



On the Tree Conjecture for the Network Creation Game

Davide Bilò¹ · Pascal Lenzner² 

Published online: 16 August 2019
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract

Selfish Network Creation focuses on modeling real world networks from a game-theoretic point of view. One of the classic models by Fabrikant et al. (2003) is the network creation game, where agents correspond to nodes in a network which buy incident edges for the price of α per edge to minimize their total distance to all other nodes. The model is well-studied but still has intriguing open problems. The most famous conjectures state that the price of anarchy is constant for all α and that for $\alpha \geq n$ all equilibrium networks are trees. We introduce a novel technique for analyzing stable networks for high edge-price α and employ it to improve on the best known bound for the latter conjecture. In particular we show that for $\alpha > 4n - 13$ all equilibrium networks must be trees, which implies a constant price of anarchy for this range of α . Moreover, we also improve the constant upper bound on the price of anarchy for equilibrium trees.

Keywords Network creation games · Price of anarchy · Tree conjecture · Algorithmic game theory

1 Introduction

Many important networks, e.g. the Internet or (online) social networks, have not been explicitly designed or planned by any central authority. Instead, they have emerged

This article is part of the Topical Collection on *Special Issue on Theoretical Aspects of Computer Science (2018)*

✉ Davide Bilò
davidebilo@uniss.it

Pascal Lenzner
pascal.lenzner@hpi.de

¹ Department of Humanities and Social Sciences, University of Sassari, Sassari, Italy

² Algorithm Engineering Group, Hasso Plattner Institute, University of Potsdam, Potsdam, Germany

via a decentralized process in which independent and selfish agents interact [49]. Thus, such networks cannot be studied via the analysis of a suitable optimization problem like variants of the Minimum Spanning Tree, the Steiner Tree or the Network Design Problems [22, 29, 36].

Nevertheless, modeling and understanding such real world networks is an important challenge for researchers in the fields of Computer Science, Network Science, Economics and Social Sciences. A significant part of this research focuses on assessing the impact of the lack of central coordination and the agents' selfish behavior on the overall network quality. This impact is measured via the *price of anarchy* [41], which is the ratio of the overall network quality of the worst possible outcome of the uncoordinated network creation process and the network quality of the centrally designed optimal network. If this ratio is constant and close to 1, then this implies that naturally grown networks are efficient in the sense that they approximate optimal centrally designed networks well. Moreover, this implies that no coordinating central authority is needed to guide the process to socially favorable outcomes. If, on the other hand, the price of anarchy is high, i.e., non-constant in the number of nodes of the network, then this is clear sign that a coordinating mechanism is needed.

Clearly, if there is no or little coordination among the egoistic agents, then it cannot be expected that the obtained networks optimize the overall network quality. The reason for this is that each agent aims to improve the network quality for herself and is mostly agnostic to the overall network quality. However, empirical observations, e.g. the famous small-world phenomenon [39, 50], low diameter and average distances of real-world networks [10], suggest that selfishly built networks are indeed very efficient in terms of the overall cost and of the individually perceived service quality of the participating agents. Thus, it is a main challenge to justify these observations analytically.

A very promising approach towards this end is to model the creation of a network as a strategic game which yields networks as equilibrium outcomes and then to investigate the overall quality of these networks. For this, a thorough understanding of the structure of such equilibrium networks is the key.

We contribute to this endeavor by providing new insights into the structure of equilibrium networks for one of the classical models of selfish network creation [27]. In this model, agents correspond to nodes in a network and can buy costly links to other nodes to minimize their total distance in the created network. Our insights yield significant progress towards settling the so-called tree conjecture [27, 45] and a simplified proof that the price of anarchy of the game is constant if the edge-price is sufficiently high.

1.1 Model and Definitions

We consider the classical *network creation game* as introduced by Fabrikant et al. [27]. There are n agents V , which correspond to nodes in a network, who want to create a connected network among themselves. Each agent selfishly strives for minimizing her cost for creating network links while maximizing her own connection quality. All edges in the network are undirected and unweighted and agents can create any incident edge for the price of $\alpha > 0$, where α is a fixed parameter of the game.

The strategy $S_u \subseteq V \setminus \{u\}$ of an agent u denotes which edges are bought by this agent, that is, agent u is willing to create (and pay for) all the edges (u, x) , for all $x \in S_u$. Let \mathbf{s} be the n -dimensional vector of the strategies of all agents. The strategy-vector \mathbf{s} induces an undirected network $G(\mathbf{s}) = (V, E(\mathbf{s}))$, where for each edge $(u, v) \in E(\mathbf{s})$ we have $v \in S_u$ or $u \in S_v$. If $v \in S_u$, then we say that agent u is the *owner* of edge (u, v) or that agent u *buys* the edge (u, v) , otherwise, if $u \in S_v$, then agent v owns or buys the edge (u, v) .¹ Since the created networks will heavily depend on α we emphasize this by writing $(G(\mathbf{s}), \alpha)$ instead of $G(\mathbf{s})$. The *cost* of an agent u in the network $(G(\mathbf{s}), \alpha)$ is the sum of her cost for buying edges, called the *creation cost*, and her cost for using the network, called the *distance cost*, which depends on agent u 's distances to all other nodes within the network. The cost of u is defined as

$$\text{cost}(G(\mathbf{s}), \alpha, u) = \alpha|S_u| + \text{distcost}(G(\mathbf{s}), u),$$

where the distance cost is defined as

$$\text{distcost}(G(\mathbf{s}), u) = \begin{cases} \sum_{w \in V} d_{G(\mathbf{s})}(u, w), & \text{if } G(\mathbf{s}) \text{ is connected} \\ \infty, & \text{otherwise.} \end{cases}$$

Here $d_{G(\mathbf{s})}(u, w)$ denotes the length of a shortest path between u and w in the network $G(\mathbf{s})$. We will mostly omit the reference to the strategy vector, since it is clear that a strategy vector directly induces a network and vice versa.

A network $(G(\mathbf{s}), \alpha)$ is in *pure Nash equilibrium* (NE), if no agent can unilaterally change her strategy to strictly decrease her cost. That is, in a NE network no agent can profit by a strategy change if all other agents stick to their strategies. Since in NE networks no agent wants to change the network, we call them *stable*.

The *social cost*, denoted $\text{cost}(G(\mathbf{s}), \alpha)$, of a network $(G(\mathbf{s}), \alpha)$ is the sum of the cost of all agents, that is, $\text{cost}(G(\mathbf{s}), \alpha) = \sum_{u \in V} \text{cost}(G(\mathbf{s}), \alpha, u)$. Let OPT_n be the minimum social cost of a n -agent network and let $\max \text{NE}_n$ be the maximum social cost of any NE network on n agents. The *price of anarchy* (PoA) [41] is the maximum over all n of the ratio $\frac{\max \text{NE}_n}{\text{OPT}_n}$.

Let $G = (V, E)$ be any undirected connected graph with n vertices. A *cut-vertex* x of G is a vertex with the property that G with vertex x removed contains at least two connected components. We say that G is *biconnected* if $n \geq 3$ and G contains no cut-vertex. A *biconnected component* H of G is a maximal induced subgraph of G which is also biconnected. Note that we rule out trivial biconnected components which contain exactly one edge. Thus, there exist at least two vertex-disjoint paths between any pair of vertices x, y in a biconnected component H , which implies that there exists a simple cycle containing x and y .

1.2 Related Work

There is a huge body of literature on selfish network creation. A good source for an overview of classical models is the survey by Jackson [32] and his book [33].

¹No edge can have two owners in any equilibrium network. Hence, we will assume throughout the paper that each edge in $E(\mathbf{s})$ has a unique owner.

Network creation games, as defined above, were introduced by Fabrikant et al. [27]. Their model essentially is a simplified unilateral variant of the connections model by Jackson & Wolinsky [34]. In this model agents have an individual intrinsic value and derive value from each other proportional to their proximity by buying links. For establishing such a connection, both participating agents have to agree and pay the individual price of the link. The authors introduce and use the solution concept of *pairwise stability* for this model. A network is pairwise stable if all edges in the network are profitable for both endpoints and if all non-edges of the network are blocked by at least one of its endpoints since it does not yield positive utility. Later Bala & Goyal [9] drastically simplified this model by introducing a version where edges are unilaterally bought by one endpoint and where agents simply want to reach all other agents in the created network.

