

ON THE DIAMETER OF HYPERBOLIC RANDOM GRAPHS*

TOBIAS FRIEDRICH[†] AND ANTON KROHMER[†]

Abstract. Large real-world networks are typically scale-free. Recent research has shown that such graphs are described best in a geometric space. More precisely, the Internet can be mapped to a hyperbolic space such that geometric greedy routing is close to optimal [M. Boguñá, F. Papadopoulos, and D. Krioukov, *Nature Commun.*, 1 (2010), pp. 1–62]. This observation has pushed the interest in hyperbolic networks as a natural model for scale-free networks. Hyperbolic random graphs follow a power law degree distribution with controllable exponent β and show high clustering [L. Gugelmann, K. Panagiotou, and U. Peter, *Proceedings of the 39th International Colloquium on Automata, Languages and Programming*, 2012, pp. 573–585]. For understanding the structure of the resulting graphs and for analyzing the behavior of network algorithms, the next question is bounding the size of the diameter. The only known explicit bound is $\mathcal{O}((\log n)^{32/((3-\beta)(5-\beta)+1)})$ [M. Kiwi and D. Mitsche, *Proceedings of ANALCO*, 2015, pp. 26–39]. We present two much simpler proofs for an improved upper bound of $\mathcal{O}((\log n)^{2/(3-\beta)})$ and a lower bound of $\Omega(\log n)$. If $\beta > 3$, we show that the latter bound is tight by proving an upper bound of $\mathcal{O}(\log n)$ for the diameter.

Key words. hyperbolic random graphs, diameter, scale-free networks

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1. Introduction. Large real-world networks are almost always sparse and non-regular. Their degree distribution typically follows a *power law*, which is synonymously used for being *scale-free*. Since the 1960s, large networks have been studied in detail and dozens of models have been suggested. In the past few years, a new line of research has emerged, which showed that scale-free networks can be modeled more realistically when incorporating *geometry*.

Euclidean random graphs. It is not new to study graphs in a geometric space. In fact, graphs with *Euclidean geometry* have been studied intensively for more than a decade. The standard Euclidean models are random geometric graphs. They result from placing n nodes independently and uniformly at random on a Euclidean space. Edges are created between pairs of nodes if and only if their distance is below some fixed threshold r . These graphs have been studied in relation to subjects such as cluster analysis, statistical physics, hypothesis testing, and wireless sensor networks [33]. The resulting graphs are more or less regular and hence do not show scale-free behavior with power law degree distribution as observed in large real-world graphs.

Hyperbolic random graphs. For modeling scale-free graphs, it is natural to apply a non-Euclidean geometry with negative curvature. Krioukov et al. [26] introduced a new graph model based on *hyperbolic geometry*. Similar to Euclidean random graphs, nodes are uniformly distributed in a hyperbolic space and two nodes are connected if their hyperbolic distance is small. The resulting graphs have many properties observed in large real-world networks. This was impressively demonstrated

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TABLE 1

Known diameter bounds for various random graphs. In all cases the diameter depends on the choice of the model parameters. Here we consider a constant average degree. For scale-free networks, we also assume a power law exponent $2 < \beta < 3$. Note that the table therefore refers to a nonstandard PA version with adjustable power law exponent $2 < \beta < 3$ (normally, $\beta = 3$).

| Random graph model | Diameter | |
|--------------------------|--|--------------------|
| Sparse Erdős–Rényi [9] | $\Theta(\log n)$ [34] | |
| d -dim. Euclidean [33] | $\Theta(n^{1/d})$ [21] | |
| Watts–Strogatz [37] | $\Theta(\log n)$ [10] | |
| Kleinberg [25] | $\Theta(\log n)$ [28] | |
| Chung–Lu [13] | $\Theta(\log n)$ [13] | } power law graphs |
| PA [3] | $\Theta(\log \log n)$ [15] | |
| Hyperbolic [26] | $\mathcal{O}((\log n)^{\frac{32}{(3-\beta)(5-\beta)}+1})$ [24] | |

by Boguñá, Papadopoulos, and Krioukov [8]. They computed a maximum likelihood fit of the Internet graph in the hyperbolic space and showed that greedy routing in this hyperbolic space finds nearly optimal shortest paths in the Internet graph. The quality of this embedding is an indication that hyperbolic geometry naturally appears in large scale-free graphs.

