On Approximate Nash Equilibria in Network Design*

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Abstract. We study a basic network design game where n self-interested agents, each having individual connectivity requirements, wish to build a network by purchasing links from a given set of edges. A fundamental cost sharing mechanism is Shapley cost sharing that splits the cost of an edge in a fair manner among the agents using the edge. In this paper we investigate if an optimal minimum-cost network represents an attractive, relatively stable state that agents might want to purchase. We resort to the concept of α-approximate Nash equilibria. We prove that for single source games in undirected graphs, any optimal network represents an $H(n)$-approximate Nash equilibrium, where $H(n)$ is the n-th Harmonic number. We show that this bound is tight. We extend the results to cooperative games, where agents may form coalitions, and to weighted games. In both cases we give tight or nearly tight lower and upper bounds on the stability of optimal solutions. Finally we show that in general source-sink games and in directed graphs, minimum-cost networks do not represent good states.

1 Introduction

Today many networks are not built and maintained by a central authority but rather by a large number of economic agents that usually have selfish interests. As a result, game-theoretic approaches for modeling network formation and agent behavior have received considerable research interest over the past years, see e.g. [2,4,5,6,7,8,9,10,11,13,16,19,21].

We study a very basic network design game that has received a lot of attention [1,4,5,6,7,12,14,17]. Let $G = (V, E, c)$ be a graph with a non-negative cost function $c : E \mapsto \mathbb{R}^0_+$. The graph may be directed or undirected. There are n selfish agents, each having to connect a set of terminals in G. A strategy $S_i \subseteq E$ of an agent i is an edge set connecting the desired terminals. Considering all agents, we obtain a combination $\mathcal{S} = (S_1, \ldots, S_n)$ of strategies. Edges used by the agents have to be paid for. A fundamental cost sharing mechanism is Shapley cost sharing, proposed by Anshelevich et al. [4] for network design games. In Shapley cost

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sharing the cost of an edge is split in a fair manner among the agents using the edge. More specifically, in an *unweighted game*, if k agents use an edge e, then each of them pays a share of $c(e)/k$. Thus, given a combination $\mathcal{S}$ of strategies, the cost of an agent i, $1 \leq i \leq n$, is $cost_i(\mathcal{S}) = \sum_{e \in S_i} c(e)/|\{j : e \in S_j\}|$. In a *weighted game*, each agent i has a positive weight w_i and pays a share proportional to its weight. For any edge $e \in S_i$, agent i pays a share of $c(e)w_i/W_e$, where $W_e = \sum_{j:e \in S_j} w_j$ is the total weight of the agents j using e in their strategies. The cost of agent i is $cost_i(\mathcal{S}) = \sum_{e \in S_i} c(e)w_i/W_e$.

Previous work has analyzed stable states in which agents have no incentive to deviate from their strategies. In a standard non-cooperative game a combination $\mathcal{S}$ of strategies forms a *Nash equilibrium* if no agent has a better strategy with a strictly smaller cost if all other agents adhere to their strategies. In cooperative games, where coordination among agents is allowed, one is interested in *strong Nash equilibria* that are resilient to deviations of coalitions of agents [3]. A combination $\mathcal{S}$ of strategies forms a strong Nash equilibrium if there exists no coalition of agents that can jointly change strategy such that every agent of the coalition achieves a strictly smaller cost. There exist two performance measures evaluating Nash equilibria relative to globally optimal solutions. The *price of anarchy* is the maximum ratio of the total cost incurred by any Nash equilibrium to the cost paid by an optimal solution [18]. The *price of stability* is the minimum ratio, i.e. the cost ratio of the best Nash equilibrium relative to the optimum [4]. Anshelevich et al. [4] showed that, for unweighted non-cooperative games, the price of anarchy is n while the price of stability is $H(n)$. Here $H(n) = \sum_{i=1}^{n} 1/i$ is the n-th Harmonic number, which is closely approximated by the natural logarithm, i.e. $\ln(n+1) \leq H(n) \leq \ln n + 1$. For unweighted cooperative games the price of anarchy is $H(n)$ [1,12].

In this paper we study if an optimal solution – which is a minimum-cost network establishing the required connections – represents an attractive, relatively stable state that agents might want to purchase. If the n agents buy an optimal solution, which extra cost does any agent incur compared to a strategy deviation? The motivation for our study is twofold. (1) In Nash equilibria there exist agents that pay a high cost compared to the average agent cost in an optimal solution. In a worst-case equilibrium this cost factor can be as high as n; even in a best-case equilibrium the factor can be $H(n)$. With this information in mind the agents might be interested in purchasing an optimal solution provided that the incentive of a strategy deviation is not too high. (2) The only known protocol to reach a good equilibrium, attaining a price of stability of $H(n)$, relies on optimal solutions. Anshelevich et al. [4] showed that if the agents start in an optimal solution, then a sequence of improving moves converges to a Nash equilibrium whose cost is at most $H(n)$ times the optimum cost. Hence, if agents start in an optimal solution, they might as well consider remaining in this solution provided that the state has favorable properties.