For the network creation game Fabrikant et al. [27] gave the first general bound of $\mathcal{O}(\sqrt{\alpha})$ on the PoA and they conjectured that above some constant edge-price α all NE networks are trees. This conjecture, called the *tree conjecture*, is especially interesting since they also showed that tree networks in NE have constant PoA. In particular, they proved that the PoA of stable tree networks is at most 5. Interestingly, the tree conjecture in its general version was later disproved by Albers et al. [1]. However, non-tree NE networks are known only when $\alpha < n$, in particular, for every $\varepsilon > 0$, there exist non-tree NE networks with $\alpha \leq n - \varepsilon$ [45]. It is believed that for $\alpha \geq n$ the tree conjecture might be true. Settling this claim is currently a major open problem and there has been a series of papers which improved bounds concerning the tree conjecture.

First, Albers et al. [1] proved that for $\alpha \geq 12n \log n$ every NE network is a tree. Then, using a technique based on the average degree of the biconnected component, this was significantly improved to $\alpha > 273n$ by Mihalák & Schlegel [48] and even further to $\alpha \geq 65n$ by Mamageishvili et al. [45]. The main idea of this average degree technique is to prove a lower and an upper bound on the average degree of the unique biconnected component in any equilibrium network. The lower bound has the form "for $\alpha > c_1 n$ the average degree is at least c_2 " and the upper bound has the form "for $\alpha > c_3 n$ the average degree is at most $f(\alpha)$ ", where c_1, c_2, c_3 are constants and f is a function which monotonically decreases in α . For large enough α both bounds contradict each other, which proves that equilibrium networks for this α cannot have a biconnected component and thus must be trees. Recently a preprint by Álvarez & Messegué [5] was announced which invokes the average degree technique with a stronger lower bound. This then yields a contradiction already for $\alpha > 17n$. For their stronger lower bound the authors use that in every minimal cycle (we call them "min cycles") of an equilibrium network all agents in the cycle buy exactly one edge of the cycle. This fact has been independently established by us [44] and we also use it. Moreover, the authors of [5] also provide an improved bound on the price of anarchy which is achieved by a better bound on the diameter of the 2-edge-connected component of any NE network. It is shown that the price of anarchy is constant for $\alpha > 9n$. Note, that this result does not imply that all NE networks are trees for $\alpha > 9n$, hence it has no implication on the tree conjecture. In a newer follow-up preprint [6], which appeared after the extended abstract [15] of the present paper was published, Álvarez & Messegué significantly improve on their price of anarchy bounds by employing an

even more refined and elaborate variant of the average degree technique. Their new analysis yields a constant price of anarchy for $\alpha > n(1 + \epsilon)$ for any constant $\epsilon > 0$. Also this newer result does not yield progress on the tree conjecture.

The currently best general upper bound of $2^{\mathcal{O}(\sqrt{\log n})}$ on the PoA is due to Demaine et al. [24] and it is known that the PoA is constant if $\alpha < n^{1-\epsilon}$ for any fixed $\epsilon \geq \frac{1}{\log n}$ [24]. Moreover, Graham et al. [31] proved that for any constant $\alpha > 2$ the PoA is in $1 + o(1)$, i.e., it tends to 1 as n increases.

All in all, the PoA was shown to be constant for almost all α , except for the range between $n^{1-\epsilon}$, for any fixed $\epsilon \geq \frac{1}{\log n}$, and $\alpha < 65n$ (or $\alpha \leq 9n$ which is claimed in [5]). See Fig. 1 for an illustration.

It is widely conjectured that the PoA is constant for all α and settling this open question is a long standing problem in the field. A constant PoA proves that agents create socially close-to-optimal networks even without central coordination. Quite recently, a constant PoA was proven by Chauhan et al. [17] for a version with non-uniform edge prices. In contrast, non-constant lower bounds on the PoA have been proven for local versions of the network creation game by Bilò et al. [11, 13] and Cord-Landwehr & Lenzner [20].

For other variants and aspects of network creation games, we refer to [2–4, 7–9, 12, 14, 16, 18, 19, 21, 23, 25, 26, 28, 30, 32–35, 38, 40, 42, 43, 46, 47].

An extended abstract of this work appeared in [15].

1.3 Our Contribution

In this paper we introduce a new technique for analyzing stable non-tree networks for high edge-price α and use it to improve on the current best lower bound for α for which all stable networks must be trees. In particular, we prove that for $\alpha > 4n - 13$ any stable network must be a tree (see Section 2). This is a significant improvement over the known bound of $\alpha > 65n$ by Mamageishvili et al. [45] and the recently claimed bound of $\alpha > 17n$ by Àlvarez & Messegué [5]. Since the PoA for stable tree networks is constant [27], our bound directly implies a constant PoA for $\alpha > 4n - 13$. Thus, compared to the highly technical proof in [6], our result yields a very simple proof that the PoA is constant for $\alpha > 4n - 13$. Last but not least, in Section 3, we provide a refined analysis of the PoA of stable tree networks and thereby improve on the best known constant upper bound for stable trees by Fabrikant et al. [27]. This result yields an improved PoA for $4n - 13 < \alpha < n^2$.

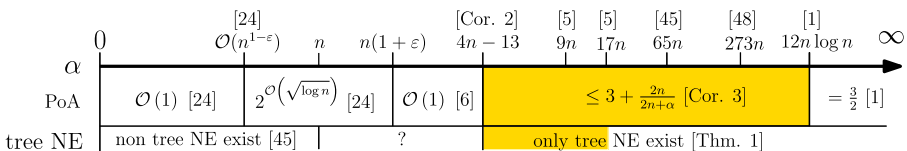


Fig. 1 Illustration of the best known bounds on the PoA and on the tree conjecture. The contribution of this paper is highlighted

Thus, by providing a novel analysis technique we make significant progress towards settling the tree conjecture in network creation games and we provide a simple proof for the PoA being constant for $\alpha > 4n - 13$. See Fig. 1.

Our new technique exploits properties of cycles in stable networks by focusing on *critical pairs*, *strong critical pairs* and *min cycles*. The latter have been introduced in our earlier work [44] and are also used in the preprint by Álvarez & Messegué [5]. However, in contrast to the last attempts for settling the tree conjecture [5, 6, 45, 48], we do not rely on the average degree technique. Instead we propose a more direct and entirely structural approach. Our main new contribution is that we reveal a rich combinatorial structure within any non-tree NE network for $\alpha > 2n - 6$. We show that in this case any agent in any min cycle with a certain minimum length owns exactly one edge of the respective min cycle. With this we prove that the biconnected component of any non-tree NE network cannot be a cycle and thus must contain two nodes which are situated in a particular configuration, which we denote as critical pair. Finally we derive a contradiction to the existence of this specific critical pair for $\alpha > 4n - 13$.

Besides yielding a much simpler proof of the PoA being constant for $\alpha > 4n - 13$, we believe that this approach may be crucial for finally settling the tree conjecture and for further improvements on the PoA. The reason for this is that our proof technique is orthogonal to the average degree technique used in [5, 6, 45, 48] and combining these two approaches seems to be a promising next step. We discuss extensions in that direction in the Conclusion.

2 Improving the Range of α of the Tree Conjecture

In this section we prove our main result, that is, we show that for $\alpha > 4n - 13$, every NE network (G, α) with $n \geq 4$ nodes must be a tree.

We proceed by first establishing properties of cycles in stable networks. Then we introduce the key concepts called critical pairs, strong critical pairs and min cycles. Finally, we provide the last ingredient, which is a critical pair with a specific additional property, and combine all ingredients to obtain the claimed result.

2.1 Properties of Cycles in Stable Networks

We begin by showing that for large values of α , stable networks cannot contain cycles of length either 3 or 4.

Lemma 1 For $\alpha > \frac{n-1}{2}$, no stable network (G, α) contains a cycle of length 3.

