $T_0, T_1, T_2, T_3, T_4, T_5$ such that each one of them satisfies the condition of Theorem 8. Then by Theorem 8, there exists $h_i : V \to L_1$ such that $||h_i(u)-h_i(v)|| \leq d_G(u,v)$ for $(u,v) \in E$ and $||h_i(u)-h_i(v)|| \geq d_G(u,v)/3$ for $(u,v) \in T_i$ for $i = 0,1,2,3,4,5$. By setting $g = \frac{1}{7}(h + h_0 + h_1 + h_2 + h_3 + h_4 + h_5)$, we obtain $||g(u)-g(v)|| \leq d_G(u,v)$ for $(u,v) \in E$ and $||g(u)-g(v)|| \geq d_G(u,v)/21$ for $(u,v) \in T$.

Claim 1 $T \setminus T'$ *can be partitioned into* $T_0, T_1, T_2, T_3, T_4, T_5$ *such that each of the* T_i*'s satisfies the condition of Theorem 8.*

Proof. It is sufficient to show that the geodesic pairs incident on each face can be partitioned into T_i^f for $i = 0,1,2,3,4,5$ such that each T_i^f satisfies the condition of Theorem 8. Let $f \in F$ and $S_f^1, S_f^2, \ldots . S_f^k$ be the partition of $C(f)$ created in the first step. Suppose there exists $(u,v) \in T \setminus T'$ such that $u \in S_f^{[l]}, v \in S_f^{[l+t]}$ for some $t \geq 3$, where $[p] = p$ if $p \leq k$ and $p - k$ otherwise. Then our procedure in the first step would not have created separate partitions for $S_f^{[l+1]}$ and $S_f^{[l+2]}$ and hence such a geodesic pair (u,v) can't exist. Therefore for all $u, v \in C(f)$ and $(u,v) \in T \setminus T'$, one of the following must hold: $u \in S_f^{[l]}, v \in S_f^{[l+1]}$ (called *type 1*) or $u \in S_f^{[l]}, v \in S_f^{[l+2]}$ (called *type 2*) for some l.

Let (u,v) be a geodesic pair (u,v) such that $u \in S_f^i$ and $v \in S_f^j$. We say that (u,v) belongs to *class* i if $\min(i,j) \mod 3 = i$. Let T_0^f, T_1^f, T_2^f be the geodesic pairs of class 0,1,2 of *type 1* incident on the face f and T_3^f, T_4^f, T_5^f be the geodesic pairs of class 0,1,2 of *type 2* incident on the face f. We set $T_i = \bigcup_{f \in F} T_i^f$ for $i = 0,1,2,3,4,5$. The geodesic pairs in each of the T_i's satisfy the condition of Theorem 8 and this completes the proof of the claim. ■

■

6 Single Source Shortest Path Embeddings

We next show how to find an embedding such that the distance between all pairs of vertices whose shortest path uses a fixed vertex v is approximately preserved. To prove this result, we make use of a well known result of Klein, Plotkin and Rao [7] on small diameter decomposition. Let (X, D) be a finite metric space. A distribution μ over (vertex) partitions of X is called (β, Δ)-lipschitz if every partition P in the support of μ satisfies $S \in P \implies diam_X(S) \leq \Delta$ and moreover for all $x, y \in X$, $\underset{P \sim \mu}{\mathcal{P}}[P(x) \neq P(y)] \leq \beta \cdot \frac{d(x,y)}{\Delta}$. Klein, Plotkin and Rao [7] showed that there exists a (c, Δ)-lipschitz partition of a planar metric where $\Delta > 0$ is arbitrary and c is an absolute constant. [1]

Theorem 10 (Klein, Plotkin and Rao [7]) *Let (X, D) be a finite planar metric and let $\Delta \in \mathbb{R}_{\geq 0}$ be a given number. Then there exists a polynomial-time computable (c, Δ)-lipschitz partition of (X, D), where c is some absolute constant.*