Known properties. A number of properties of hyperbolic random graphs have been studied. Gugelmann, Panagiotou, and Peter [22] compute exact asymptotic expressions for the expected number of vertices of degree k and prove a constant lower bound for the clustering coefficient. They confirm that the clustering is nonvanishing and that the degree sequence follows a power law distribution with controllable exponent β . For $2 < \beta < 3$, hyperbolic random graphs have a giant component of size $\Omega(n)$ [6] and an average distance of $\Theta(\log \log n)$ [11], similar to other scale-free networks like that of Chung and Lu [13]. Other studied properties include the clique number [19], bootstrap percolation [12], and tree width [4], as well as algorithms for efficient generation of hyperbolic random graphs [11, 35] and embedding real networks in the hyperbolic plane [2, 5, 31, 32, 36].

Diameter. The *diameter* of a graph G is the longest shortest path between any two nodes in (the giant component of) G . It is a fundamental structural property of a random graph because it sets a worst-case lower bound on the number of steps required for all communication processes. Imagine, for instance, a simple broadcast protocol in which each activated node activates all neighbors. Starting with one active node, it takes $\Omega(D)$ iterations of this process to activate all nodes in the giant component, where D is the diameter.

In contrast to the average distance, the diameter is determined by just one single long path. Due to this sensitivity to small changes, the diameter is notoriously hard to analyze. Even subtle changes to the graph model can make an exponential difference in the diameter, as can be seen when comparing Chung–Lu (CL) random graphs [13] and preferential attachment (PA) graphs [3] in the range of the power law exponent $2 < \beta < 3$. On the one hand, it has been shown that a CL graph can be embedded in a PA graph and they behave effectively the same [18]. On the other hand, the diameter of CL graphs is $\Theta(\log n)$ [13], while for PA graphs it is $\Theta(\log \log n)$ [15]. Table 1 provides an overview of existing results in other random graph models. The only known upper bounds on the diameter of hyperbolic random graphs are $\mathcal{O}((\log n)^{\frac{32}{(3-\beta)(5-\beta)}+1})$ by Kiwi and Mitsche [24] and a polylogarithm with no explicit constant by Bringmann, Keusch, and Lengler [11]. Our conference paper [20] gave an incorrect proof of a logarithmic upper bound on the diameter for the case $2 < \beta < 3$. In particular,

Lemma 14 of [20] used wrong probabilities p_i due to a sign error. Therefore, the best known bounds for the diameter are so far the polylogarithmic upper bounds by Kiwi and Mitsche [24].

2. Our contribution. We improve upon the previous results as follows. First, we present a much simpler proof which also shows a polylogarithmic upper bound for the diameter, but with a better (that is, smaller) exponent.

THEOREM 1. *Let $2 < \beta < 3$. The diameter of the giant component in the hyperbolic random graph is $\mathcal{O}((\log n)^{\frac{2}{3-\beta}})$ with probability $1 - \mathcal{O}(n^{-2})$.*

The proof of Theorem 1 is presented in section 4. It serves as an introduction to the proof of a logarithmic upper bound for the diameter presented in section 5. There we show with more advanced techniques that for large power law exponents the following theorem holds.

THEOREM 2. *Let $\beta > 3$. Then, the diameter of the hyperbolic random graph is $\mathcal{O}(\log n)$ with probability $1 - \mathcal{O}(n^{-2})$.*

The logarithmic upper bound is optimal. In particular, we show that Theorem 2 is tight by presenting the following matching lower bound.