Proof Let (G, α) be a stable network for a fixed value of $\alpha > \frac{n-1}{2}$. For the sake of contradiction, assume that G contains a cycle C of length 3. Assume that $V(C) = \{u_0, u_1, u_2\}$ and that C contains the three edges (u_0, u_1) , (u_1, u_2) , and (u_2, u_0) . Let

$$V_i = \{x \in V \mid d_G(u_i, x) < d_G(u_j, x), \forall j \neq i\}.$$

Observe that, for every $i \in \{0, 1, 2\}$ we have $|V_i| \geq 1$, as $u_i \in V_i$. Furthermore, all the V_i 's are pairwise disjoint. Without loss of generality, assume that $|V_0| = \max\{|V_0|, |V_1|, |V_2|\}$ and that u_1 buys the edge (u_1, u_2) . Consider the strategy change in which agent u_1 deletes the edge (u_1, u_2) . The building cost of the agent decreases by α while her distance cost increases by at most $|V_2|$. Since $|V_2| \leq |V_0|$, from $|V_0| + |V_1| + |V_2| \leq n$ we obtain $|V_2| \leq \frac{n-1}{2}$. Since G is stable, $\frac{n-1}{2} - \alpha \geq 0$, i.e., $\alpha \leq \frac{n-1}{2}$, a contradiction. \square

Lemma 2 For $\alpha > n - 2$, no stable network (G, α) contains a cycle of length 4.

Proof Let (G, α) be a stable network for a fixed value of $\alpha > n - 2$. For the sake of contradiction, assume that G contains a cycle C of length 4. Assume that $V(C) = \{u_0, u_1, u_2, u_3\}$ and that C contains the four edges (u_0, u_1) , (u_1, u_2) , (u_2, u_3) , and (u_3, u_0) . For the rest of this proof, we assume that all indices are modulo 4 in order to simplify notation. Let

$$V_i = \{x \in V \mid d_G(u_i, x) < d_G(u_j, x), \forall j \neq i\}.$$

Note that for every $i \in \{0, 1, 2, 3\}$ we have $|V_i| \geq 1$, as $u_i \in V_i$. Let

$$Z_i = \{x \in V \mid d_G(u_i, x) = d_G(u_{i-1}, x) \text{ and } d_G(u_i, x), d_G(u_{i-1}, x) < d_G(u_j, x), \forall j \neq i, i - 1\}.$$

Observe that in the families of the sets V_i and Z_i every pair of sets is pairwise disjoint.

We now rule out the case in which an agent owns two edges of C . Without loss of generality, assume that agent u_0 owns the two edges (u_0, u_1) and (u_0, u_3) . Consider the strategy change in which agent u_0 swaps² the edge (u_0, u_1) with the edge (u_0, u_2) and, at the same time, deletes the edge (u_0, u_3) . The creation cost of agent u_0 decreases by α , while her distance cost increases by $|V_1| + |V_3| - |V_2|$. Since (G, α) is stable, agent u_0 has no incentive in deviating from her current strategy. Therefore, $|V_1| + |V_3| - |V_2| - \alpha \geq 0$, i.e.,

$$\alpha \leq |V_1| + |V_3| - |V_2| \leq n - |V_0| - |V_2| - |V_2| \leq n - 3,$$

where the last but one inequality follows from the pairwise disjointness of all V_i sets, which implies $|V_0| + |V_1| + |V_2| + |V_3| \leq n$. Since, $\alpha > n - 2$, no agent can own two edges of C . Therefore, to prove the claim, we need to show that no agent can own a single edge of C .

Without loss of generality, assume that for every $i \in \{0, 1, 2, 3\}$ agent u_i owns the edge (u_i, u_{i+1}) . Moreover, again without loss of generality, assume that

$$|V_1| + |Z_2| = \min_{0 \leq i \leq 3} \{|V_i| + |Z_{i+1}|\}.$$

²A swap of edge (a, b) to edge (a, c) by agent a who owns edge (a, b) consists of deleting edge (a, b) and buying edge (a, c) .

Since

$$\sum_{0 \leq i \leq 3} (|V_i| + |Z_{i+1}|) \leq n,$$

we have that $|V_1| + |Z_2| \leq \frac{n}{4}$.

Consider the strategy change in which agent u_0 deletes the edge (u_0, u_1) . The creation cost of agent u_0 decreases by α , while her distance cost increases by $2|V_1| + |Z_2| \leq \frac{n}{2}$.

Since (G, α) is stable, agent u_0 has no incentive to deviate from her current strategy. Therefore, $\frac{n}{2} - \alpha \geq 0$, i.e., $\alpha \leq \frac{n}{2} \leq n - 2$, when $n \geq 4$. We have obtained a contradiction. \square

Definition 1 (Directed Cycle) Let C be a cycle of (G, α) of length k . We say that C is *directed* if there is an ordering u_0, \dots, u_{k-1} of its k vertices such that, for every $i = 0, \dots, k - 1$, $(u_i, u_{(i+1) \bmod k})$ is an edge of C which is bought by agent u_i .

We now show that if α is large enough, then directed cycles cannot be contained in a stable network as a biconnected component.

Lemma 3 For $\alpha > n - 2$, no stable network (G, α) with $n \geq 6$ vertices contains a biconnected component which is also a directed cycle.

Proof Let (G, α) be a stable network for a fixed value of $\alpha > n - 2$. Let H be a biconnected component of G . For the sake of contradiction, assume that H is a directed cycle of length k . We can apply Lemma 2 to exclude the case in which $k = 4$. Similarly, since $\alpha > n - 2 \geq \frac{n-1}{2}$ for every $n \geq 3$, we can use Lemma 1 to exclude the case in which $k = 3$.

Let u_0, \dots, u_{k-1} be the k vertices of H and, w.l.o.g., assume that every agent u_i is buying an edge towards agent $u_{(i+1) \bmod k}$. To simplify notation, in the rest of this proof we assume that all indices are modulo k . Let

$$V_i = \{x \in V \mid d_G(u_i, x) < d_G(u_j, x), \forall j \neq i\}.$$

Observe that V_i is a partition of V . We divide the proof into two cases.

The first case occurs when H is a cycle of length $k \geq 6$. Without loss of generality, assume that

$$|V_2| = \max_{0 \leq i \leq k-1} |V_i|.$$

In this case, consider the strategy change of agent u_0 when she swaps the edge (u_0, u_1) with the edge (u_0, u_2) . The distance cost of agent u_0 increases by

$$|V_1| - |V_2| - |V_3| \leq -1.$$

Thus, agent u_0 has an improving strategy, a contradiction.

The second and last case occurs when H is a cycle of length $k = 5$. If $|V_i| \neq |V_j|$ for some $i, j \in \{0, 1, 2, 3, 4\}$, then there exists an $i \in \{0, 1, 2, 3, 4\}$ such that $|V_i| < |V_{i+1}|$. Without loss of generality, let $|V_1| < |V_2|$. The distance cost of agent u_0 when she swaps the edge (u_0, u_1) with the edge (u_0, u_2) increases by $|V_1| - |V_2| \leq -1$.

Thus agent u_0 has an improving strategy, a contradiction. If $|V_0| = |V_1| = |V_2| = |V_3| = |V_4|$, then the increase in the overall cost incurred by agent u_0 when she deletes the edge (u_0, u_1) would be equal to

$$3|V_1| + |V_2| - \alpha = \frac{4}{5}n - \alpha.$$

Since G is stable and n is a multiple of 5, $\frac{4}{5}n - \alpha \geq 0$, i.e., $\alpha \leq \frac{4}{5}n \leq n - 2$, for every $n \geq 10$, a contradiction. \square

2.2 Critical Pairs

The next definition introduces the concept of a (strong) critical pair. As we will see, (strong) critical pairs are the first key ingredient for our analysis. Essentially, we will show that stable networks cannot have critical pairs, if α is large enough.

Definition 2 (Critical Pair) Let (G, α) be a non-tree network and let H be a biconnected component of G . We say that $\langle v, u \rangle$ is a *critical pair* if all of the following five properties hold:

1. Agent $v \in V(H)$ buys two distinct non-bridge edges, say (v, v_1) and (v, v_2) , with $v_1, v_2 \in V(H)$;
2. Agent $u \in V(H)$, with $u \neq v$ buys at least one edge (u, u') with $u' \in V(H)$ and $u' \neq v$;
3. $d_G(v, u) \geq 2$;
4. there is a shortest path between v and u in G which uses the edge (v, v_1) ;
5. there is a shortest path between v and u' in G which does not use the edge (u, u') .

The critical pair $\langle v, u \rangle$ is *strong* if there is a shortest path between u and v_2 which does not use the edge (v, v_2) . See Fig. 2 for an illustration.

