From Graph Theory to Network Science: The Natural Emergence of Hyperbolicity

Tobias Friedrich
Hasso Plattner Institute, University of Potsdam, Potsdam, Germany https://hpi.de/friedrich/
friedrich@hpi.de

Abstract

Network science is driven by the question which properties large real-world networks have and how we can exploit them algorithmically. In the past few years, hyperbolic graphs have emerged as a very promising model for scale-free networks. The connection between hyperbolic geometry and complex networks gives insights in both directions:

(1) Hyperbolic geometry forms the basis of a natural and explanatory model for real-world networks. Hyperbolic random graphs are obtained by choosing random points in the hyperbolic plane and connecting pairs of points that are geometrically close. The resulting networks share many structural properties for example with online social networks like Facebook or Twitter. They are thus well suited for algorithmic analyses in a more realistic setting.

(2) Starting with a real-world network, hyperbolic geometry is well-suited for metric embeddings. The vertices of a network can be mapped to points in this geometry, such that geometric distances are similar to graph distances. Such embeddings have a variety of algorithmic applications ranging from approximations based on efficient geometric algorithms to greedy routing solely using hyperbolic coordinates for navigation decisions.

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1 Introduction

In the field of algorithms, a major discrepancy between theory and practice derives from the fact that the analysis usually assumes worst-case instances. However, real-world instances behave very differently. In fact, numerous NP-hard problems can be solved in reasonable time even on large real-world instances, using techniques such as

branch-and-bound [30], branch-and-reduce [5], or reductions to integer linear programming [26], which in itself is an NP-hard problem that is rather well-behaved on practical instances. One approach to bridge this gap between theory and practice is to employ an average-case analysis by bounding the expected run time under the assumption that the input is randomly drawn from a certain distribution. This was already pointed out by Karp [42] in 1983, who noted that: "One way to validate or compare imperfect[1] algorithms for NP-hard combinatorial problems is simply to run them on typical instances and see how often they fail. [...] While probabilistic assumptions are always open to question, the approach seems to have considerable explanatory power." In 1986, Levin [45] laid the foundation for average-case complexity theory by providing an average-case complete problem. For more on this topic, see the survey by Bogdanov and Trevisan [19].

[1] Karp calls an algorithm "imperfect" if it potentially outputs the wrong answer or runs too long.

Probability distributions on graphs.

The practical relevance and "considerable explanatory power" [42] of an average-case analysis of course heavily depends on the assumed probability distribution of the input. Thus, when focusing on graph problems, the considered graph model should mimic at least the most important properties of typical real-world networks. Two fundamental properties that have been observed in large real-world networks of many domains are the following.

Heterogeneity: Large real-world networks are highly heterogeneous, i.e., they usually have some high-degree and many low-degree vertices. In fact, many real-world networks are *scale-free*, which means that the number of vertices of degree at most x is roughly proportional to $x^{-\beta}$ for some constant β. One then says that such graphs have a *power-law* degree distribution with *power-law exponent* β, which usually lies between 2 and 3.

Interdependency: Edges in real-world networks are typically not independent, i.e., vertices with a common neighbor tend to be rather similar and thus more likely to be connected than two random vertices. Formally, this can be measured using the so-called *clustering coefficient*, which comes in two flavors, local and global. The global clustering coefficient is the ratio of triangles among triples of vertices that have at least two edges; and the local clustering coefficient of a single vertex is the ratio of connected neighbors among all pairs of neighbors (one then usually considers these local clustering coefficients averaged over all vertices).

We note that some types of real-world networks are homogeneous in the sense that most vertices have roughly the same degree (e.g., the degrees of most vertices in a typical road network lie between 1 and 4 [52]). However, large networks from many domains (social networks, electricity maps, biological networks, co-author graphs, romantic relationships, etc. [41, 29, 46, 47]) are highly heterogeneous. Intuitively, this reflects the fact that the entities represented by nodes typically differ in importance, influence, or popularity.