Theorem 11 *Let $G = (V, E)$ be a planar graph with edge-length $l : E \to \mathbf{R}_{\geq 0}$ and $v \in V$. Let $T = \{(s_i, t_i)\}_{i=1}^k$ be the set of pair of vertices such that $d(s_i, t_i) = d(v, s_i) + d(v, t_i)$ for $i = 1, 2, \ldots, k$. Then there exists an embedding $g : V \to L_1$ and a constant $\beta > 1$ such that $d(s_i, t_i)/\beta \leq ||g(s_i) - g(t_i)|| \leq d(s_i, t_i)$ for all i. Furthermore such an embedding can be computed in polynomial time.*

Proof. We first prove the theorem for the special case when $d(v, s_i) = d(v, t_i)$ for all $(s_i, t_i) \in T$. Let $B_i = \{x | d(v, x) \leq 2^{i+1}\}$ for $i \geq 0$. Since $d(v, s_j) = d(v, t_j)$ for each $(s_j, t_j) \in T$, there exists an i such that $(s_j, t_j) \in B_{i+1} \setminus B_i$. Let $T_i = \{(s_j, t_j) | 2^i \leq d(v, s_j) = d(v, t_j) \leq 2^{i+1}\}$ and G_i be the graph obtained by setting the edge length of all the edges contained inside $P_i = B_{i-2}$ and $Q_i = V \setminus B_{i+2}$ to 0. More formally, for all $(u, w) \in E$ such that $u, w \in P_i$ or $u, w \in Q_i$, we set $l_{(u,w)} = 0$ to obtain G_i from G. Claim 2 shows that distance between the (s_i, t_i) pairs in G_i are within a constant factor of the distance in G.

Claim 2 *$d_G(s_j, t_j)/4 \leq d_{G_i}(s_j, t_j) \leq d_G(s_j, t_j)$ for all $(s_j, t_j) \in T_i$.*

Proof. $d_{G_i}(s_j, t_j) \leq d_G(s_j, t_j)$ follows directly from construction since each G_i is formed by setting the length of some edges of G to 0. Since $(s_j, t_j) \in T_i$, we have $d(s_j, t_j) = d(s_j, v) + d(t_j, v) \geq 2^{i+1}$ and $d(s_j, t_j) = d(s_j, v) + d(t_j, v) \leq 2^{i+2}$. If the shortest path between (s_j, t_j) in G_i doesn't use any vertex in P_i or Q_i, then $d(s_j, t_j)$ remains unchanged and the statement of claim follows trivially. Suppose the (s_j, t_j) shortest path in uses a vertex in P_i or Q_i. Then we have:

$$\begin{aligned} d(P_i, s_j) = d(B_{i-2}, s_j) \geq d(B_{i-2}, V \setminus B_{i-1}) \geq 2^{i-1} \\ d(Q_i, s_j) = d(V \setminus B_{i+2}, s_j) \geq d(V \setminus B_{i+2}, B_{i+1}) \geq 2^{i+1} \\ d(P_i, t_j) = d(B_{i-2}, t_j) \geq d(B_{i-2}, V \setminus B_{i-1}) \geq 2^{i-1} \\ d(Q_i, t_j) = d(V \setminus B_{i+2}, t_j) \geq d(V \setminus B_{i+2}, B_{i+1}) \geq 2^{i+1} \end{aligned}$$

We have $d_{G_i}(s_j, t_j) \geq \min\{d(P_i, s_j) + d(P_i, t_j), d(Q_i, s_j) + d(Q_i, t_j)\} \geq 2^i \geq d_G(s_j, t_j)/4$ and the claim follows. ∎

We now use Theorem 10 to construct a distribution over (vertex) partitions $\mathcal{P}$ of G_i by setting $\Delta = 2^{i-1}$. Suppose the partition is (c, Δ)-lipschitz for some constant $c > 0$. We construct a cut metric $\mathcal{C}_i$ using $\mathcal{P}$ as follows: for each partition $P = \{P_1, P_2, \ldots, P_k\} \in \mathcal{P}$ with weight $\mu(P)$, we include $P_1, P_2, \ldots, P_k$ in $\mathcal{C}_i$, each with weight $\frac{\mu(P) \cdot \Delta}{c}$. Claim 3 shows that $\mathcal{C}_i$ preserves distances between the pair of vertices in T_i.