In the rest of this section, when we say that two vertices v and u of G form a critical pair, we will denote by v_1, v_2 , and u' the vertices corresponding to the critical

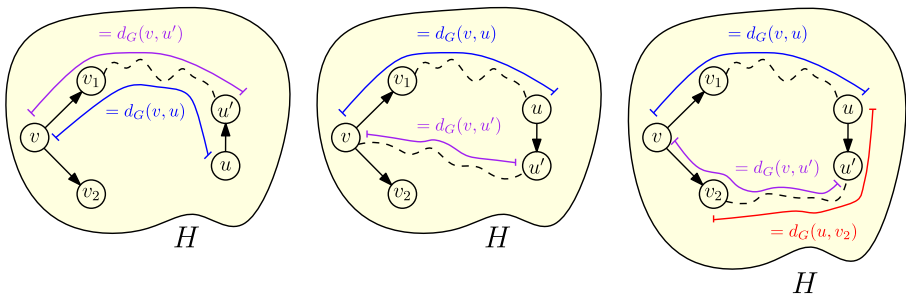


Fig. 2 Illustrations of a critical pair $\langle v, u \rangle$. Edge-ownership is depicted by directing edges away from their owner. **Left:** Edge (u, u') belongs to a shortest path tree T rooted at v and u' is the parent of u in T . **Middle:** Edge (u, u') does not belong to any shortest path tree T rooted at v . Note that in this case (v, v_2) can also be on the shortest path from v to u' . **Right:** Illustration of a strong critical pair $\langle v, u \rangle$

pair $\langle v, u \rangle$ that satisfy all the conditions given in Definition 2. We can observe the following.

Observation 4 *If $\langle v, u \rangle$ is a critical pair of a network (G, α) , then there exists a shortest path tree T of G rooted at v , where either the edge (u, u') is not an edge of T or u' is the parent of u in T .*

Observation 5 *If $\langle v, u \rangle$ is a critical pair, then for every shortest path tree T of (G, α) rooted at u , either the edge (v, v_1) is not an edge of T or v_1 is the parent of v in T . Furthermore, if $\langle v, u \rangle$ is a strong critical pair, then there is a shortest path tree of G rooted at u such that the edge (v, v_2) is not contained in the shortest path tree.*

The next technical lemma provides useful bounds on the distance cost of the nodes involved in a critical pair.

Lemma 6 *Let (G, α) be a stable network and let a, b be two distinct vertices of G such that a buys an edge (a, a') , with $a' \neq b$. If $d_G(a, b) \geq 2$ and there exists a shortest path tree T of G rooted at b such that either (a, a') is not an edge of T or a' is the parent of a in T , then*

$$\text{distcost}(G, a) \leq \text{distcost}(G, b) + n - 3.$$

Furthermore, if a is buying also the edge (a, a'') , with $a'' \neq a'$, $a'' \neq b$, and (a, a'') is not an edge of T , then

$$\text{distcost}(G, a) \leq \text{distcost}(G, b) + n - 3 - \alpha.$$

Proof Consider the strategy change in which agent a swaps the edge (a, a') with the edge (a, b) and deletes any other edge she owns and which is not contained in T , if any. Let T' be a shortest path tree rooted at b of the graph obtained after the swap. Observe that $d_{T'}(b, x) \leq d_G(b, x)$, for every $x \in V$. Furthermore, as $d_G(a, b) \geq 2$, while $d_{T'}(a, b) = 1$, we have $d_{T'}(a, b) \leq d_G(a, b) - 1$. Therefore,

$$\sum_{x \in V} d_{T'}(b, x) \leq \text{distcost}(G, b) - 1.$$

Moreover, the distance from a to every $x \neq a$ is at most $1 + d_{T'}(b, x)$. Finally, the distance from a to herself, which is clearly 0, is exactly 1 less than the distance from b to a in T' . Therefore the distance cost of a in T' is less than or equal to

$$\text{distcost}(G, b) - 1 + (n - 1) - 1 = \text{distcost}(G, b) + n - 3.$$

If besides performing the mentioned swap agent a additionally saves at least α in cost by deleting at least one additional edge which is not in T , then

$$\text{distcost}(G, a) \leq \text{distcost}(G, b) + n - 3 - \alpha.$$

This is true since G is stable, which implies that the overall cost of a in G cannot be larger than the overall cost of a after the strategy change. \square

Now we employ Lemma 6 to prove the structural property that stable networks cannot contain strong critical pairs if α is large enough.

Lemma 7 *For $\alpha > 2n - 6$, no stable network (G, α) contains a strong critical pair.*

Proof Let (G, α) be a non-tree stable network for a fixed value of $\alpha > 2n - 6$ and, for the sake of contradiction, let (v, u) be a strong critical pair. Using Observation 4 together with Lemma 6 (where $a = u, a' = u'$, and $b = v$), we have that

$$\text{distcost}(G, u) \leq \text{distcost}(G, v) + n - 3.$$

Furthermore, using Observation 5 together with Lemma 6 (where $a = v, a' = v_1, a'' = v_2$, and $b = u$), we have that

$$\text{distcost}(G, v) \leq \text{distcost}(G, u) + n - 3 - \alpha.$$

By summing up both the left-hand and the right-hand side of the two inequalities we obtain $0 \leq 2n - 6 - \alpha$, i.e., $\alpha \leq 2n - 6$, a contradiction. □

2.3 Min Cycles

We now introduce the second key ingredient for our analysis: min cycles.

Definition 3 (Min Cycle) Let (G, α) be a non-tree network and let C be a cycle in G . We say that C is a min cycle if, for every two vertices $x, x' \in V(C)$, $d_C(x, x') = d_G(x, x')$.

First, we show that every edge of every biconnected graph is contained in some min cycle. This was also proven in [44] and [5].

Lemma 8 *Let H be a biconnected graph. Then, for every edge e of H , there is a min cycle that contains the edge e .*

Proof Since H is biconnected, there exists at least one cycle containing the edge e . Among all the cycles in H that contain the edge e , let C be a cycle of minimum length. We claim that C is a min cycle. For the sake of contradiction, assume that C is not a min cycle. This implies that there are two vertices $x, y \in V(C)$ such that $d_H(x, y) < d_C(x, y)$. Among all pairs $x, y \in V(C)$ of vertices such that $d_H(x, y) < d_C(x, y)$, let x', y' be the one that minimizes the value $d_H(x', y')$ (ties are broken arbitrarily). Let Π be a shortest path between x' and y' in G . By the choice of x' and y' , Π is edge disjoint from C . Let P and P' be the two edge-disjoint paths between x' and y' in C and, w.l.o.g., assume that e is contained in P . (See Fig. 3 for an illustration.) Let ℓ and ℓ' be the length of P and P' , respectively. Clearly, the length of C is equal to $\ell + \ell'$. Since $d_C(x', y') \leq \ell'$, we obtain $d_H(x', y') < \ell'$. Therefore, the cycle obtained by concatenating P and Π has a length equal to $\ell + d_H(x', y') < \ell + \ell'$, and therefore, it is strictly shorter than C , a contradiction. □

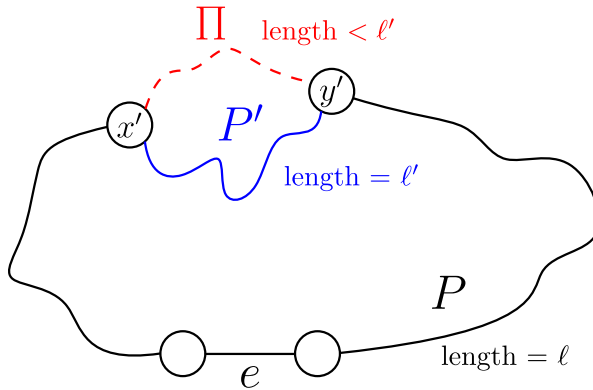


Fig. 3 The cycle C containing edge e and the paths P , P' and Π

Now we proceed with showing that stable networks contain only min cycles which are directed and not too short. For this, we employ our knowledge about strong critical pairs.

Lemma 9 For $\alpha > 2n - 6$, every min cycle of a non-tree stable network (G, α) with $n \geq 4$ vertices is directed and has a length of at least 5.

Proof Let (G, α) be a non-tree stable network for a fixed $\alpha > 2n - 6$ and let C be a min cycle of G . Since $2n - 6 \geq \frac{n-1}{2}$ for every $n \geq 4$, using Lemma 1, we have that C cannot be a cycle of length equal to 3. Furthermore, since $2n - 6 \geq n - 2$ for every $n \geq 4$, using Lemma 2, we have that C also cannot be a cycle of length equal to 4. Therefore, C is a cycle of length greater than or equal to 5.