The interdependency between edges is also not surprising. For example, two autonomous systems in the Internet that both have a direct connection to a third autonomous system are likely to be geographically close, which increases the chance that they also have a direct connection. In a similar fashion, two researchers who collaborated with the same third researcher are probably working on similar topics and are thus more likely to collaborate than two random researchers. One can therefore expect the clustering coefficients of real-world networks to be bounded away from 0 (e.g., the local clustering coefficients of collaboration networks are mostly above 0.5 [52]). In the following we briefly discuss different random graph models with heterogeneity and interdependency in mind.

Random graphs.

The earliest and most-studied model is the Erdős-Rényi random graph [31]. In this model, an input graph $G(n, p)$ with n vertices is generated by connecting each vertex pair independently with probability p. Erdős-Rényi graphs are popular among researchers mainly for two reasons. First, for $p = \frac{1}{2}$, it produces each (labeled) graph with n vertices with the same probability, which seems like a desirable property. Second, it is simple, which makes it accessible to rigorous and very detailed mathematical analysis. It is thus not surprising, that, on the one hand, Erdős-Rényi graphs were used in the early days of average-case analysis, and are, on the other hand, still the object of current research. To name two examples, Angluin and Valiant [8] showed in 1977 that on Erdős-Rényi random graphs the NP-hard problem Hamiltonian Circuit can be solved in expected time $O(n \log^2 n)$ if the probability p

is sufficiently large. Also the the W[1]-hard k-Clique problem admits an average-case FPT-algorithm on Erdős-Rényi graphs, bringing together the fields of average-case and parameterized complexity [33].

Unfortunately, graphs generated with the Erdős-Rényi model lack the above-mentioned desired properties. *Asymptotically almost surely* (i.e, with probability $\to 1$ for $n \to \infty$), all vertices have roughly the same degree, leading to almost regular graphs, and their clustering coefficients tend to 0 for $n \to \infty$.

Heterogeneous random graphs.

To account for the heterogeneity of real-world networks, different models have been introduced. The Barabási-Albert model [9] (also called *preferential attachment*) adds one vertex at a time, connecting it to already existing vertices with probability proportional to their degree. This model actually has an explanatory character in the sense that a reasonable assumption (namely that already popular nodes are more attractive to new nodes) leads to the power-law degree distribution observed in practice. On the downside, this procedural description of the model introduces strong stochastic dependencies, which makes a mathematical analysis of the resulting graphs rather difficult [21]. The Chung-Lu model [28, 27] produces scale-free graphs by assigning weights to the nodes (following a power law) and connecting every pair of vertices with a probability proportional to the product of their weights [3, 4]. Though the Chung-Lu model cannot explain the emergence of a power-law degree distribution in real-world networks, it is much more accessible to a mathematical analysis due to the fact that edges are chosen independently. Further similar random models are inhomogeneous random graphs by van der Hofstad [40] and Norros-Reittu graphs [48].

However, despite this abundance of theoretical models for power-law networks, all of them fall short in describing real-world networks as their clustering coefficient tends to 0 for $n \to \infty$ while it is bounded away from 0 for most real-world networks.

Random graphs with interdependencies.

There are a number of random graph models that lead to graphs with non-vanishing clustering coefficients. An example of such models are *geometric random graphs* [38] (also called *random unit disk graphs*). Such a graph is obtained by assigning random coordinates to each vertex and connecting two vertices if and only if they are close (with respect to Euclidean distance). It is not surprising that the geometric locality (two vertices close to a third vertex are also close to each other) leads to high clustering coefficients in the resulting graphs [49]. Though random geometric graphs are well suited to represent sensor networks [51], they are less suited for many other real-world networks. As in the Erdős-Rényi model, the resulting degree distribution is rather homogeneous [49]. Moreover, two other properties often observed in large real-world networks, namely sparsity and a small diameter, cannot be achieved together by random unit disk graphs for the following reason: If the generated graph is sparse, the vertices have to occupy an area linear in the number of vertices. This inevitably leads to a polynomial diameter (e.g., $\sqrt{n}$ in the 2-dimensional Euclidean plane).

Watts and Strogatz [56] proposed a model leading to sparse graphs with high clustering and logarithmic diameter and coined the term *small-world network* for this type of network. Their model starts with a regular graph with high clustering and randomly rewires edges (i.e., it deletes and adds edges randomly). The resulting graphs inherit the high clustering from the initial graph while obtaining a small diameter due to the edges added independently

at random (as in the Erdős-Rényi model). On the downside, this model also leads to a homogeneous degree distribution.