Claim 3 *For any $(u, w) \in G_i$, we have $\delta_{\mathcal{C}_i}(u, w) \leq d_{G_i}(u, w)$. Furthermore, for any $(s_j, t_j) \in T_i$, we have $\delta_{\mathcal{C}_i}(s_j, t_j) \geq \frac{d_{G_i}(s_j, t_j)}{4c}$.*

[1] In fact Klein, Plotkin and Rao [7] showed that such a partition exists for any minor-closed family of graphs.

Proof. By the definition of (c, Δ)-lipschitz partition, we have that for any $u, w \in G_i$, probability that u and w are in separate partitions is at most $\frac{c \cdot d_{G_i}(u,w)}{\Delta}$. Hence, for any $u, w \in G_i$, we have:

$$\delta_{\mathcal{C}_i}(u,w) = \sum_{C \in \mathcal{C}_i : |C \cap \{u,w\}|=1} w(C) = \underset{P \sim \mu}{\mathcal{P}}[P(u) \neq P(w)] \cdot \frac{\Delta}{c} \leq \frac{c \cdot d_{G_i}(u,w)}{\Delta} \cdot \frac{\Delta}{c} \leq d_{G_i}(u,w).$$

We now show the other part. Since $d_{G_i}(s_j, t_j) \geq 2^i > \Delta$, we have that in every partition $P \in \mathcal{P}$, (s_j, t_j) are in different subsets of P. This implies that the total weight of cuts in $\mathcal{C}_i$ separating (s_j, t_j) is at least $\frac{\Delta}{c}$. Hence, $\delta_{\mathcal{C}_i}(s_j, t_j) \geq \frac{2^{i-1}}{c} \geq \frac{d_{G_i}(s_j,t_j)}{4c}$ and the claim follows. ■

Claim 4 *Let $\mathcal{C} = \bigcup_{i \geq 2} \mathcal{C}_i$. For any $(u,w) \in G$, we have $\delta_{\mathcal{C}}(u,w) \leq 3 \cdot d_G(u,w)$. Furthermore, for any $(s_j, t_j) \in T$, we have $\delta_{\mathcal{C}}(s_j, t_j) \geq \frac{d_G(s_j,t_j)}{16 \cdot c}$.*

Proof. By Theorem 10, if an edge has length 0 in G_i, then it is not separated by any cut in $\mathcal{C}_i$. Furthermore, any edge $(u,w) \in E$ can be a part of at most three G_i's. Hence, using Claim 3, we obtain $\delta_{\mathcal{C}}(u,w) \leq 3 \cdot d_G(u,w)$. By Claim 3, for any $(s_j,t_j) \in T_i$, $\delta_{\mathcal{C}_i}(s_j,t_j) \geq d_{G_i}(s_j,t_j)/4c$. Using Claim 2, we have $d_{G_i}(s_j,t_j) \geq d_G(s_j,t_j)/4$. Hence, $\delta_{\mathcal{C}}(s_j,t_j) \geq \delta_{\mathcal{C}_i}(s_j,t_j) \geq d_G(s_j,t_j)/16c$. ■

We have shown that the theorem holds when $d_G(v,s_j) = d_G(v,t_j)$ for all $(s_j,t_j) \in T$. We now prove a more general version of Claim 4 when $d_G(v,s_j) \neq d_G(v,t_j)$.

Claim 5 *For any $(s_j,t_j) \in T$, $\delta_{\mathcal{C}}(s_j,t_j) \geq \frac{d_G(s_j,t_j)}{16 \cdot c} - 3|d_G(v,s_j) - d_G(v,t_j)|$.*

Proof. Without loss of generality, we may assume that $d_G(v,s_j) \geq d_G(v,t_j)$. Let s'_j be the vertex on a shortest path from v to s_j such that $d_G(v,s'_j) = d_G(v,t_j)$. By Claim 4, we have $\delta_{\mathcal{C}}(s'_j,t_j) \geq d_G(s'_j,t_j)/16c$. All the cuts in $\mathcal{C}$ which contain exactly one of s'_j and t_j contribute to $\delta_{\mathcal{C}}(s'_j,t_j)$ and they can be partitioned into two groups: one in which s'_j and s_j are in the same partition and the other in which s_j and t_j are in the same partition. Let's call them $\delta_{\mathcal{C}}(s'_j s_j, t_j)$ and $\delta_{\mathcal{C}}(s'_j, s_j t_j)$ respectively. Observe that,

$$\delta_{\mathcal{C}}(s'_j,t_j) = \delta_{\mathcal{C}}(s'_j s_j,t_j) + \delta_{\mathcal{C}}(s'_j, s_j t_j) \text{ and } \delta_{\mathcal{C}}(s'_j s_j, t_j) \leq \delta_{\mathcal{C}}(s_j,t_j).$$