For the sake of contradiction, assume that C is not directed. This means that C contains an agent, say v , that is buying both her incident edges in C . We prove the contradiction thanks to Lemma 7, by showing that C contains a strong critical pair.

If C is an odd-length cycle, then v has two distinct antipodal vertices $u, u' \in V(C)$ which are adjacent in C .³ Without loss of generality, assume that u is buying the edge towards u' . Clearly, $d_G(v, u) \geq 2$. Furthermore, since C is a min cycle, it is easy to check that $\langle v, u \rangle$ is a strong critical pair.

If C is an even-length cycle, then let $u \in V(C)$ be the (unique) antipodal vertex of v and let u' be a vertex that is adjacent to u in C . Observe that $d_G(v, u), d_G(v, u') \geq 2$. Again using the fact that C is a min cycle, we have the following:

- If u is buying the edge towards u' , then $\langle v, u \rangle$ is a strong critical pair.
- If u' is buying the edge towards u , then $\langle v, u' \rangle$ is a strong critical pair.

In both cases, we have proved that C contains a strong critical pair. □

³In a cycle of length ℓ , two vertices of the cycle are *antipodal* if their distance is $\lfloor \ell/2 \rfloor$.

Let (G, α) be a non-tree stable network with $n \geq 6$ vertices for a fixed $\alpha > 2n - 6$ and let H be a biconnected component of G . Since $2n - 6 \geq n - 2$ for every $n \geq 4$, Lemma 3 implies that H cannot be a directed cycle. At the same time, if H is a cycle, then it is also a min cycle and therefore, Lemma 9 implies that H must be directed, which contradicts Lemma 3. Therefore, we have proved the following.

Corollary 1 *For $\alpha > 2n - 6$, no non-tree stable network (G, α) with $n \geq 6$ vertices contains a cycle as one of its biconnected components.*

2.4 Combining the Ingredients

Towards our main result, we start with proving that every stable network must contain a critical pair which satisfies an interesting structural property. This lemma is the third and last ingredient that is used in our analysis.

Lemma 10 *For $\alpha > 2n - 6$, every non-tree stable network (G, α) with $n \geq 6$ vertices contains a critical pair $\langle v, u \rangle$. Furthermore, there exists a path P between v and v_2 in G such that (a) the length of P is at most $2d_G(u, v)$ and (b) P uses none of the edges (v, v_1) and (v, v_2) . Here, vertices v_1 and v_2 are defined as in Definition 2.*

Proof Let (G, α) be a network of $n \geq 6$ vertices which is stable for a fixed $\alpha > 2n - 6$, and let H be any biconnected component of G . By Corollary 1, we have that H cannot be a cycle. As a consequence, H contains at least $|V(H)| + 1$ edges and, therefore, it has a vertex, say v , that buys at least two edges of H .

Let v_1 and v_2 be the two distinct vertices of H such that v buys the edges (v, v_1) and (v, v_2) . Let C_i be the min cycle that contains the edge (v, v_i) , whose existence is guaranteed by Lemma 8. Lemma 9 implies that C_i is a directed cycle of length greater than or equal to 5. Therefore, since (v, v_1) is an edge of C_1 bought by agent v , C_1 cannot contain the edge (v, v_2) , which is also bought by v . Similarly, since (v, v_2) is an edge of C_2 bought by agent v , C_2 cannot contain the edge (v, v_1) , which is also bought by v .

Let T be a shortest path tree rooted at v which gives priority to the shortest paths using the edges (v, v_1) or (v, v_2) . More precisely, for every vertex x , if there is a shortest path from v to x containing the edge (v, v_1) , then x is a descendant of v_1 in T . Furthermore, if no shortest path from v to x contains the edge (v, v_1) , but there is a shortest path from v to x containing the edge (v, v_2) , then x is a descendant of v_2 in T .

Consider the directed version of C_i in which each edge is directed from their owner agent towards the other end vertex. Let u_i be, among the vertices of C_i which are also descendants of v_i in T , the one which is in maximum distance from v with respect to the directed version of C_i . Finally, let (u_i, u'_i) be the edge of C_i which is bought by agent u_i . Clearly, u'_i is not a descendant of v_i in T . Therefore, by construction of T , $d_G(v, u'_i) \leq d_G(v, u_i)$, otherwise u'_i would have been a descendant of v_i in T , or there would have been a min cycle containing both edges (v, v_1) and (v, v_2) (which are both bought by agent v), thus contradicting Lemma 9.

Without loss of generality, assume that $d_G(v, u_2) \leq d_G(v, u_1)$. Let $u = u_1$ and $u' = u'_1$. We show that $\langle v, u \rangle$ is a critical pair. By Lemma 9, C_1 is a cycle of length $k \geq 5$. As C_1 is a min cycle, $k = d_G(v, u) + 1 + d_G(v, u')$. Moreover, since $d_G(v, u') \leq d_G(v, u)$, we have that $d_G(v, u) \geq \frac{k-1}{2} \geq 2$. Therefore $u' \neq v$. Next, the shortest path in T between v and u uses the edge (v, v_1) which is owned by agent v . Furthermore, the shortest path in T between v and u' does not use the edge (u, u') . Therefore, $\langle v, u \rangle$ is a critical pair.

Now, consider the path P which is obtained from C_2 by removing the edge (v, v_2) . Recalling that C_2 does not contain the edge (v, v_1) , it follows that P is a path between v and v_2 which uses none of the two edges (v, v_1) and (v, v_2) . Therefore, recalling that $d_G(v, u'_2) \leq d_G(v, u_2)$, the overall length of P is less than or equal to

$$\begin{aligned} d_G(v, u'_2) + 1 + d_G(v_2, u_2) &\leq d_G(v, u_2) + 1 + d_G(v, u_2) - 1 \\ &\leq 2d_G(v, u_1) \\ &= 2d_G(v, u). \end{aligned}$$

□

Finally, we prove our main result. For this and in the rest of the paper, given a vertex x of a network (G, α) and a subset U of vertices of G , we denote by $d_G(x, U)$ the sum of the shortest path lengths from x to all nodes in U , i.e.,

$$d_G(x, U) = \sum_{x' \in U} d_G(x, x').$$

Theorem 1 *For $\alpha > 4n - 13$, every stable network (G, α) with $n \geq 4$ vertices is a tree.*

Proof First of all, it is easy to check that for $\alpha > 3$ every stable network with $n = 4$ vertices is a tree. Moreover, the same holds true for $n = 5$ and for $\alpha > 7$.

Let $\alpha > 4n - 13$ be a fixed value and let (G, α) be a stable network with $n \geq 6$ vertices. Since $4n - 13 \geq 2n - 6$, for every $n \geq 4$, we have that if (G, α) is not a tree, then, by Lemma 10, it contains a critical pair $\langle v, u \rangle$ satisfying the conditions stated in Lemma 10. Moreover, Lemma 7 implies that $\langle v, u \rangle$ cannot be a strong critical pair. As a consequence, every shortest path from u to v_2 uses the edge (v, v_2) . Since $\langle v, u \rangle$ is a critical pair, this implies that there is a shortest path from u to v_2 which uses both the edges (v_1, v) and (v, v_2) . To finish our proof, we show that this contradicts the assumed stability of (G, α) . This implies that (G, α) must be a tree.

Let $T(u)$ be a shortest path tree of G rooted at u having v_1 as the parent of v and v as the parent of v_2 . Observe that, by definition of a critical pair, there is a shortest path between v and u containing the edge (v, v_1) . Therefore, $T(u)$ is well defined. Furthermore, let X be the set of vertices which are descendants of v_2 in $T(u)$. Note that since $v_2 \in X$, we have $|X| \geq 1$.