2 Combining heterogeneity and interdependency.

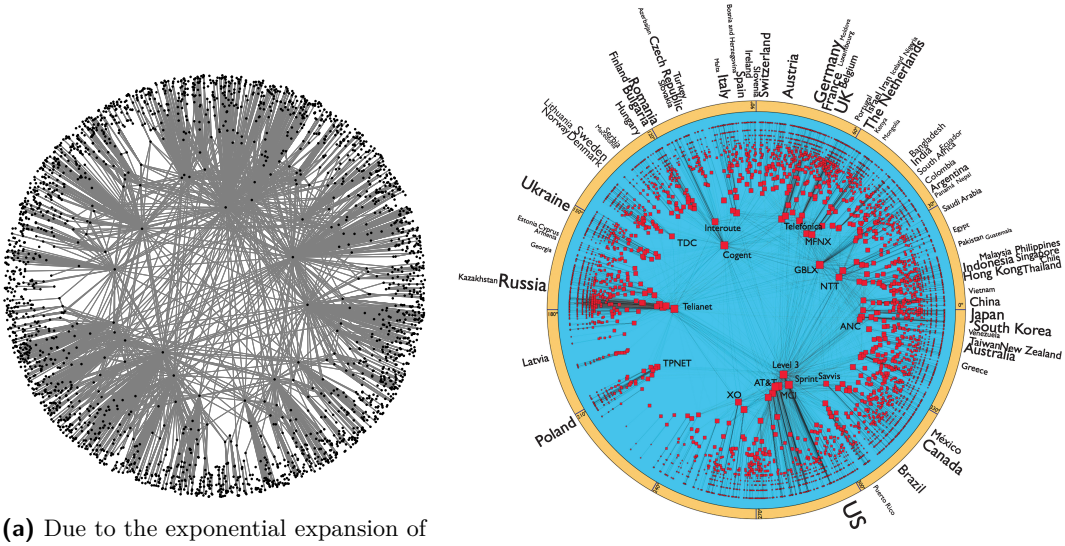
None of the aforementioned models fulfill both properties (heterogeneity and interdependency) at the same time. A natural model that leads to graphs with both features are *hyperbolic random graphs* as introduced by Krioukov et al. [44]. We strongly believe that hyperbolic random graphs are an excellent model to describe and study real-world networks. This believe is supported by an empirical analysis of a few hundred real-world networks [14]. In the following we briefly describe the model, discuss previous results establishing additional desirable properties and thereby debate why we believe that hyperbolic random graphs are well suited for representing large real-world networks.

Definition and basic properties.

Hyperbolic random graphs are generated in the same way as geometric random graphs, replacing the Euclidean with the hyperbolic plane, i.e., the vertices are assigned to random positions within a disk of the hyperbolic plane and two vertices are connected if and only if their hyperbolic distance is small. As in the Euclidean case, the geometry leads to a non-vanishing clustering coefficient [44, 39, 25]. However, the hyperbolic plane expands exponentially, i.e., the area and the circumference of a disk grows exponentially in its radius. Thus, when distributing the vertices evenly within a disk in the hyperbolic plane, most vertices will end up close to the boundary of the disk; see Figure 1a. As distances are much larger between vertices close to the boundary, these vertices have low degree, while the few vertices close to the center have high degree. In fact, this way of distributing the vertices leads to a power-law degree distribution with power-law exponent $\beta = 3$ [44, 39]. Moreover, one can obtain arbitrary exponents β, with $\beta > 2$, by assuming a hyperbolic space with different negative curvature when sampling the radial coordinates of the vertices [18]. Thus, the power-law exponent is a parameter of the model. Similarly, the expected average degree of the vertices can be controlled by changing the threshold distance, below which vertices are still connected.

Beyond these two fundamental properties of a heterogeneous degree distribution and interdependency between edges, hyperbolic random graphs show other desirable properties. In contrast to the Euclidean plane, the exponential expansion of hyperbolic space makes it possible to have a graph with low average degree and only logarithmic diameter (intuitively speaking, spreading the vertices over a region with linear area no longer implies that this region has polynomial diameter as in the Euclidean plane). In fact, hyperbolic random graphs have polylogarithmic diameter [35, 36, 43] and the average distance between pairs of vertices is $\Theta(\log \log n)$ [1].