THEOREM 3. *Let $\beta > 2$. Then, there exists a component in the hyperbolic random graph with diameter $\Omega(\log n)$ with probability $1 - \mathcal{O}(n^{1-\frac{\beta}{2}})$. If $\beta < 3$, this is the giant component.*

Let us briefly discuss these results. First, even though we prove all diameter bounds on the giant component for the case $2 < \beta < 3$, our proofs will make apparent that the giant component is in fact the component with the largest diameter in the graph. Second, the statements in Theorems 1 and 2 hold with probability $1 - \mathcal{O}(n^{-2})$. It is, however, straightforward to modify our proofs to show that these statements hold with probability $1 - \mathcal{O}(n^{-c})$ for *any* constant c . Note that this does not hold for Theorem 3.

It is an open problem to close the gap between the lower bound $\Omega(\log n)$ and the upper bound $\mathcal{O}(\log^{\frac{2}{3-\beta}} n)$ on the diameter in the case $2 < \beta < 3$. We conjecture that the diameter in this case is $\Theta(\log n)$ as well. A major indicator for this is that the bound $\mathcal{O}(\log^{\frac{2}{3-\beta}} n)$ becomes worse as $\beta \rightarrow 3$, whereas for $\beta > 3$ we have a tight result. On the other hand, similar to most scale-free random graph models, hyperbolic random graphs have a distinct phase transition at $\beta = 3$. This makes such a behavior unlikely but not impossible. We discuss the difficulties in proving a tight bound for the case $2 < \beta \leq 3$ in more detail at the end of section 5.

Techniques used. Our formal analysis of the diameter deals with a number of technical challenges. First, in contrast to proving a bound on the average distance, it is not possible to average over all path lengths. In fact, it is not even sufficient to exclude a certain kind of path with probability $1 - \mathcal{O}(n^{-c})$ because this has to hold for all possible $\Omega(n!)$ paths.

A second major challenge is the fact that a probabilistic analysis of shortest paths typically uncovers the probability space in a consecutive fashion. Successively revealing the positions of nodes on the path introduces strong stochastic dependencies that are difficult to handle with probabilistic tail bounds [16]. Instead of studying the stochastic dependence structure in detail, we use the geometry and model the hyperbolic random graph as a Poisson point process (PPP). This allows us to analyze different areas in the graph independently, which in turn supports our stochastic analysis of shortest paths.

We then bound the length of a shortest path by a multiplicative drift argument known from evolutionary computation [27]. We show that the length of $\mathcal{O}(\log n)$ shortest paths follows an Erlang distribution and is thereby still $\mathcal{O}(\log n)$. This result may be of independent interest, as it relaxes some of the conditions that are usually required to apply the drift theorem.

3. Notation and preliminaries. In this section, we briefly introduce hyperbolic random graphs. For a more thorough investigation of this model, we refer to [22, 26].

Let \mathbb{H}_2 be the hyperbolic plane. Following [26], we use the *native* representation. Here, a point $v \in \mathbb{H}_2$ is represented by polar coordinates (r_v, φ_v) , and r_v is the hyperbolic distance of v to the origin.¹

To construct a hyperbolic random graph G with parameters n, α , and C , consider a circle D_R with radius $R = 2 \ln n + C$ that is centered at the origin of \mathbb{H}_2 . Inside D_R , n points are distributed independently as follows. For each point v , draw φ_v uniformly at random from $[0, 2\pi)$, and draw r_v according to the probability density function

$$\rho(r) := \frac{\alpha \sinh(\alpha r)}{\cosh(\alpha R) - 1} \approx \alpha e^{\alpha(r-R)}.$$

Next, connect two points u, v if their hyperbolic distance is at most R , i.e., if

$$(1) \quad d(u, v) := \cosh^{-1}(\cosh(r_u) \cosh(r_v) - \sinh(r_u) \sinh(r_v) \cos(\Delta\varphi_{u,v})) \leq R.$$

By $\Delta\varphi_{u,v}$ we describe the small relative angle between two nodes u, v , i.e., $\Delta\varphi_{u,v} := \cos^{-1}(\cos(\varphi_u - \varphi_v)) \leq \pi$.