Hence $\delta_{\mathcal{C}}(s_j,t_j) \geq \delta_{\mathcal{C}}(s'_j,t_j) - \delta_{\mathcal{C}}(s'_j, s_j t_j)$. By Claim 4, we have,

$$\delta_{\mathcal{C}}(s'_j, s_j t_j) \leq \delta_{\mathcal{C}}(s'_j, s_j) \leq 3 d_G(s'_j, s_j) = 3|d_G(v,s_j) - d_G(v,t_j)|$$

Using $\delta_{\mathcal{C}}(s'_j,t_j) \geq d_G(s'_j,t_j)/16c$, we obtain:

$$\delta_{\mathcal{C}}(s_j,t_j) \geq \delta_{\mathcal{C}}(s'_j,t_j) - 3|d_G(v,s_j) - d_G(v,t_j)| \geq d_G(s_j,t_j)/16c - 3|d_G(v,s_j) - d_G(v,t_j)|.$$

■

We augment $\mathcal{C}$ by the following single source cuts: let $V = \{v_1, v_2, \ldots, v_n\}$ such that $d_G(v,v_1) \leq d_G(v,v_2) \leq \ldots \leq d_G(v,v_n)$. Let $R_i = \{v_1, v_2, \ldots, v_i\}$ and $w(R_i) = 3 \cdot (d_G(v_{i+1}) - d_G(v_i))$ for $i = 1, 2, 3, \ldots, n$. Let $\mathcal{R} = \{(R_1, w(R_1)), \ldots, (R_n, w(R_n))\}$ and $\mathcal{C}' = \mathcal{C} \cup \mathcal{R}$.

Claim 6 *For any $(u,w) \in G$, we have $\delta_{\mathcal{C}'}(u,w) \leq 6 \cdot d_G(u,w)$. Furthermore, for any $(s_j,t_j) \in T$, we have $\delta_{\mathcal{C}'}(s_j,t_j) \geq d_G(s_j,t_j)/16c$.*

Proof. Observe that $\delta_R(u,w) = 3 \cdot (d_G(v,u) - d_G(v,w)) \leq 3 \cdot d_G(u,w)$. Using Claim 5, for any $(u,w) \in E$ we have, $\delta_{\mathcal{C}'}(u,w) = \delta_{\mathcal{C}}(u,w) + \delta_{\mathcal{R}}(u,w) \leq 3 \cdot d_G(u,w) + 3 \cdot d_G(u,w) = 6 \cdot d_G(u,w)$. Using Claim 5, for any $(s_j,t_j) \in T$ we have $\delta_{\mathcal{C}'}(s_j,t_j) = \delta_{\mathcal{C}}(s_j,t_j) + \delta_{\mathcal{R}}(s_j,t_j) \geq d_G(s_j,t_j)/16c - 3|d_G(v,s_j) - d_G(v,t_j)| + 3|d_G(v,s_j) - d_G(v,t_j)| = d_G(s_j,t_j)/16c$. ■

Let $\mathcal{C}'' = \mathcal{C}'/6$. Then for any $(u,w) \in G$, we have $\delta_{\mathcal{C}'}(u,w) \leq d_G(u,w)$ and for any $(s_j,t_j) \in T$, we have $\delta_{\mathcal{C}'}(s_j,t_j) \geq d_G(s_j,t_j)/96c$. By using the equivalence between the cut-metric and L_1-embedding, we have the desired $g : V \to L_1$ with $\beta = 96c$. ■