Such a more realistic random graph model now opens up the possibility to explain why real-world instances tend to be algorithmically well-behaved, by performing an average-case analysis. There is not much work in this direction. One particular example is bidirectional breadth-first search. Borassi and Natale [22] observed that bidirectional search performs sublinear in practice. This was the motivation for Bläsius et al. [12], who proved that the runtime of bidirectional search is sublinear runtime with high probability for hyperbolic random graphs.



(a) Due to the exponential expansion of the hyperbolic plane, most vertices lie close to the disk's boundary. Nodes close to the center are rare, but highly connected, leading to the power-law degree distribution.

(b) The embedding of the Internet by Boguná et al. [20] into the hyperbolic plane enables greedy routing with high success ratio and low stretch, i.e., for 97% of the vertex pairs, greedy routing succeeds and resulting paths are on average only 10% longer than the shortest paths.

Figure 1 A hyperbolic random graph (a) and an embedding of the Internet into the hyperbolic plane (b).

Component structure and connectivity.

Note that there is no explicit mechanism ensuring that a hyperbolic random graph is connected, and in fact, it usually consists of multiple connected components. As some applications are only interested in connected graphs, it is good to know that hyperbolic random graphs (at least for $2 < \beta < 3$) have a so-called *giant component* [17], i.e., a connected component with a linear number of vertices, while all other components have only polylogarithmic size [43].

Related to the component structure is the question of how densely connected each component is. Despite the fact that we assume real-world networks to have constant average degree, we still expect to find large highly connected subgraphs forming communities. For hyperbolic random graphs with power-law exponent $2 < \beta < 3$, it has been shown that the largest clique has polynomial size $\Theta(\sqrt{n^{3-\beta}})$ and that there are $\sqrt{n^{(3-\beta)k}} \cdot \Theta(k)^{-k}$ cliques of size k [34, 15]. On the other hand, different parts of the graph are loosely connected in the sense that hyperbolic random graphs have small balanced separators. To be more precise, if $2 < \beta < 3$, there exists a hierarchy of balanced separators each having size $\Theta(\sqrt{n^{3-\beta}})$ [10], leading to sublinear treewidth of $\Theta(\sqrt{n^{3-\beta}})$ (which is tight due to the matching bound for the clique size). On the one hand, this result enables faster algorithms by using dynamic programming on the tree decomposition. On the other hand, it gives structural insights distinguishing hyperbolic random graphs further from the Barabási-Albert model having linear treewidth [37]. We note that real-world networks typically have rather small treewidth [2, Tables I and V], which gives another indication that hyperbolic random graphs are well suited for representing large real-world networks.

Generating algorithms and related models.

Algorithmically, hyperbolic random graphs can naively be generated in $\Theta(n^2)$ time [7], which has been improved to $O(n^{1.5} \log n)$ [55] and even down to linear [24, 54, 50, 23] by using geometric data structures. The algorithm of Bringmann et al. [24, 23] additionally makes use of the relationship between hyperbolic random graphs and so-called geometric inhomogeneous random graphs (GIRGs), which can be seen as a combination of the Chung-Lu model with a geometry. In a similar way, hyperbolic random graphs have been related to Barabási-Albert graphs additionally equipped with a geometry [32].

The hyperbolic metric of real-world networks.

Beyond the very promising hyperbolic random graph model (showing properties that one expects in large real-world networks of many domains), the metric defined by most networks appears to be very similar to the metric of the hyperbolic plane. To support this observation, we name three examples. Boguná [20] embedded the Internet network into the hyperbolic plane by assigning a hyperbolic coordinate to every autonomous system; see Figure 1b. They observed that greedy routing solely based on these coordinates is almost maximally efficient, i.e., it finds short paths between almost any two pairs of vertices. Verbeek and Suri [53] show that graphs with low quasi-cyclicity (which appears to be low for many networks) admit a metric embedding into a hyperbolic space of constant dimension with constant additive distortion. Finally, Albert et al. [6] computed the so-called Gromov hyperbolicity (which basically measures how close a graph metric is to the metric of the hyperbolic plane) for several biological and social networks. They found out that these networks are indeed hyperbolic in this sense, i.e., their Gromov hyperbolicity is small. Though all three results relate networks to hyperbolic geometry in different ways, they all support the above claim that the metric of networks is similar to the hyperbolic metric.