Since $\langle v, u \rangle$ is a critical pair, thanks to Observation 4, we can use Lemma 6 (where $a = u, a' = u',$ and $b = v$) to obtain

$$\text{distcost}(G, u) \leq \text{distcost}(G, v) + n - 3. \tag{1}$$

Furthermore, observe that

$$\begin{aligned} \text{distcost}(G, u) &= \sum_{x \in X} (d_G(u, v) + d_G(v, x)) + d_G(u, V \setminus X) \\ &= d_G(u, v)|X| + d_G(v, X) + d_G(u, V \setminus X). \end{aligned} \tag{2}$$

Therefore, by substituting $\text{distcost}(G, u)$ in (1) with (2) we obtain the following

$$d_G(u, v)|X| + d_G(v, X) + d_G(u, V \setminus X) \leq \text{distcost}(G, v) + n - 3. \tag{3}$$

Let $T'(u)$ be the tree obtained from $T(u)$ by the the swap of the edge (v, v_1) with the edge (v, u) . The distance cost incurred by agent v if she swaps the edge (v, v_1) with the edge (v, u) is at most

$$\begin{aligned} d_{T'(u)}(v, V) &= d_{T'(u)}(v, X) + d_{T'(u)}(v, V \setminus X) \\ &= d_{T(u)}(v, X) + d_{T'(u)}(v, V \setminus X) \\ &\leq d_{T(u)}(v, X) + \sum_{x \in V \setminus (X \cup \{v\})} (1 + d_{T(u)}(u, x)) \\ &\leq d_G(v, X) + \sum_{x \in V \setminus X} (1 + d_G(u, x)) - 2 \\ &= d_G(v, X) + n - |X| + d_G(u, V \setminus X) - 2. \end{aligned}$$

Since (G, α) is stable, agent v cannot decrease her distance cost by swapping any of the edges she owns. Therefore, we obtain

$$\text{distcost}(G, v) \leq d_G(v, X) + n - |X| + d_G(u, V \setminus X) - 2. \tag{4}$$

By summing both the left-hand and the right-hand sides of the two inequalities (3) to (4) and simplifying we obtain

$$d_G(u, v)|X| \leq 2n - 5 - |X|. \tag{5}$$

Consider the network (G', α) induced by the strategy vector in which agent v deviates from her current strategy by swapping the edge (v, v_1) with the edge (v, u) and, at the same time, by deleting the edge (v, v_2) . By Lemma 10, there exists a path P between v and v_2 in G , of length at most $2d_G(u, v)$, such that P uses none of the edges (v, v_1) and (v, v_2) . As a consequence, using both (1) and (5) in the second to

last inequality of the following chain, the distance cost of v with respect to (G', α) is upper bounded by

$$\begin{aligned}
 d_{G'}(v, V) &\leq \sum_{x \in X} (2d_G(u, v) + d_G(v_2, x)) + \sum_{x \in V \setminus (X \cup \{v\})} (1 + d_G(u, x)) \\
 &\leq 2d_G(u, v)|X| + d_G(v_2, X) + n - |X| + d_G(u, V \setminus X) - 2 \\
 &\leq 2d_G(u, v)|X| + d_G(v, X) - |X| + n - |X| + d_G(u, V \setminus X) - 2 \\
 &= 2d_G(u, v)|X| + d_G(u, X) - d_G(u, v)|X| + n - 2|X| \\
 &\quad + d_G(u, V \setminus X) - 2 \\
 &= d_G(u, v)|X| + n - 2|X| + \text{distcost}(G, u) - 2 \\
 &\leq 2n - 5 - |X| + n - 2|X| + \text{distcost}(G, v) + n - 3 - 2 \\
 &= \text{distcost}(G, v) + 4n - 10 - 3|X| \\
 &\leq \text{distcost}(G, v) + 4n - 13.
 \end{aligned}$$

By her strategy change, agent v will save α in edge cost and her distance cost will increase by at most $4n - 13$. Thus, if $\alpha > 4n - 13$, then this yields a strict cost decrease for agent v which contradicts the stability of (G, α) . \square

With the results from Fabrikant et al. [27] Theorem 1 yields:

Corollary 2 For $\alpha > 4n - 13$ the PoA is at most 5.

In Section 3 we improve the upper bound of 5 on the PoA for stable tree networks from Fabrikant et al. [27]. With this, we establish the following:

Corollary 3 For every $\alpha > 4n - 13$ the PoA is at most $3 + \frac{2n}{2n+\alpha}$.

3 Improved Price of Anarchy for Stable Tree Networks

In this section we show a better bound on the PoA of stable tree networks. To prove the bound, we need to introduce some new notation first. Let T be a tree on n vertices and, for a vertex v of T , let $T - v$ be the forest obtained by removing vertex v together with all its incident edges from T . We say that v is a *centroid* of T if every tree in $T - v$ has at most $n/2$ vertices. It is well known that every tree has at least one centroid vertex [37].

Lemma 11 Let (T, α) be a stable tree network rooted at a centroid c of T , and let $u, v \in V(T)$, with $u, v \neq c$, be two vertices such that u buys the edge towards v in T . Then $d_T(c, u) < d_T(c, v)$, i.e., u is the parent of v in T . Furthermore, if \bar{T} denotes the subtree of T rooted at v , then v is a centroid of \bar{T} .

Proof We show that $d_T(c, u) < d_T(c, v)$ by proving that if $d_T(c, u) > d_T(c, v)$, then (T, α) is not stable. So, assume that $d_T(c, u) > d_T(c, v)$. Consider the forest

obtained from T after the removal of the edge (u, v) and let T_v be the tree of the forest that contains vertex v . Since v is closer to c than u in T and u is not a vertex of T_v , it follows that c is a vertex of T_v . See Fig. 4.

Since u and v are both in the same tree, say T'' , of the forest $T - c$, it follows that the tree induced by all the vertices of T which are not contained in T'' , say T' , is entirely contained in T_v and has $n' \geq n/2$ vertices, as c is a centroid of T . Now consider the strategy change in which agent u swaps the edge (u, v) with the edge (u, c) . Observe that after the swap of the edge (u, v) with the edge (u, c) , the distance from u to each vertex in T' decreases by $d_T(v, c)$, while the distance from u to each vertex in $T_v \setminus T'$ increases by at most $d_T(v, c)$. Therefore, if we denote by n_v the number of vertices of T_v , then the cost of agent u increases by at most

$$\begin{aligned} d_T(v, c)(n_v - n') - d_T(v, c)n' &= d_T(v, c)(n_v - 2n') \\ &\leq d_T(v, c)(n - 1 - 2n/2) \\ &\leq -1. \end{aligned}$$

Therefore, (T, α) is not stable.

We now prove that v is a centroid of \bar{T} . Let \bar{V} be the set of vertices of \bar{T} . Observe that the claim trivially holds if $|\bar{V}| \leq 2$. Therefore, we assume that $|\bar{V}| \geq 3$. Notice that

$$\text{distcost}(G, u) = d_T(u, \bar{V}) + d_T(u, V \setminus \bar{V}) = |\bar{V}| + d_T(v, \bar{V}) + d_T(u, V \setminus \bar{V}).$$

Since (T, α) is stable, $d_T(v, \bar{V}) = \min_{v' \in \bar{V}} d_T(v', \bar{V})$, otherwise u would have incentive to change her strategy by swapping the edge (u, v) with the edge (u, v^*) such that $v^* \in \arg \min_{v' \in \bar{V}} d_T(v', \bar{V})$.

Let x_1, \dots, x_k be the k neighbors of v in \bar{T} . Clearly, x_1, \dots, x_k are also the k children of v in T . Let (v, x_i) be any edge of \bar{T} adjacent to v . Consider the forest F obtained by removing the edge (v, x_i) from \bar{T} . Let X_i be the set of vertices of the

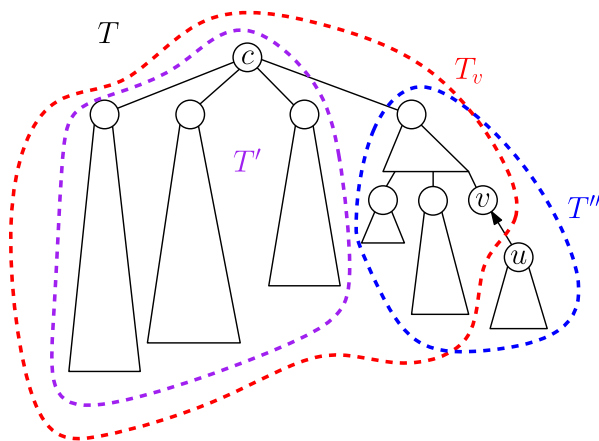


Fig. 4 Illustration of the subtrees T_v , T' and T'' used in the proof of Lemma 11

tree of F that contains x_i . Let $Y_i = \bar{V} \setminus X_i$ be the set of vertices of the tree of F that contains v . We have that

$$d_T(v, \bar{V}) = d_T(v, Y_i) + d_T(v, X_i) = d_T(v, Y_i) + |X_i| + d_T(x_i, X_i).$$

Similarly,

$$d_T(x_i, \bar{V}) = d_T(x_i, Y_i) + d_T(x_i, X_i) = |Y_i| + d_T(v, Y_i) + d_T(x_i, X_i).$$

Since $d_T(v, \bar{V}) \leq d_T(x_i, \bar{V})$, it follows that

$$d_T(v, Y_i) + |X_i| + d_T(x_i, X_i) \leq |Y_i| + d_T(v, Y_i) + d_T(x_i, X_i)$$

and hence we have $|X_i| \leq |Y_i|$. Therefore, for every $i = 1, \dots, k$, we have that

$$|X_i| \leq 1 + \sum_{j=1, j \neq i}^k |X_j| = |\bar{V}| - |X_i|,$$

which implies that $|X_i| \leq |\bar{V}|/2$. Hence, v is a centroid of \bar{T} . □

We now show a useful bound on the number of vertices contained in each of the subtrees of a stable tree network rooted at a centroid.