7 Constrained Embedding

Let $G = (V, E)$ be a planar graph and f be its infinite face. Let $V(f) = \{v_1, v_2, \ldots, v_l = v_1\}$ be the vertices on the cycle of f in clockwise ordering. Suppose that we are given a cut-metric $\mathcal{C} = \{(C_1, w_1), (C_2, w_2), \ldots, (C_m, w_m)\}$ w.r.t $V(f)$ such that each cut $C_i \subseteq V(f)$ corresponds to a contiguous subset of vertices on cycle of f. We say that $\mathcal{C}$ is a **cut-metric w.r.t** f. Suppose we wish to extend $\mathcal{C}$ to V, i.e. we wish to find a cut-metric $\mathcal{D}$ w.r.t V [2] such that $\mathcal{D} = \cup_{i=1}^m \mathcal{D}_i$, $\mathcal{D}_i = \{(D_i^1, w_i^1), (D_i^2, w_i^2), \ldots, (D_i^k, w_i^{k_i})\}$, $\sum_{j=1}^{k_i} w_i^j = w_i$, $C_i \subseteq D_i^j$, $(V(f) \setminus C_i) \cap D_i^j = \emptyset$ for $i = 1, 2, \ldots, m$, and $\delta_{\mathcal{D}}(u, v) \le d_G(u, v)$ for $(u, v) \in E$. We call $\mathcal{D}$ an **extension** of $\mathcal{C}$ to G w.r.t face f. Lemma 4 shows that this is always possible if $\delta_{\mathcal{C}}(u, v) \le d_G(u, v)$ for $u, v \in V(f)$. Note that by definition of $\mathcal{D}$, it follows that $\delta_{\mathcal{D}}(u, v) = \delta_{\mathcal{C}}(u, v)$ for all $u, v \in V(f)$.

Lemma 4 *Let $G = (V, E)$ be a planar graph and f be its infinite face. Let $\mathcal{C}$ be a cut-metric w.r.t face f such that $\delta_{\mathcal{C}}(u, v) \le d_G(u, v)$ for all $u, v \in f$ and C_i corresponds to a contiguous set of vertices on f. Then there exists an extension $\mathcal{D}$ of $\mathcal{C}$ to G w.r.t f.*

Proof. We set up a multicommodity flow instance in the planar dual of G such that all the sink-source pairs are on the infinite face and the cut-condition is satisfied. We use Theorem 1 to find a feasible flow and the fact that circuits in a planar graph correspond to (central) cuts in the (planar) dual to finish the proof. Let $\mathcal{C} = \{(C_1, w_1), (C_2, w_2), \ldots, (C_m, w_m)\}$, $V(f) = \{v_1, v_2, \ldots, v_l = v_1\}$ be the vertices on cycle of f and $e_i = (v_i, v_{i+1}), 1 \le i \le l - 1$ be the edges. Let $G^D = (V^D, E^D)$ be the planar dual of G and f^D be the dual vertex corresponding to the infinite face f. For each edge $e^D \in E^D$, we set the capacity of edge e^D as $c(e^D) := l(e)$. Let $e_1^D, \ldots, e_{l-1}^D$ be the edges incident on f^D in G^D. We split the vertex f^D into $l - 1$ vertices $f_1^D, \ldots, f_{l-1}^D$ such that e_i^D is the only edge incident on vertex f_i^D. Note that each of the vertices f_i^D lie on a single face of G^D.

Each $C_i \in \mathcal{C}$ separates exactly two of the edges on f, say e_j, e_k (since each C_i is a contiguous subset of vertices on f). We set up a multicommodity flow instance in G^D as follows: for each $C_i \in \mathcal{C}$, we introduce a demand edge (f_j^D, f_k^D) with demand value w_i. By Lemma 1, to check that the cut-condition is satisfied for the instance, we only need to verify it for central cuts. In this case, each central cut corresponds to a (v_i, v_j) path in G for some $1 \le i < j \le l - 1$. The total capacity of the supply edges across such a cut is the length of shortest path between (v_i, v_j) in G and total demand across such a cut is equal to $\delta_{\mathcal{C}}(v_i, v_j)$. Since for each $v_i, v_j \in V(f)$, $\delta_{\mathcal{C}}(u, v) \le d_G(u, v)$, the cut-condition is satisfied and we have a feasible flow satisfying all the demands. Each flow path in G^D corresponds to a set of edges in G, which in turn correspond to a cut in G (recall that a circuit in a planar graph corresponds to a cut in its (planar) dual). Let $\mathcal{D}$ be such a set of cuts. Since the total flow through any edge in G^D is at most $c(e^D) = l(e)$, the total weight of cuts in G separating e is at most $l(e)$. Since each $D \in \mathcal{D}$ corresponds to a path in G^D, $D \cap (V(f) \setminus C_i) = \emptyset$. Hence $\mathcal{D}$ is a valid extension of $\mathcal{C}$ and this completes the proof of the lemma. ■