We address the above issues by studying *approximate Nash equilibria* in which the equilibrium constraint is relaxed [5,7]. In a non-cooperative game a combination $\mathcal{S}$ of strategies forms an α-approximate Nash equilibrium, for some

$\alpha \geq 1$, if no agent can improve its cost by a factor of more than α assuming that all the other agents adhere to their strategies. Formally, for no agent i exists a strategy change S'_i such that $cost_i(S_1, \ldots, S'_i, \ldots, S_n) < cost_i(\mathcal{S})/\alpha$. In cooperative games a combination $\mathcal{S}$ of strategies is an α-approximate strong Nash equilibrium if no coalition of agents can change strategy such all agents of the coalition improve their cost by a factor of more than α. More specifically, for no non-empty coalition I of agents exists a strategy change S'_I such that $cost_i(\mathcal{S}'_I, \mathcal{S}_{-I}) < cost_i(\mathcal{S})/\alpha$ holds for all $i \in I$. Here $\mathcal{S}_{-I}$ is the vector of the original strategies of agents $i \notin I$.

We evaluate the quality of optimal solutions for a variety of settings. The main conclusion is that optimal solutions represent good states for single source games in undirected graphs. This holds true for unweighted games, considering both non-cooperative and cooperative agent behavior, as well as for weighted games. On the other hand, in general source-sink games and in directed graphs, optimal solutions do not represent satisfying states.

Previous work. Research on the network design game defined above was initiated by Anshelevich et al. [5]. In this first paper the authors considered general cost sharing schemes that are not restricted to Shapley cost sharing. The cost of an edge may be split in an arbitrary way among agents. Anshelevich et al. [5] considered undirected graphs. First they studied single source games in which each agent i has to connect one terminal t_i to a common source s, $1 \leq i \leq n$. They showed that the cost of an optimal solution can be shared among the agents such that the resulting strategies form a Nash equilibrium. Anshelevich et al. [5] also studied general source-sink games where each agent has to connect an arbitrary set of terminals. Here the cost on an optimal solution can shared such that the agents' strategies form a 3-approximate Nash equilibrium.

In a second paper Anshelevich et al. [4] investigated network design with Shapley cost sharing. They first focused on unweighted games and showed that in directed and undirected graphs the price of anarchy is n while the price of stability is upper bounded by $H(n)$. This upper bound of $H(n)$ is tight for directed graphs. Additionally Anshelevich et al. [4] studied weighted games and showed a lower bound of $\Omega(\max\{n, \log W\})$ on the price of stability, where W is the total weight of all the agents.

Further work on unweighted games was presented by Chekuri et al. [6] and Fiat et al. [14]. Both papers address single source games in undirected graphs. Chekuri et al. [6] showed that the price of anarchy is $O(\sqrt{n}\log^2 n)$ if agents join the game sequentially and perform best-response moves. Fiat et al. [14] proved that the price of stability is $O(\log\log n)$ if each vertex of the graph is the terminal of some agent. Chen and Roughgarden [7] further investigated weighted games in directed graphs. They assume that each agent has to connect a terminal pair (s_i, t_i) and proved that, for any $\alpha = \Omega(\log w_{\max})$, the price of stability of $O(\alpha)$-approximate Nash equilibria is $O((\log W)/\alpha)$. Here $w_{\max}$ is the maximum weight of any agent. In particular, there exists an $O(\log W)$-approximate Nash equilibrium whose cost is within a constant factor of optimal. Cooperative network design games were studied in [3,12]. It shows that the price of anarchy drops to $H(n)$. Finally, approximate pure Nash equilibria for a different class of graphical games were recently studied by Nguyen and Tardos [20].

Our contribution. We evaluate the stability of optimal solutions in network design games with Shapley cost sharing, complementing the existing results for this classical cost sharing mechanism. In Section 2 we present a comprehensive study of unweighted games. We focus mostly on single source games in undirected graphs. First, for non-cooperative games, we prove that any optimal solution represents an $H(n)$-approximate Nash equilibrium. We show that this bound is tight. There exist games in which an optimal solution does not form an α-approximate Nash equilibrium for any $\alpha < H(n)$.