Lemma 12 *Let (T, α) be a stable tree network rooted at a centroid c of T and let v be a leaf of T . Let c_1, \dots, c_k be the vertices along the path in T between $c_1 = c$ and $c_k = v$, where c_{i+1} is the child of c_i , and, finally, for every $i = 1, \dots, k$, let*

$$n_i = |\{x \in V \mid d_T(c_i, x) < d_T(c_j, x), j \neq i\}|.$$

We have that

$$\sum_{j=1}^i n_j \geq n \cdot \sum_{j=1}^i 1/2^j.$$

Proof The proof is by induction on i . The base is when $i = 1$. Since $c = c_0$ is a centroid of T , the subtree of T rooted at c_1 contains at most $n/2$ vertices. Therefore $n_1 \geq n/2$.

Now, assume that $\sum_{j=1}^i n_j \geq n \sum_{j=1}^i 1/2^j$. We prove the claim for $i + 1$. Using Lemma 11, we have that c_{i+1} is a centroid of the subtree of T rooted at c_{i+1} . As a consequence, if $m = \sum_{j=1}^i n_j$, we have that $n_{i+1} \geq \frac{n-m}{2}$. Therefore,

$$\begin{aligned} \sum_{j=1}^{i+1} n_j &= m + n_{i+1} \geq m + \frac{n-m}{2} = \frac{m+n}{2} \\ &\geq n \cdot \sum_{j=1}^i 1/2^{j+1} + \frac{n}{2} = n \cdot \sum_{j=1}^{i+1} \frac{1}{2^j}. \end{aligned}$$

□

We can finally prove our upper bound on the PoA of stable tree networks.

Theorem 2 For $\alpha \geq 2$, the PoA restricted to the class of stable tree networks of n vertices is upper bounded by $3 + \frac{2n^2 - 8n - 4\alpha}{2n^2 + (\alpha - 2)n}$.

Proof Let (T, α) be a stable tree network rooted at a centroid c of T and let v be any leaf of T . Let c_1, \dots, c_{r+2} be the vertices along the path in T between $c_1 = c$ and $c_{r+2} = v$, where c_{i+1} is the child of c_i , and, finally, for every $i = 1, \dots, r + 2$, let

$$n_i = |\{x \in V \mid d_T(c_i, x) < d_T(c_j, x), j \neq i\}|.$$

Consider the strategy change in which agent v buys the edge (v, c_1) and let $k = \lfloor r/2 \rfloor$. The creation cost of agent v clearly increases by α , while her distance cost decreases by at least

$$\sum_{i=1}^k ((r + 2 - 2i)n_i).$$

Since, $r + 2 - 2i$ is strictly positive and monotonically decreasing with respect to i , using Lemma 12 we can observe that the distance cost of agent v is minimized when, for every $i = 1, \dots, k$, n_{r-i} is minimum, i.e., when $n_{r-i} \geq \frac{n}{2^i}$. Therefore, the distance cost decrease of v is lower bounded by

$$\begin{aligned} \sum_{i=1}^k ((r + 2 - 2i)n_i) &\geq \sum_{i=1}^k \left((r + 2 - 2i)n/2^i \right) \\ &= (r + 2)n \sum_{i=1}^k \frac{1}{2^i} - 2n \sum_{i=1}^k \frac{i}{2^i} \\ &= (r + 2)n \left(1 - \frac{1}{2^k} \right) \\ &\quad - 2n \left(\frac{k/2^{k+2} - (k + 1)/2^{k+1} + 1/2}{(1/2 - 1)^2} \right) \\ &= (r + 2)n - \frac{r + 2}{2^k}n - \frac{2k}{2^k}n + \frac{4(k + 1)}{2^k}n - 4n \\ &= (r - 2)n + n \left(\frac{2k + 2 - r}{2^k} \right) \\ &\geq (r - 2)n, \end{aligned}$$

where last inequality holds because $r \leq 2k + 1$.

Since (T, α) is stable, agent v has no incentive to buy the edge (v, c_1) . Hence, $\alpha - (r - 2)n \geq 0$, i.e., $r \leq \alpha/n + 2$. This implies that the length of the path from the centroid to any leaf of T is at most $\alpha/n + 4$. Thus, the diameter of T is less than or equal to $2\alpha/n + 8$. Since in every tree of n vertices, there are $2(n - 1)$ distinct pairs

of vertices at distance 1, while the other $(n-1)(n-2)$ pairs are at distance of at most $2\alpha/n + 8$, the upper bound on the social cost of T is

$$\begin{aligned} \text{cost}(T) &= \alpha(n-1) + \left(\frac{2\alpha}{n} + 8\right)(n-1)(n-2) + 2(n-1) \\ &= \frac{8n^2 + (3\alpha - 14)n - 4\alpha}{n} \cdot (n-1). \end{aligned}$$

The cost of the social optimum on n nodes, which for $\alpha \geq 2$ is the star [27], is

$$\text{cost}(S_n) = \alpha(n-1) + 2(n-1)(n-2) + 2(n-1) = (2n + \alpha - 2)(n-1).$$

Therefore,

$$\frac{\text{cost}(T)}{\text{cost}(S_n)} = \frac{8n^2 + (3\alpha - 14)n - 4\alpha}{2n^2 + (\alpha - 2)n} = 3 + \frac{2n^2 - 8n - 4\alpha}{2n^2 + (\alpha - 2)n}.$$

□

4 Conclusion

In this paper we have opened a new line of attack on settling the tree conjecture and on proving a constant price of anarchy for the network creation game for all α . Our technique is orthogonal to the known approaches using bounds on the average degree of vertices in a biconnected component. We are confident that our methods can be refined and/or combined with the average degree technique to obtain even better bounds – ideally proving or disproving the conjectures. For further progress on these problems we propose as first refinement to prove that any non-tree NE network for $\alpha > 2n - 6$ must contain a strong critical pair, which would prove the following conjecture:

Conjecture 1 For $\alpha > 2n - 6$ all NE networks must be trees.

Also the facts that strong critical pairs cannot exist in any NE network (Lemma 7) and that the edge ownership configuration in any min cycle in any NE is heavily restricted (Lemma 9) should be utilized. Using these, we have already established some structural implications in our earlier work [44]. In particular, both mentioned facts imply that a particular substructure, called a wheel of min-cycles, exists in any non-tree NE network for $\alpha > 2n - 6$. Thus, this yields further insight into the structure of the biconnected component of any NE network. These additional constraints could be employed to obtain better lower bounds on the average degree of the biconnected component of any non-tree NE network and hence a further refinement of the average degree technique. Moreover, besides focusing on strong critical pairs, which essentially describe a particular structural relationship of two nodes in the network, it seems possible to apply our technique to other types of node-pairs or even to specific configurations of more than two nodes. For example, considering two nodes which both buy at least two edges and analyzing suitable potential strategy changes

involving both nodes may yield a contradiction to the respective network being in NE for $\alpha \leq 2n - 6$.

Another interesting approach is to modify our techniques to cope with the so-called max-version of the network creation game [24], where agents try to minimize their maximum distance to all other nodes, instead of minimizing the sum of distances. Also for the max-version it is still open for which α all stable networks must be trees.