Embeddings in the hyperbolic plane.

A common application of a random model describing real-world instances reasonably well is the possibility to perform a meaningful average-case analysis. However, acknowledging hyperbolic random graphs as a reasonable model for real-world networks opens up another line of research: viewing a given real-world network as a hyperbolic random graph but without known coordinates. It is then a natural question, whether we can retrieve the missing geometric information, i.e., whether we can embed the graph into the hyperbolic plane such that most edges are short and most non-adjacent vertices are far apart. There are a number of algorithms for embedding a network into the hyperbolic space. Algorithms with quasilinear runtime are known for maximum likelihood embeddings [11, 16] and for optimizing greedy routing [13].

References

1 Mohammed Amin Abdullah, Michel Bode, and Nikolaos Fountoulakis. Typical distances in a geometric model for complex networks. *CoRR*, abs/1506.07811, 2015.

2 Aaron B. Adcock, Blair D. Sullivan, and Michael W. Mahoney. Tree decompositions and social graphs. *Internet Mathematics*, 12(5):315–361, 2016. doi:10.1080/15427951.2016.1182952.

3 William Aiello, Fan Chung, and Linyuan Lu. A random graph model for massive graphs. In *32nd STOC*, pages 171–180, 2000. doi:10.1145/335305.335326.

4 William Aiello, Fan Chung, and Linyuan Lu. A random graph model for power law graphs. *Experimental Mathematics*, 10(1):53–66, 2001. doi:10.1080/10586458.2001.10504428.

5 Takuya Akiba and Yoichi Iwata. Branch-and-reduce exponential/FPT algorithms in practice: A case study of vertex cover. *Theoretical Computer Science*, 609:211–225, 2016. doi:10.1016/j.tcs.2015.09.023.

6 Réka Albert, Bhaskar DasGupta, and Nasim Mobasheri. Topological implications of negative curvature for biological and social networks. *Physical Review E*, 89:032811, 2014. doi:10.1103/PhysRevE.89.032811.

7 Rodrigo Aldecoa, Chiara Orsini, and Dmitri Krioukov. Hyperbolic graph generator. *Computer Physics Communications*, 196:492–496, 2015. doi:10.1016/j.cpc.2015.05.028.

8 D. Angluin and L.G. Valiant. Fast probabilistic algorithms for hamiltonian circuits and matchings. *Journal of Computer and System Sciences*, 18(2):155–193, 1979. doi:10.1016/0022-0000(79)90045-X.

9 Albert-László Barabási and Réka Albert. Emergence of scaling in random networks. *Science*, 286(5439):509–512, 1999. doi:10.1126/science.286.5439.509.

10 Thomas Bläsius, Tobias Friedrich, and Anton Krohmer. Hyperbolic random graphs: Separators and treewidth. In *24th ESA*, pages 15:1–15:16, 2016. doi:10.4230/LIPIcs.ESA.2016.15.

11 Thomas Bläsius, Tobias Friedrich, Anton Krohmer, and Sören Laue. Efficient embedding of scale-free graphs in the hyperbolic plane. In *24th ESA*, pages 16:1–16:18, 2016. doi:10.4230/LIPIcs.ESA.2016.16.

12 Thomas Bläsius, Cedric Freiberger, Tobias Friedrich, Maximilian Katzmann, Felix Montenegro-Retana, and Marianne Thieffry. Efficient shortest paths in scale-free networks with underlying hyperbolic geometry. In *45th ICALP*, pages 20:1–20:14, 2018.

13 Thomas Bläsius, Tobias Friedrich, Maximilian Katzmann, and Anton Krohmer. Hyperbolic embeddings for near-optimal greedy routing. In *20th ALENEX*, pages 199–208, 2018.