Then, in Section 2, we investigate cooperative games where agents may coordinate their actions. We consider a general scenario where coalitions of up to c agents may be formed, for any $1 \leq c \leq n$. We prove that any optimal solution is a $2c(\ln(n/c) + 2)$-approximate strong Nash equilibrium. The analysis is considerably more involved than that for non-cooperative games. More specifically we show that, given any tree establishing the required connections and any coalition I of agents, there exists one agent $i \in I$ whose cost shares along the path from t_i to the source do not grow too much. For cooperative games, allowing coalitions of up to c agents, we give a nearly matching lower bound: There exist games in which an optimal solution does not represent an α-approximate strong Nash equilibrium for $\alpha < c' \ln(n/c')$, where $c' = \min\{c, \lfloor n/e \rfloor\}$. Hence, for $c < \lfloor n/e \rfloor$ the bound is $\alpha < c \ln(n/c)$; for large coalitions of size $c \geq \lfloor n/e \rfloor$, the bound is $\lfloor n/e \rfloor$ and hence linear in n. This behavior is consistent with our upper bound. Moreover, in Section 2, we consider general source-sink games, in which each agent has to connect an individual set of terminals, as well as directed graphs. In both cases we show negative results, even for non-cooperative games. There are general source-sink games for which an optimal solution is an $\Omega(n)$-approximate Nash equilibrium. In directed graphs the approximation guarantee α can even be unbounded.

In Section 3 we study weighted games. We consider single source games in undirected graphs. We show that in non-cooperative games, any optimal solution is an α-approximate Nash equilibrium, where $\alpha = w_{\max} \sum_{k=0}^{n-1} 1/(w_{\max}+k)$. This bound is again tight. Optimal solutions generally do not form α-approximate Nash equilibria, for $\alpha < w_{\max} \sum_{k=0}^{n-1} 1/(w_{\max} + k)$. The latter expression is upper bounded by $w_{\max}(\ln(W/w_{\max}) + 1)$. Here $w_{\max}$ and W denote again the maximum and total weight of the agents.

Here we finally relate our results to those of Chen and Roughgarden [7] mentioned above for non-cooperative games. In this paper we evaluate the quality of optimal solutions, which are solutions of specific interest, and develop explicit bounds not resorting to O-notation. On the other hand, Chen and Roughgarden develop asymptotic trade-offs. For unweighted games these trade-offs imply the existence of an $O(\log n)$-approximate Nash equilibrium whose cost is within a constant factor of the optimum cost. The protocol starts in an optimal solution and then performs a sequence of improving deviations. Our results show that the protocol can, and indeed will, remain in the optimal solution.

2 Unweighted Games

In this section we study games with classical Shapley cost sharing, i.e. agents have uniform weights. If an edge e is used by k agents, then each agent has to pay a share of $c(e)/k$. We first consider the standard setting where agents are non-cooperating entities. Then we consider the setting where agents cooperate and may form coalitions. For both scenarios we focus on single source games in undirected graphs. More specifically given an undirected graph $G = (V, E, c)$, each agent i, $1 \leq i \leq n$, has to connect a terminal t_i to a common source s, where $t_i, s \in V$. Finally we address general source-sink games and games in directed graphs.

2.1 Non-cooperative Games

We first prove an upper bound on the quality of optimal solutions and then give a matching lower bound.

Theorem 1. *In single source games, any optimal solution represents an $H(n)$-approximate Nash equilibrium.*

Proof. Let E_{OPT} be the edge set used by an optimal solution to establish the required connections. As we study single source games, E_{OPT} forms a tree. Consider the combination $\mathcal{S}$ of strategies in which every agent i connects its terminal t_i to the common source s using only edges of E_{OPT}. Let P_i be the simple path used by agent i and let $cost_i(P_i)$ denote the corresponding cost paid by i within $\mathcal{S}_{OPT}$. We observe that path P_i is unique in E_{OPT}.

Now suppose that an agent i changes strategy and selects a different path Q_i, $Q_i \neq P_i$, in order to connect t_i to s. Let $cost_i(Q_i)$ be the associated cost incurred by agent i when performing this strategy change. We will show

$$cost_i(P_i) \leq H(n) cost_i(Q_i), \tag{1}$$

which establishes the theorem.