References

1. Albers, S., Eilts, S., Even-Dar, E., Mansour, Y., Roditty, L.: On Nash equilibria for a network creation game. *ACM Trans. Econ. Comput. (TEAC)* **2**(1), 2 (2014)
2. Alon, N., Demaine, E.D., Hajiaghayi, M.T., Leighton, T.: Basic network creation games. *SIAM J. Discret. Math.* **27**(2), 656–668 (2013)
3. Álvarez, C., Blesa, M.J., Duch, A., Messegué, A., Serna, M.: Celebrity games. *Theor. Comput. Sci.* **648**, 56–71 (2016)
4. Álvarez, C., Messegué, A.: Max celebrity games. In: *International workshop on algorithms and models for the web-graph (WAW)*, pp. 88–99. Springer (2016)
5. Álvarez, C., Messegué, A.: Network creation games: Structure vs anarchy. arXiv:1706.09132 (2017)
6. Álvarez, C., Messegué, A.: On the constant price of anarchy conjecture. arXiv:1809.08027 (2018)
7. Andelman, N., Feldman, M., Mansour, Y.: Strong price of anarchy. *Games Econ. Behav.* **65**(2), 289–317 (2009)
8. Anshelevich, E., Bhardwaj, O., Usher, M.: Friend of my friend: Network formation with two-hop benefit. *Theory Comput. Syst.* **57**(3), 711–752 (2015)
9. Bala, V., Goyal, S.: A noncooperative model of network formation. *Econometrica* **68**(5), 1181–1229 (2000)
10. Barabási, A.: *Network science*. Cambridge University Press, Cambridge (2016)
11. Bilò, D., Gualà, L., Leucci, S., Proietti, G.: Network creation games with traceroute-based strategies. In: *International colloquium on structural information and communication complexity (SIROCCO)*, pp. 210–223. Springer (2014)
12. Bilò, D., Gualà, L., Leucci, S., Proietti, G.: The max-distance network creation game on general host graphs. *Theor. Comput. Sci.* **573**, 43–53 (2015)
13. Bilò, D., Gualà, L., Leucci, S., Proietti, G.: Locality-based network creation games. *ACM Transactions on Parallel Computing (TOPC)* **3**(1), 6 (2016)
14. Bilò, D., Gualà, L., Proietti, G.: Bounded-distance network creation games. *ACM Trans. Econ. Comput. (TEAC)* **3**(3), 16 (2015)
15. Bilò, D., Lenzner, P.: On the tree conjecture for the network creation game. In: *35th symposium on theoretical aspects of computer science (STACS)*, pp. 14:1–14:15 (2018)
16. Brandes, U., Hofer, M., Nick, B.: Network creation games with disconnected equilibria. In: *International workshop on internet and network economics (WINE)*, pp. 394–401. Springer (2008)
17. Chauhan, A., Lenzner, P., Melnichenko, A., Molitor, L.: Selfish network creation with non-uniform edge cost. In: *International symposium on algorithmic game theory (SAGT)*, pp. 160–172. Springer (2017)
18. Chauhan, A., Lenzner, P., Melnichenko, A., Münn, M.: On selfish creation of robust networks. In: *International symposium on algorithmic game theory (SAGT)*, pp. 141–152. Springer (2016)
19. Corbo, J., Parkes, D.: The price of selfish behavior in bilateral network formation. In: *24th ACM symposium on principles of distributed computing (PODC)*, pp. 99–107. ACM (2005)
20. Cord-Landwehr, A., Lenzner, P.: Network creation games: think global—act local. In: *International symposium on mathematical foundations of computer science (MFCS)*, pp. 248–260. Springer (2015)
21. Cord-Landwehr, A., Mäcker, A., auf der Heide, F.M.: Quality of service in network creation games. In: *International conference on web and internet economics (WINE)*, pp. 423–428. Springer (2014)
22. Cormen, T.H., Leiserson, C.E., Rivest, R.L., Stein, C.: *Introduction to algorithms*. MIT Press (2009)
23. de Keijzer, B., Janus, T.: On strong equilibria and improvement dynamics in network creation games. *Internet Mathematics*, 2019 (2019)

24. Demaine, E.D., Hajiaghayi, M.T., Mahini, H., Zadimoghaddam, M.: The price of anarchy in network creation games. *ACM Trans. Algorithms (TALG)* **8**(2), 13 (2012)
25. Demaine, E.D., Hajiaghayi, M., Mahini, H., Zadimoghaddam, M.: The price of anarchy in cooperative network creation games. *ACM SIGecom Exchanges* **8**(2), 2 (2009)
26. Ehsani, S., Fadaee, S.S., Fazli, M., Mehrabian, A., Sadeghabad, S.S., Safari, M., Saghafian, M.: A bounded budget network creation game. *ACM Trans. Algorithms (TALG)* **11**(4), 34 (2015)
27. Fabrikant, A., Luthra, A., Maneva, E., Papadimitriou, C.H., Shenker, S.: On a network creation game. In: 22nd symposium on principles of distributed computing (PODC), pp. 347–351. ACM (2003)
28. Friedrich, T., Ihde, S., Keßler, C., Lenzner, P., Neubert, S., Schumann, D.: Efficient best response computation for strategic network formation under attack. In: International symposium on algorithmic game theory (SAGT), pp. 199–211. Springer (2017)
29. Garey, M.R., Johnson, D.S.: Computers and intractability, vol. 29. wh Freeman, New York (2002)
30. Goyal, S., Jabbari, S., Kearns, M., Khanna, S., Morgenstern, J.: Strategic network formation with attack and immunization. In: International conference on web and internet economics (WINE), pp. 429–443. Springer (2016)
31. Graham, R., Hamilton, L., Levavi, A., Loh, P.-S.: Anarchy is free in network creation. In: International workshop on algorithms and models for the web-graph (WAW), pp. 220–231. Springer (2013)
32. Jackson, M.O.: A survey of network formation models: stability and efficiency. *Group Formation in Economics: Networks Clubs, and Coalitions*, pp. 11–49 (2005)
33. Jackson, M.O.: *Social and economic networks*. Princeton University Press, Princeton (2010)
34. Jackson, M.O., Wolinsky, A.: A strategic model of social and economic networks. *J. Econ. Theory* **71**(1), 44–74 (1996)
35. Janus, T., de Keijzer, B.: On strong equilibria and improvement dynamics in network creation games. In: International conference on web and internet economics (WINE), pp. 161–176. Springer (2017)
36. Johnson, D.S., Lenstra, J.K., Kan, R.AHG.: The complexity of the network design problem. *Networks* **8**(4), 279–285 (1978)
37. Kariv, O., Hakimi, S.L.: An algorithmic approach to network location problems. ii: The p-medians. *SIAM J Appl Math* **37**(3), 539–560 (1979)
38. Kawald, B., Lenzner, P.: On dynamics in selfish network creation. In: 25th ACM symposium on parallelism in algorithms and architectures (SPAA), pp. 83–92. ACM (2013)
39. Kleinberg, J.: The small-world phenomenon: An algorithmic perspective. In: 32nd ACM symposium on theory of computing (STOC), pp. 163–170. ACM (2000)
40. Kliemann, L.: The price of anarchy for network formation in an adversary model. *Games* **2**(3), 302–332 (2011)
41. Koutsoupias, E., Papadimitriou, C.: Worst-case equilibria. In: 16th symposium on theoretical aspects of computer science (STACS), pp. 404–413. Springer (1999)
42. Lenzner, P.: On dynamics in basic network creation games. In: International symposium on algorithmic game theory (SAGT), pp. 254–265. Springer (2011)
43. Lenzner, P.: Greedy selfish network creation. In: International workshop on internet and network economics (WINE), pp. 142–155. Springer (2012)
44. Lenzner, P.: On selfish network creation. PhD thesis, Humboldt University of Berlin (2014)
45. Mamageishvili, A., Mihalák, M., Müller, D.: Tree Nash equilibria in the network creation game. *Internet Math.* **11**(4-5), 472–486 (2015)
46. Meirum, E.A., Mannor, S., Orda, A.: Network formation games with heterogeneous players and the internet structure. In: 15th ACM conference on economics and computation (EC), pp. 735–752. ACM (2014)
47. Mihalák, M., Schlegel, J.C.: Asymmetric swap-equilibrium: a unifying equilibrium concept for network creation games. In: International symposium on mathematical foundations of computer science (MFCS), pp. 693–704. Springer (2012)
48. Mihalák, M., Schlegel, J.C.: The price of anarchy in network creation games is (mostly) constant. *Theory Comput. Syst.* **53**(1), 53–72 (2013)
49. Papadimitriou, C.: Algorithms, games, and the internet. In: 33rd ACM symposium on theory of computing (STOC), pp. 749–753. ACM (2001)
50. Travers, J., Milgram, S.: The small world problem. *Phys. Today* **1**, 61–67 (1967)