Theorem 12 *Let $G = (V, E)$ be a planar graph with edge-length $l : E \to \mathbf{R}_{\ge 0}$ and F be its set of faces. Furthermore, let S be a α-loose cycle and $f_2 \in F$ be the unique non-geodesic face contained inside $\overline{R(S)}$. Let $G_1 = (V_1, E_1) \subseteq G$ be the graph induced by the vertices and edges in $\overline{R(S)}$ and F_1 be the set of faces of G_1[3]. Let $\mathcal{C}$ be a cut-metric w.r.t S and $\alpha \ge 12\beta$ be a constant such that $d_G(u, v)/\alpha \le \delta_{\mathcal{C}}(u, v) \le d_G(u, v)$ for $u, v \in S$[4]. Then there exists an extension of $\mathcal{C}$ to G_1, say $\mathcal{Z}$, such that for all $u, v \in f \in F_1$, we have $d_G(u, v)/\alpha \le \delta_{\mathcal{Z}}(u, v) \le d_G(u, v)$.*

Proof. As mentioned before, we abuse notation and let S also denote the set of vertices and edges incident on the cycle S. Consider two copies of G_1, say $H_1 = (V_1, E_1)$ and $H_2 = (V_1, E_1)$

[2] i.e. each cut in $\mathcal{D}$ corresponds to a partition of V

[3] note that $S, f_2 \in F_1$ and $\overline{R(S)}$ is the closed region bounded by cycle S

[4] β is the constant from Theorem 11

with length functions l_1, l_2 defined as follows: $l_1(u,v) := l(u,v)$ for all $(u,v) \in S$ and $l(u,v)/\alpha$ otherwise; $l_2(u,v) := 0$ for all $(u,v) \in S$ and $l(u,v)$ otherwise. By definition of a α-loose cycle, it immediately follows that for any $u, v \in S$, $d_{H_1}(u,v) \geq d_G(u,v)$. Lemma 4 shows that the cuts in $\mathcal{C}$ can be extended to H_1 such that $\delta_{\mathcal{C}}(u,v) \leq d_{H_1}(u,v)$ for each $(u,v) \in E_1$. Let this cut-metric be $\mathcal{C}'$. For notational convenience, we create an (equivalent) L_1 embedding $h : V_1 \to L_1$ from $\mathcal{C}'$ by forming a new coordinate for each $C \in \mathcal{C}'$ and setting $h(u) = 0$ if $u \in C$ and w_C otherwise.

Consider the graph H_2 with metric l_2. Since the length of all the edges in S have been set to zero in l_2, we may treat all the vertices on the cycle S as a single node, say v_S. If $d_{H_2}(u,v) \neq d_G(u,v)$ for some $u, v \in V_1$, then it must be the case that $d_{H_2}(u,v) = d_{H_2}(u, v_S) + d_{H_2}(v_S, u)$. Let T_1 be the set of all pairs of vertices (u,v) in H_2 such that the shortest path between u and v in H_2 uses the vertex v_S. We use Theorem 11 to find an embedding $g_1 : V_1 \to L_1$ such that $d_{H_2}(u,v)/\beta \leq ||g_1(u) - g_1(v)|| \leq d_{H_2}(u,v)$ for all $(u,v) \in T_1$.