Let $v_1, \ldots, v_\ell$, $\ell \geq 2$, be the vertices where P_i and Q_i separate and merge again; Figure 1 shows an example. More specifically, starting at t_i, paths P_i and Q_i first traverse a common subpath $P_i(1) = Q_i(1)$ until reaching vertex v_1 where the two paths separate. Vertex v_1 may be equal to t_i, in which case paths $P_i(1) = Q_i(1)$ are empty. After v_1 path P_i traverses a subpath $P_i(2)$ while Q_i uses a subpath $Q_i(2)$. These subpaths use disjoint edge sets and meet again only at vertex v_2. In general, suppose that P_i and Q_i merge at a vertex v_j, with j being even. Then P_i and Q_i traverse a common subpath $P_i(j+1) = Q_i(j+1)$ until reaching v_{j+1}, where P_i and Q_i separate into disjoint subpaths $P_i(j+2)$ and $Q_i(j+2)$, meeting again at v_{j+2}. Finally, let $P_i(\ell+1) = Q_i(\ell+1)$ be the subpath between v_ℓ and s. For any odd number j, the subpath $P_i(j) = Q_i(j)$ may be empty. For any even j, the subpath $Q_i(j)$ contains at least one edge that does not belong to E_{OPT} because the optimal solution does not contain cycles. Let $Q'_i(j)$, with $Q'_i(j) \subseteq Q_i(j)$, be the set of edges not contained in E_{OPT}.

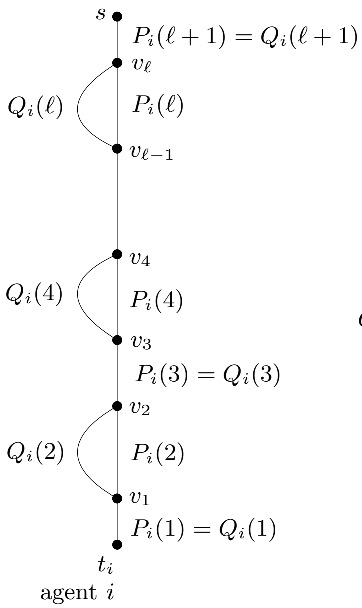

Fig. 1. Paths P_i and Q_i

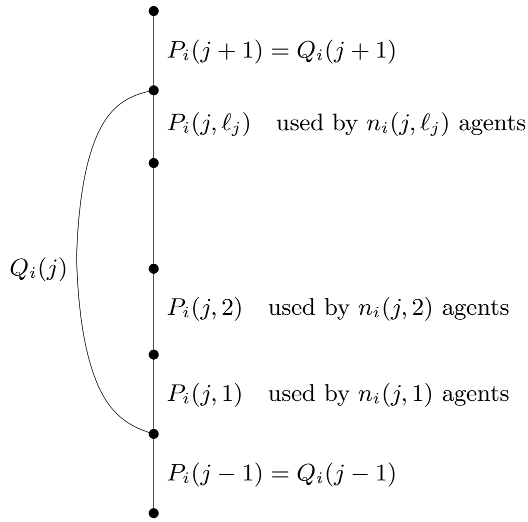

Fig. 2. Subpaths $P_i(j)$ and $Q_i(j)$

Let $cost_i(P_i(j))$ and $cost_i(Q_i(j))$ denote the costs paid by agent i on $P_i(j)$ and $Q_i(j)$, respectively, $1 \leq j \leq \ell + 1$. We have $cost_i(P_i) = \sum_{j=1}^{l+1} cost_i(P_i(j))$ and $cost_i(Q_i) = \sum_{j=1}^{l+1} cost_i(Q_i(j))$, where $cost_i(P_i(j)) = cost_i(Q_i(j))$ for any odd index j. We will prove $cost_i(P_i(j)) \leq H(n) cost_i(Q_i(j))$, for any even j, which implies inequality (1).

Consider a fixed even j and partition $P_i(j)$ into a sequence of maximal subpaths $P_i(j,1), \ldots, P_i(j,\ell_j)$ such that, for any $1 \leq k \leq \ell_j$, the number of agents using a given edge e of $P_i(j,k)$ in E_{OPT} is the same for all the edges of this subpath, cf. Figure 2. Let $n_i(j,k)$ be the number of agents using the edges of $P_i(j,k)$ within E_{OPT}, for any $1 \leq k \leq \ell_j$. Since the subpaths are maximal we have $n_i(j,1) < \ldots < n_i(j,\ell_j)$. As E_{OPT} is a minimum cost tree we have $cost(P_i(j,k)) \leq cost(Q'_i(j))$, where $cost(P_i(j,k))$ and $cost(Q'_i)$ denote the total edge costs of subpath $P_i(j,k)$ and edge set $Q'_i(j)$, respectively, $1 \leq k \leq \ell_j$. If we had $cost(P_i(j,k)) > cost(Q'_i(j))$, then in E_{OPT} we could replace $P_i(j,k)$ by $Q'_i(j)$ obtaining a solution with a strictly smaller cost. The connectivity requirements would still be maintained as agents using $P_i(j,k)$ in E_{OPT} could traverse subpaths $P_i(j,k-1), \ldots, P_i(j,1)$ and $Q_i(j)$ to reach v_j, from where they could again follow their original path to source s. In E_{OPT} agent i pays a share of $cost(P_i(j,k))/n_i(j,k)$ for $P_i(j,k)$, where $cost(P_i(j,k))$ is the total cost of edges on $P_i(j,k)$. Summing over all k and making use of the fact that the sequence $n_i(j,k)$ is strictly increasing with $n_i(j,1) \geq 1$, we obtain that the total cost paid by agent i on $P_i(j)$ is $cost_i(P_i(j)) = \sum_{k=1}^{\ell_j} cost(P_i(j,k))/n_i(j,k) \leq$