This results in a graph whose degree distribution follows a power law with exponent $\beta = 2\alpha + 1$ if $\alpha \geq \frac{1}{2}$, and $\beta = 2$ otherwise [22]. Since most real-world networks have been shown to have a power law exponent $\beta > 2$, we assume throughout the paper that $\alpha > \frac{1}{2}$. Gugelmann, Panagiotou, and Peter [22] proved that the average degree in this model is then $(1 + o(1)) \frac{2\alpha^2 e^{-C/2}}{\pi(\alpha-1/2)^2}$.

We now present a handful of lemmas useful for analyzing the hyperbolic random graph. Most of them can be found in [22]. We begin with an upper bound for the angular distance between two connected nodes. Consider two nodes with radial coordinates r, y . Denote by $\theta(r, y)$ the maximal radial distance such that these two nodes are connected. By (1),

$$(2) \quad \theta(r, y) = \arccos \left(\frac{\cosh(y) \cosh(r) - \cosh(R)}{\sinh(y) \sinh(r)} \right).$$

This convoluted expression is closely approximated by the following lemma.

LEMMA 4 (see [22]). *Let $0 \leq r, y \leq R$ and $y + r \geq R$. Then,*

$$\theta(r, y) = \theta(y, r) = 2e^{\frac{R-r-y}{2}} (1 \pm \Theta(e^{R-r-y})).$$

For most computations on hyperbolic random graphs, we need expressions for the probability that a sampled point falls into a certain area. To this end, Gugelmann, Panagiotou, and Peter [22] define the *probability measure* of a set $S \subseteq D_R$ as

$$\mu(S) := \int_S f(y) dy,$$

¹In this model, unlike some others (e.g., Poincaré halfplane), the radial hyperbolic distance is preserved by the embedding into the Euclidean plane.

where $f(r) := \frac{\rho(r)}{2\pi} = \frac{\alpha \sinh(\alpha r)}{2\pi(\cosh(\alpha R)-1)}$ is the probability mass of a point $p = (r, \varphi)$. By substituting the definitions of cosh and sinh, f is closely approximated by

$$(3) \quad f(r) = \frac{\alpha}{2\pi} e^{\alpha(r-R)} \cdot (1 + \Theta(e^{-\alpha R} - e^{-2\alpha r})).$$

We define the *ball* with radius x around a point (r, φ) as

$$B_{r,\varphi}(x) := \{(r', \varphi') \mid d((r', \varphi'), (r, \varphi)) \leq x\}.$$

For brevity, we write $B_r(x)$ instead of $B_{r,0}(x)$. Note that $D_R = B_0(R)$. Using these definitions, we can formulate the following lemma.

LEMMA 5 (see [22]). *For any $0 \leq r \leq R$ we have*

$$(4) \quad \mu(B_0(r)) = e^{-\alpha(R-r)}(1 + o(1)),$$

$$(5) \quad \mu(B_r(R) \cap B_0(R)) = \frac{2\alpha e^{-r/2}}{\pi(\alpha - 1/2)} \cdot (1 \pm \mathcal{O}(e^{-(\alpha-1/2)r} + e^{-r})).$$

Let us also mention a useful result from [7]. It essentially states that when two nodes are connected in the hyperbolic random graph, they remain connected even when moved closer to the center of the disk D_R . This means that a node v that has a smaller radial coordinate than v' —but the same angular coordinate—is connected to all neighbors of v' , and possibly more. The neighborhood of a node v is thus monotone in the radial coordinate r_v .

LEMMA 6 (see [7]). *Consider two nodes $u = (r_u, \varphi_u), v = (r_v, \varphi_v)$ in the hyperbolic random graph. If $d(u, v) \leq x$ and $r_u, r_v \leq x$, then it holds that*

$$d(u', v') \leq x,$$

where $u' = (r'_u, \varphi_u), v' = (r'_v, \varphi_v)$ with $r'_u \leq r_u$ and $r'_v \leq r_v$.