Let T_2 be the set of all geodesic pairs in G_2 (see Section 5 for the definition of geodesic pairs). In this case, we use Theorem 9 to find an embedding $g_2 : V_1 \to L_1$ such that $d_{H_2}(u,v)/21 \leq ||g_2(u) - g_2(v)|| \leq d_{H_2}(u,v)$ for all $(u,v) \in T_2$. Let T_3 be the set of all pair of vertices (u,v) such that $u, v \in f_2$ i.e. the set of all pairs of points on the non-geodesic face f_2. We use Theorem 3 to find an embedding $g_3 : V_1 \to L_1$ such that $||g_3(u) - g_3(v)|| = d_{H_2}(u,v)$ for all $(u,v) \in T_3$ and $||g_3(u) - g_3(v)|| \leq d_{H_2}(u,v)$ for all $(u,v) \in E_1$.

Let $g = (g_1 + g_2 + g_3)/3$. Since $||g_i(u) - g_i(v)|| \leq d_{H_2}(u,v)$ for $i = 1,2,3$, we have $||g_i(u) - g_i(v)|| \leq d_{H_2}(u,v)$ for $(u,v) \in E_1$. Let T be the set of all pair of vertices u, v which lie on the same face, i.e. $T = \{(u,v) | u, v \in f \text{ for some} f \in F_1\}$. Since f_2 is the only non-geodesic face in $F_1 \setminus S$, for any $u, v \in T$, the shortest path between u, v in H_2 either goes through v_S or (u,v) is a geodesic pair or $u, v \in f_2$. Hence, $T \subseteq T_1 \cup T_2 \cup T_3$ and we have $||g(u) - g(v)|| \geq d_{H_2}(u,v)/\beta_1$ where $\beta_1 = \max\{3 \cdot \beta, 3 \cdot 21, 3 \cdot 1\} = 3\beta$. Let $z := h + \dfrac{\alpha - 1}{\alpha} \cdot g$.

Claim 7 $||z(u) - z(v)|| \leq l(u,v)$ *for* $(u,v) \in E_1$.

Proof. Consider an edge $(u,v) \in S$. Since $l_1(u,v) = l(u,v)$ and $l_2(u,v) = 0$ we have:

$$||z(u) - z(v)|| = ||h(u) - h(v)|| + \frac{\alpha - 1}{\alpha} \cdot ||g(u) - g(v)|| \leq l_1(u,v) + \frac{\alpha - 1}{\alpha} \cdot l_2(u,v) = l(u,v)$$

Now consider an edge $(u,v) \in E_1 \setminus S$. Since $l_1(u,v) = \dfrac{l(u,v)}{\alpha}$ and $l_2(u,v) = l(u,v)$,

$$||z(u) - z(v)|| = ||h(u) - h(v)|| + \frac{\alpha - 1}{\alpha} \cdot ||g(u) - g(v)|| \leq \frac{1}{\alpha} \cdot l(u,v) + \frac{\alpha - 1}{\alpha} \cdot l(u,v) = l(u,v).$$

■

Claim 8 $||z(u) - z(v)|| \geq \dfrac{d_G(u,v)}{\alpha}$ *for* $(u,v) \in T$.

Proof. Let $(u,v) \in T$. We consider two cases depending on whether the shortest u, v path in H_2 uses the vertex v_S. If the shortest u, v path doesn't use the vertex v_S, then we have $d_G(u,v) = d_{H_2}(u,v)$ and,

$$||z(u) - z(v)|| \geq \frac{\alpha - 1}{\alpha} \cdot ||g(u) - g(v)|| \geq \frac{1}{2} \cdot \frac{d_{H_2}(u,v)}{3\beta} \geq \frac{d_G(u,v)}{\alpha}$$

Suppose that the shortest u, v path uses the vertex v_S. Recall that v_S was formed by identifying all the vertices in S as a single vertex. We uncontract v_S and let $u - u_1 - v_1 - v$ be the shortest u, v path where $u_1, v_1 \in S$ are the first and last vertices of S on the path. Note that the shortest u_1-v_1 path may contain some of the vertices in $G \setminus G_1$. Since $\delta_{\mathcal{C}}(u_1, v_1) \geq d_G(u_1, v_1)/\alpha$, we have,

$$||h(u) - h(v)|| \geq \frac{d_G(u_1, v_1)}{\alpha} - d_{H_1}(u_1, u) - d_{H_1}(v_1, v).$$