$\sum_{k=1}^{\ell_j} cost(P_i(j,k))/k \leq H(n)cost(Q'_i(j))$. As $Q'_i(j)$ is not part of E_{OPT}, agent i has to fully cover its edge cost when traversing $Q_i(j)$ and hence $cost(Q'_i(j)) \leq cost_i(Q_i(j))$. We conclude $cost_i(P_i(j)) \leq H(n)cost_i(Q_i(j))$. □

Theorem 2. *There exists a single source game in which the unique optimal solution does not represent an α-approximate Nash equilibrium, for any $\alpha < H(n)$.*

Proof. Consider a graph consisting of $n+1$ vertices $v_1, \ldots, v_{n+1}$ and n edges $e_i = \{v_i, v_{i+1}\}$, $1 \leq i \leq n$, cf. Figure 3. Associated with each v_i, $1 \leq i \leq n$, is one agent that wishes to connect this vertex to the source $s = v_{n+1}$. Each edge e_i, $1 \leq i \leq n$, has cost 1. Additionally there is an edge $e_0 = \{v_1, v_{n+1}\}$ of cost $1+\epsilon$, where $\epsilon > 0$ is an arbitrarily small constant. The unique optimal solution consists of the set of edges e_i, $1 \leq i \leq n$. In this solution, agent 1 pays a cost of $H(n)$. On the other hand choosing edge e_0, agent 1 incurs a cost of only $1+\epsilon$. □

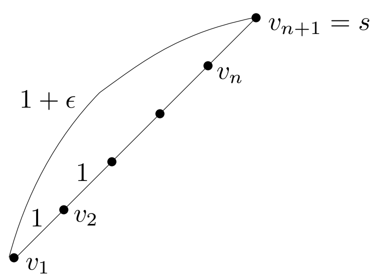

Fig. 3. A single source game without cooperation

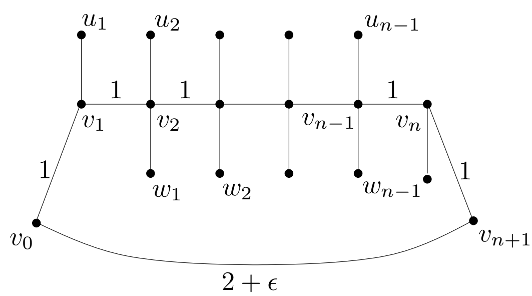

Fig. 4. A source-sink game

2.2 Cooperative Games

We study general cooperative games in which coalitions of up to c agents may be formed, for any $1 \leq c \leq n$.

Theorem 3. *In single source games, any optimal solution represents an α-approximate strong Nash equilibrium, where $\alpha = 2c(\ln(n/c) + 2)$, if coalitions up to size c are allowed.*

In order to establish the theorem, we first prove a property of trees T in which agents connect terminals to the root of T using the edges of the tree. The property holds for any tree T but when using the property in the proof of Theorem 3, T will be an optimal solution of a single source game. So let T be an arbitrary tree with root s. There are n agents, each of which has to connect a terminal of T to s using the edges of T. Let A denote the set of agents i whose terminal t_i is different from s. For any agent $i \in A$, let P_i be the path from t_i to s in T. We partition P_i into maximal subpaths $P_i(1), \ldots, P_i(l_i)$ such that, for any subpath, the number of agents using the edges of the subpath does not vary. Let $n_i(j)$ be the number of agents using the edges of $P_i(j)$, $1 \le j \le l_i$. Define

$$N_i(T) = \sum_{j=1}^{l_i} \frac{1}{n_i(j)},$$

which intuitively is the sum of the fractions paid by agent i on $P_i(1), \ldots, P_i(l_i)$, ignoring edge costs. The following lemma states that in any non-empty coalition $I \subseteq A$ there exists an agent i whose value $N_i(T)$ is logarithmic in $|A|/|I|$.

Lemma 1. *Let T be an arbitrary tree and A be the set of agents whose terminal is not equal to the root of T. For any $I \subseteq A$, $I \neq \emptyset$, there exists an $i \in I$ satisfying $N_i(T) \le 2\ln(\frac{2|A|}{|I|}) + 1$.*

Proof. Due to space limitations we present the main ideas of the proof. A complete proof is given in the full version of the paper.