14 Thomas Bläsius, Tobias Friedrich, Maximilian Katzmann, Anton Krohmer, and Jonathan Striebel. Towards a systematic evaluation of generative network models. In *15th WAW*, pages 99–114, 2018.

15 Thomas Bläsius, Tobias Friedrich, and Anton Krohmer. Cliques in hyperbolic random graphs. *Algorithmica*, 80:2324–2344, 2018.

16 Thomas Bläsius, Tobias Friedrich, Anton Krohmer, and Sören Laue. Efficient embedding of scale-free graphs in the hyperbolic plane. *IEEE/ACM Transactions on Networking*, 26(1-57570FN):920–933, 2018.

17 Michel Bode, Nikolaos Fountoulakis, and Tobias Müller. On the largest component of a hyperbolic model of complex networks. *Electronic Journal of Combinatorics*, 22(3):P3.24, 2015.

18 Michel Bode, Nikolaos Fountoulakis, and Tobias Müller. The probability of connectivity in a hyperbolic model of complex networks. *Random Structures & Algorithms*, 49(1):65–94, 2016. doi:10.1002/rsa.20626.

19 Andrej Bogdanov and Luca Trevisan. Average-case complexity. *Foundations and Trends in Theoretical Computer Science*, 2(1):1–106, 2006. doi:10.1561/0400000004.

20 Marián Boguná, Fragkiskos Papadopoulos, and Dmitri Krioukov. Sustaining the internet with hyperbolic mapping. *Nature Communications*, 1:62, 2010. doi:10.1038/ncomms1063.

21 Béla Bollobás and Oliver Riordan. The diameter of a scale-free random graph. *Combinatorica*, 24:5–34, 2004. doi:10.1007/s00493-004-0002-2.

22 Michele Borassi and Emanuele Natale. KADABRA is an ADaptive Algorithm for Betweenness via Random Approximation. In *24th ESA*, volume 57, pages 20:1–20:18, 2016.

23 Karl Bringmann, Ralph Keusch, and Johannes Lengler. Sampling geometric inhomogeneous random graphs in linear time. In *25th ESA*, pages 20:1–20:15, 2017.

24 Karl Bringmann, Ralph Keusch, and Johannes Lengler. Geometric inhomogeneous random graphs. *Theoretical Computer Science*, 2019.

25 Elisabetta Candellero and Nikolaos Fountoulakis. Clustering and the hyperbolic geometry of complex networks. *Internet Mathematics*, 12(1-2):2–53, 2016. doi:10.1080/15427951.2015.1067848.

26 Markus Chimani, Maria Kandyba, Ivana Ljubić, and Petra Mutzel. Obtaining optimal k-cardinality trees fast. In *20th ALENEX*, pages 27–36, 2008. doi:10.1137/1.9781611972887.3.

27 Fan Chung and Linyuan Lu. The average distances in random graphs with given expected degrees. *Proceedings of the National Academy of Sciences*, 99(25):15879–15882, 2002.

28 Fan Chung and Linyuan Lu. Connected components in random graphs with given expected degree sequences. *Annals of Combinatorics*, 6(2):125–145, 2002. doi:10.1007/PL00012580.

29 Aaron Clauset, Cosma Rohilla Shalizi, and M. E. J. Newman. Power-law distributions in empirical data. *SIAM Review*, 51(4):661–703, 2009. doi:10.1137/070710111.

30 Daniel Delling, Andrew V. Goldberg, Ilya Razenshteyn, and Renato F. Werneck. Exact combinatorial branch-and-bound for graph bisection. In *14th ALENEX*, pages 30–44, 2012. doi:10.1137/1.9781611972924.3.

31 P. Erdős and A. Rényi. On random graphs I. *Publicationes Mathematicae*, 6:290–297, 1959.

32 L. Ferretti, M. Cortelezzi, and M. Mamino. Duality between preferential attachment and static networks on hyperbolic spaces. *Europhysics Letters*, 105(3):38001, 2014.

33 Nikolaos Fountoulakis, Tobias Friedrich, and Danny Hermelin. On the average-case complexity of parameterized clique. *Theoretical Computer Science*, 576:18–29, 2015. doi:10.1016/j.tcs.2015.01.042.