Using the fact that $\alpha \geq 12\beta = 4\beta_1$, we have:

$$\begin{aligned}
||z(u) - z(v)|| &\geq \frac{d_G(u_1, v_1)}{\alpha} - d_{H_1}(u_1, u) - d_{H_1}(v_1, v) + \frac{\alpha - 1}{\alpha} \cdot \frac{d_{H_2}(u, u_1) + d_{H_2}(v_1, v)}{\beta_1} \\
&\geq \frac{d_G(u_1, v_1) - d_G(u_1, u) - d_G(v_1, v)}{\alpha} + \frac{1}{2} \cdot \frac{d_G(u, u_1) + d_G(v_1, v)}{\beta_1} \\
&\geq \frac{d_G(u, u_1) + d_G(u_1, v_1) + d_G(v_1, v)}{\alpha} \\
&= \frac{d_G(u, v)}{\alpha}. \ \blacksquare
\end{aligned}$$

Using the equivalence of cut-metric and L_1-embedding, we can construct a cut-metric $\mathcal{Z}$ from $z : V_1 \to L_1$ satisfying the conditions of the theorem. ■

8 Putting Everything Together

Theorem 13 *Let $G = (V, E)$ be a planar graph with length function $l : E \to \mathbb{R}_{\geq 0}$, F be its set of faces and $T = \{(u, v) | u, v \in f \in F\}$. Then there exists a $z : V \to L_1$ such that $||z(u) - z(v)|| \leq l(u, v)$ for $(u, v) \in E$ and $||z(u) - z(v)|| \geq d_G(u, v)/c$ for $(u, v) \in T$, where $c = 144\beta^2$.*

Proof. We prove the theorem by using induction on the number of vertices. We first compute a α-good length function l' by setting $\alpha = 12\beta$ in Theorem 7. If there are no non-geodesic faces w.r.t l', then we use Theorem 9 to get an L_1 embedding with distortion at most 21. If there exists a non-geodesic face w.r.t l', we use the decomposition guaranteed by α-good length function and find an innermost non-geodesic α-loose face. Let $G_1 = (V_1, E_1)$ be the graph obtained by removing all the vertices in $I(S_f)$. Since f is non-geodesic w.r.t l', there exists a vertex in $I(S_f)$. Hence, the number of vertices in G_1 is strictly smaller than G and we inductively compute an embedding $z_1 : V \to L_1$ satisfying the conditions of the theorem. Using the equivalence between L_1-embedding and cut-metric, we compute a cut-metric equivalent to z_1 , say $\mathcal{Z}_1$. Let $\mathcal{Z}_1^f$ be the cut-metric induced by $\mathcal{Z}_1$ on the cycle S_f. We use Theorem 12 to compute a cut-metric which extends $\mathcal{Z}_1^f$ to vertices in $G \setminus G_1$, say $\mathcal{Z}_f$. We obtain the final cut-metric by setting $\mathcal{Z} = (\mathcal{Z}_1 \setminus \mathcal{Z}_1^f) \cup \mathcal{Z}_f$. Let $z : V \to L_1$ be the equivalent embedding to $\mathcal{Z}$. By induction hypothesis and statement of Theorem 12, $||z(u) - z(v)|| \leq l'(u, v)$ for $(u, v) \in E$ and $||z(u) - z(v)|| \geq d_{G'}(u, v)/\alpha$ for $(u, v) \in T$, where $d_{G'}$ is the shortest path metric on G w.r.t l'. The statement of the theorem then follows by noting that l' is constructed by reducing the length of edges w.r.t l by a factor of at most α. ■

9 Conclusions

In this paper, we proved a $\mathcal{O}(1)$ flow-cut gap when G is planar and both end points of every demand edge is incident on one of the faces. Although our result does not directly imply any bounds on the (half)-integral flow-cut gap, we believe that it should be possible to exploit the laminar structure of flows in such instances to prove such a bound. Inductive arguments have been used successfully for proving better flow-cut gaps for planar instances, for example series-parallel graphs [3] and k-outer planar graphs [4]. We believe that the techniques developed in this paper could be useful for extending such an approach to a more general setting.

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