We prove a slightly stronger bound on $N_i(T)$. Given T and A, a vertex $v \neq s$ in T is called a *branching vertex* if v has at least two children rooting subtrees both of which contain terminals. Let B be the set of branching vertices. We will prove

$$N_i(T) \le 2\ln\left(\frac{|A|+|B|}{|I|}\right) + 1. \qquad (2)$$

The lemma then follows because $|B| < |A|$.

We prove (2) by induction on the number m of edges of T. In the base case we have $m = 1$. The tree consists of a single edge $\{v, s\}$ and A is the set of agents that have to connect v to s. For any $I \subseteq A$, $I \neq \emptyset$, and any $i \in I$ there holds $N_i(T) = 1/|A| \le 1 \le 2\ln(\frac{|A|+|B|}{|I|}) + 1$.

Next consider a tree T with $m > 1$ edges. If there is an agent $i \in I$ whose terminal t_i is equal to a child of s, then the analysis is simple. For this agent we have $N_i(T) \le 1$ and as above we conclude $N_i(T) \le 2\ln((|A| + |B|)/|I|) + 1$ because $|A| \ge |I|$ and $|B| \ge 0$. In the following we assume that, for no agent $i \in I$, the terminal t_i is equal to a child of s. We distinguish two cases depending on whether s has a degree of 1 or a degree larger than 1.

Suppose that s has degree 1. Let $\{s', s\}$ be the edge adjacent to s in T, and let T' be the tree rooted at s'. Let $A' \subseteq A$, be the set of agents i whose terminal t_i is a vertex of T' but not equal to the root s'. There holds $I \subseteq A'$ because we assume that, for no agent of I, the terminal is equal to a child of s. For any $i \in I$, consider the path P_i from t_i to s and the path P_i' from t_i to s'. Obviously P_i consists of P_i' followed by edge $\{s', s\}$. Partition both P_i and P_i' into maximal subpaths $P_i(1), \ldots, P_i(l_i)$ and $P_i'(1), \ldots, P_i'(l_i')$, respectively, such that the edges of a subpath are used by a non-varying number of agents. Let $n_i(j)$ and $n_i'(j)$ be the number of agents using $P_i(j)$ and $P_i'(j)$, respectively. We have $P_i(j) = P_i'(j)$ and hence $n_i(j) = n_i'(j)$, for $j = 1, \ldots, l_i' - 1$. If the number $n_i'(l_i')$ of agents using $P_i'(l_i')$ is equal to the number of agents using edge $\{s', s\}$, then $l_i = l_i'$ and $P_i(l_i)$ consists of $P_i'(l_i)$ followed by $\{s', s\}$. Otherwise $l_i = l_i' + 1$ as well as $P_i(l_i') = P_i'(l_i')$ and $P_i(l_i) = \{s', s\}$.

By induction hypothesis, there exists an agent $i \in I$ satisfying $N_i(T') \leq 2\ln((|A'|+|B'|)/|I|)+1$, where B' is the set of branching vertices in T'. In the following we consider this fixed agent i. If $n_i'(l_i')$ is equal to the number of agents using $\{s', s\}$, then we are done: As argued in the last paragraph $n_i(j) = n_i'(j)$, for $j = 1, \ldots, l_i' - 1$, and $l_i = l_i'$ which implies $n_i(l_i) = n_i'(l_j')$. Hence $N_i(T) = N_i(T') \leq 2\ln((|A'|+|B'|)/|I|)+1 \leq 2\ln((|A|+|B|)/|I|)+1$ because $|A'| \leq |A|$ and $|B'| \leq |B|$.

If on the other hand $n_i'(l_i')$ is not equal to the number of agents using $\{s', s\}$, then (a) there exists an agent in A whose terminal is equal to s' or (b) s' is a branching vertex. In case (a) we have $|A| > |A'|$ and in case (b) we have $|B| > |B'|$. Hence in both cases $|A|+|B| > |A'|+|B'|$. Again $n_i(j) = n_i'(j)$, for $j = 1, \ldots, l_i' - 1$, Since $l_i = l_i' + 1$ and $P_i(l_i') = P_i'(l_i')$, there holds $n_i(l_i') = n_i'(l_i')$ and $n_i(l_i) = 1/|A|$ because edge $\{s', s\}$ is used by all the agents of A. We obtain

$$\begin{aligned} N_i(T) = N_i(T') + \tfrac{1}{|A|} \quad &\leq \quad 2\ln\left(\tfrac{|A'|+|B'|}{|I|}\right) + 1 + \tfrac{2}{2|A|} \\ &\leq 2\left(\ln\left(\tfrac{|A'|+|B'|}{|I|}\right) + \tfrac{1}{|A|+|B|}\right) + 1 \\ &\leq 2\Big(\ln(|A'|+|B'|) + \tfrac{1}{|A'|+|B'|+1} - \ln(|I|)\Big) + 1. \end{aligned}$$

The second inequality holds because $|A| > |B|$ and hence $2|A| > |A|+|B|$. The third inequality follows since $|A|+|B| \geq |A'|+|B'|+1$. For any positive integer K there holds $\ln K + 1/(K+1) \leq \ln(K+1)$. Setting $K = |A'|+|B'|$ we obtain as desired $N_i(T) \leq 2(\ln(|A'|+|B'|+1) - \ln(|I|)) + 1 \leq 2\ln((|A|+|B|)/|I|)+1$.