34 Tobias Friedrich and Anton Krohmer. Cliques in hyperbolic random graphs. In *34th INFOCOM*, pages 1544–1552, 2015.

35 Tobias Friedrich and Anton Krohmer. On the diameter of hyperbolic random graphs. In *42nd ICALP*, pages 241–252, 2015.

36 Tobias Friedrich and Anton Krohmer. On the diameter of hyperbolic random graphs. *SIAM Journal on Discrete Mathematics*, 32(2):1314–1334, 2018.

37 Yong Gao. Treewidth of Erdős-Rényi random graphs, random intersection graphs, and scale-free random graphs. *Discrete Applied Mathematics*, 160(4–5):566–578, 2012. doi:10.1016/j.dam.2011.10.013.

38 E. Gilbert. Random plane networks. *Journal of the Society for Industrial and Applied Mathematics*, 9(4):533–543, 1961. doi:10.1137/0109045.

39 Luca Gugelmann, Konstantinos Panagiotou, and Ueli Peter. Random hyperbolic graphs: degree sequence and clustering. In *39th ICALP*, pages 573–585, 2012.

40 Remco van der Hofstad. *Random Graphs and Complex Networks:*, volume 1 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, 2016. doi:10.1017/9781316779422.

41 Matthew O. Jackson. *Social and Economic Networks*. Princeton University Press, 2010.

42 Richard M. Karp. The probabilistic analysis of combinatorial optimization algorithms. In *Proceedings of the International Congress of Mathematicians*, pages 1601–1609, 1983.

43 Marcos Kiwi and Dieter Mitsche. A bound for the diameter of random hyperbolic graphs. In *12th ANALCO*, pages 26–39, 2015. doi:10.1137/1.9781611973761.3.

44 Dmitri Krioukov, Fragkiskos Papadopoulos, Maksim Kitsak, Amin Vahdat, and Marián Boguñá. Hyperbolic geometry of complex networks. *Physical Review E*, 82(3):036106, 2010. doi:10.1103/PhysRevE.82.036106.

45 Leonid A. Levin. Average case complete problems. *SIAM Journal on Computing*, 15(1):285–286, 1986. doi:10.1137/0215020.

46 Michael Mitzenmacher. A brief history of generative models for power law and lognormal distributions. *Internet Mathematics*, 1(2):226–251, 2003. doi:10.1080/15427951.2004.10129088.

47 M. E. J. Newman. Power laws, Pareto distributions and Zipf's law. *Contemporary Physics*, 46(5):323–351, 2005. doi:10.1080/00107510500052444.

48 Ilkka Norros and Hannu Reittu. On a conditionally Poissonian graph process. *Advances in Applied Probability*, 38(1):59–75, 2006.

49 M. Penrose. *Random Geometric Graphs*. Oxford scholarship online. Oxford University Press, 2003.

50 Manuel Penschuck. Generating Practical Random Hyperbolic Graphs in Near-Linear Time and with Sub-Linear Memory. In *16th SEA*, pages 26:1–26:21, 2017.

51 G. J. Pottie and W. J. Kaiser. Wireless integrated network sensors. *Communications of the ACM*, 43(5):51–58, 2000. doi:10.1145/332833.332838.

52 Ryan Rossi and Nesreen Ahmed. The network data repository with interactive graph analytics and visualization. In *29th AAAI*, pages 4292–4293.

53 Kevin Verbeek and Subhash Suri. Metric embedding, hyperbolic space, and social networks. In *30th SoCG*, pages 501–510, 2014. doi:10.1145/2582112.2582139.

54 M. von Looz, M. S. Özdayi, S. Laue, and H. Meyerhenke. Generating massive complex networks with hyperbolic geometry faster in practice. In *20th IEEE HPEC*, pages 1–6, 2016. doi:10.1109/HPEC.2016.7761644.

55 Moritz von Looz, Henning Meyerhenke, and Roman Prutkin. Generating random hyperbolic graphs in subquadratic time. In *26th ISAAC*, pages 467–478, 2015.

56 Duncan J. Watts and Steven H. Strogatz. Collective dynamics of 'small-world' networks. *Nature*, 939:440–442, 1998. doi:10.1038/30918.