The analysis of the case that the root s of T has a degree larger than 1 is omitted here. The main idea is to partition T into two trees T_1 and T_2 such that for any agent i whose terminal is in T_j, $j \in \{1, 2\}$, there holds $N_i(T) = N_i(T_j)$. Using induction hypothesis one can then show that there exists an agent i with $N_i(T) \leq 2\ln((|A|+|B|)/|I|)+1$. □

Proof (of Theorem 3). Consider any optimal solution and let E_{OPT} be the corresponding edge set. Moreover, let $\mathcal{S}$ be the combination of strategies in which every agent i connects its terminal t_i to the common source s using only edges of E_{OPT}. In order to prove the theorem we show that if any non-empty coalition I of at most c agents changes strategy, then there exist an agent $i \in I$ whose cost before and after strategy change satisfies $\frac{1}{\alpha} cost_i(\mathcal{S}) \leq cost_i(\mathcal{S}_I, \mathcal{S}_{-I})$, where $\alpha = 2c(\ln(n/c)+2)$.

If a coalition I contains an agent i whose terminal t_i is equal to the source s, then there is nothing to show because for this agent $cost_i(\mathcal{S}) = 0$ and the desired inequality trivially holds. Hence in the following we consider non-empty coalitions I not containing an agent i whose terminal is equal to s.

Let A be the set of agents whose terminal is not equal to s. Consider any non-empty coalition $I \subseteq A$ of size at most c. The optimal solution E_{OPT} forms a tree and hence by Lemma 1 there exists an agent $i \in I$ with $N_i(E_{OPT}) \leq 2\ln(2|A|/|I|)+1$. Fix this agent i. We will prove that if I performs any strategy change, for this agent i the desired inequality holds.

For agent i let P_i be the path connecting t_i to s in E_{OPT}. Let Q_i be the path used by the agent when I changes strategy. As in the proof of Theorem 1 we partition P_i and Q_i into subpaths $P_i(1), \ldots, P_i(l+1)$ and $Q_i(1), \ldots, Q_i(l+1)$ along the vertices $v_1 \ldots, v_l$ where P_i and Q_i separate and merge. Let $cost_i(P(j))$ be the cost incurred by agent i for $P_i(j)$ before strategy change, $1 \leq j \leq l+1$. Similarly, let $cost_i(Q(j))$ be the cost paid by the agent for $Q_i(j)$ after strategy change, $1 \leq j \leq l+1$. We have $P_i(j) = Q_i(j)$, for any odd number j, and hence $\frac{1}{|I|} cost_i(P(j)) \leq cost_i(Q(j))$ because at most $|I|-1$ additional agents of I can join edges of $P_i(j)$ after strategy change. Since $|I| \leq c$ this implies $\frac{1}{\alpha} cost_i(P(j)) \leq cost_i(Q(j))$ for any odd number j. In the following we show that the last inequality also holds for any even j.

For any even j we partition $P_i(j)$ into maximal subpaths $P_i(j,1), \ldots, P_i(j,l_j)$ such that all the edges of a subpath $P_i(j,k)$ are used by the same number $n_i(j,k)$ of agents, $1 \leq k \leq l_j$, considering the time before strategy change. Let $Q_i'(j) \subseteq Q_i(j)$ be the non-empty set of edges not contained in E_{OPT}. For any path π let $cost(\pi)$ be the total cost of the edges of π. There holds $cost(P_i(j,k)) \leq cost(Q_i'(j))$, for any $1 \leq k \leq l_j$ and $cost(Q_i'(j)) \leq cost(Q_i(j))$. Hence

$$cost_i(P_i(j)) = \sum_{k=1}^{l_j} cost(P_i(j,k))/n_i(j,k) \leq cost(Q_i'(j)) \sum_{k=1}^{l_j} 1/n_i(j,k).$$

Consider the partitioning of P_i into maximal subpaths such that the edges of a subpath are used by the same number of agents. Paths $P_i(j,1), \ldots, P_i(j,l_j)$ are a subsequence of this partition and hence $\sum_{k=1}^{l_j} 1/n_i(j,k) \leq N_i(E_{OPT})$. Moreover $cost_i(Q_i'(j)) \geq cost(Q_i'(j))/|I|$ because the cost of the edges of $Q_i'(j)$, which are not part of E_{OPT}, must be fully covered by the coalition I and agent i pays a share of at least $1/|I|$. Thus $cost(P_i(j)) \leq |I| cost_i(Q_i'(j)) N_i(E_{OPT}) \leq |I| \cdot N_i(E_{OPT}) cost_i(Q_i(j))$. By our choice of agent i and Lemma 1, $N_i(E_{OPT}) \leq 2\ln(2|A|/|I|) + 1 \leq 2\ln(2n/|I|) + 1$. We obtain $cost(P_i(j)) \leq |I|(2\ln(2n/|I|) + 1) cost_i(Q_i(j)) \leq c(2\ln(2n/c)+1) cost_i(Q_i(j))$. The last inequality holds because $|I|(2\ln(2n/|I|)+1)$ is increasing in $|I|$. We conclude $cost(P_i(j)) \leq 2c(\ln(n/c) + 2) cost_i(Q_i(j))$ and, as desired, $\frac{1}{\alpha} cost(P_i(j)) \leq cost(Q_i(j))$. □

Theorem 4. *There exists a single source game, allowing coalitions of size up to c, in which the unique optimal solution does not represent an α-approximate strong Nash equilibrium, for any $\alpha < c' \ln(n/c')$, where $c' = \min\{c, \lfloor n/e \rfloor\}$,*

The proof is given in the full version of the paper.

2.3 Source-Sink Games and Directed Graphs

A natural question is if the results of the previous sections can be extended to (a) general source-sink games in which each agent has to connect an individual set of terminals or (b) to directed graphs. Unfortunately, this is not the case. Even for non-cooperative games we can show high lower bounds on the approximation factor α.

Theorem 5. *There exists a general source-sink game in which the unique optimal solution represents an α-approximate Nash equilibrium with $\alpha = \Omega(n)$.*

Proof. Consider the graph depticted in Figure 4, shown at the end of Section 2.1. There are n vertices $v_1, \ldots, v_n$ which are connected by edges $e_i = \{v_i, v_{i+1}\}$, $1 \leq i \leq n-1$. Furthermore, there are vertices $u_1, \ldots, u_{n-1}$, and $w_1, \ldots, w_{n-1}$ with corresponding edges $\{u_i, v_i\}$ and $\{v_{i+1}, w_i\}$, $1 \leq i \leq n-1$. Agent i, $1 \leq i \leq n-1$, has to connect u_i and w_i. There are two additional vertices v_0 and v_{n+1} with associated edges $e_0 = \{v_0, v_1\}$ and $e_n = \{v_n, v_{n+1}\}$. Agent n has to connect terminals v_0 and v_{n+1}. All edges mentioned so far have a cost of 1. Finally, there is an edge $e' = \{v_0, v_{n+1}\}$ of cost $2+\epsilon$. The unique optimal solution purchases all the edges e_i, $0 \leq i \leq n$, in addition to $\{u_i, v_i\}$ and $\{v_{i+1}, w_i\}$, $1 \leq i \leq n-1$. In this solution agent n pays a cost of $2 + (n-1)/2 \geq n/2$, whereas purchasing edge e' incurs a cost of $2+\epsilon$. □

Theorem 6. *For any C, there exist single source games in directed graphs in which an optimal solution does not form a C-approximate Nash equilibrium.*

The proof is given in the full version of the paper.

3 Weighted Games

We scale the agents' weights such that the minimum weight is equal to 1 and hence $w_i \geq 1$, for all the agents. Let $w_{\max} = \max_{1 \leq i \leq n} w_i$ be the maximum weight of any agent. We consider single source games in undirected graphs and extend Theorems 1 and 2. Again we give tight upper and lower bounds on the value of α such that any optimal solution represents an α-approximate Nash equilibrium. The proofs of the following Theorems 7 and 8 are presented in the full version of this paper. The expression $\alpha = w_{\max} \sum_{k=0}^{n-1} 1/(w_{\max}+k)$ is upper bounded by $w_{\max}(\ln(W/w_{\max}) + 1)$.

Theorem 7. *In single source games, any optimal solution represents an α-approximate Nash equilibrium, where $\alpha = w_{\max} \sum_{k=0}^{n-1} 1/(w_{\max}+k)$.*

Theorem 8. *There exists a single source game in which the unique optimal solution does not represent an α-approximate Nash equilibrium, for any $\alpha < w_{\max} \sum_{k=0}^{n-1} 1/(w_{\max}+k)$.*

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