Finally, we often make use of the well-known inequality $1 + x \leq e^x$ that holds for all $x \in \mathbb{R}$. Let us mention here that for $x \rightarrow 0$ this inequality is actually a close approximation, as seen in the following lemma.

LEMMA 7. *Let $0 < x < 1$ and let ε be such that $1 - x = e^{-\varepsilon}$. Then, $1 - x \geq e^{-(1+\varepsilon)x}$.*

Proof. Since $1 - x = e^{-\varepsilon}$, it suffices to show that $e^{-\varepsilon} \geq e^{-(1+\varepsilon)x}$. Thus,

$$\begin{aligned} e^{-\varepsilon} &\geq e^{-(1+\varepsilon)x}, \\ \Leftrightarrow \varepsilon &\leq (1 + \varepsilon)x, \\ \Leftrightarrow (1 - x) &\leq \frac{1}{1+\varepsilon}. \end{aligned}$$

Thus, by estimating $1 + \varepsilon \leq e^\varepsilon$, the statement holds if $1 - x \leq e^{-\varepsilon}$. But this is true by assumption. □

Notice that when x is very small, that is, $x = o(1)$, it must hold that $\varepsilon = o(1)$. Thus, we have that $1 - x \geq e^{-(1+o(1))x}$ (i.e., the estimation $1 - x \approx e^{-x}$ is accurate in that case).

3.1. The Poisson point process. We often want to argue about the probability that an area $S \subseteq D_R$ contains one or more nodes. To this end, we usually apply the simple formula²

$$(6) \quad \Pr[\exists v \in S] = 1 - (1 - \mu(S))^n \geq 1 - \exp(-n \cdot \mu(S)).$$

Unfortunately, this formula becomes significantly more complicated once the positions of some nodes are already known. This introduces conditions on $\Pr[\exists v \in S]$ which are hard to grasp analytically. For instance, assume we condition on the event that all nodes are in some area $S \subset D_R$. Then, the probability that a node is sampled in $D_R \setminus S$ is always 0.

To circumvent this technical problem, we use a PPP [33]. It describes a different way of distributing nodes inside D_R . Let the random variable $\mathcal{P}_n = \{(r_1, \varphi_1), (r_2, \varphi_2), \dots, (r_N, \varphi_N)\}$ denote the set of nodes produced by the PPP. Then, \mathcal{P}_n is fully characterized by the following two properties:

- If two areas S, S' are disjoint, then the number of nodes in \mathcal{P}_n that fall within S and S' are independent random variables.
- The expected number of points in \mathcal{P}_n that fall within S is $\int_S n\mu(S)$.

One can show that the above properties imply that the number of nodes inside S follows a Poisson distribution with mean $n\mu(S)$. In particular, we obtain that the number of nodes $N = |\mathcal{P}_n|$ inside D_R is distributed as $\text{Po}(n)$, i.e., $\mathbb{E}[N] = n$ and

$$\Pr(N = n) = \frac{e^{-n}n^n}{n!} = \Theta\left(n^{-\frac{1}{2}}\right).$$

Moreover, by conditioning on $N = n$, we recover the original distribution of nodes in D_R . Thus, let P be any property that holds with probability at most $\mathcal{O}(n^{-c})$ on a hyperbolic random graph whose node set was sampled using \mathcal{P}_n . Then, P also holds with probability at most $\mathcal{O}(n^{\frac{1}{2}-c})$ in hyperbolic random graphs. This makes the PPP an extremely useful tool as any result that holds with a high enough polynomial probability directly translates to hyperbolic random graphs with an error term of $n^{\frac{1}{2}}$.

A useful side effect of this model is that (6) changes to an equality, that is, we have $\Pr[\exists v \in S] = 1 - \exp(-n \cdot \mu(S))$.

Let us finally mention that conditioning a PPP on the existence of a point does not change its distribution; see, e.g., [23, Proposition 5]. More formally, denote by \mathcal{P}_n^x the random point set created by a PPP when conditioning on the probability that a given point $x \in \mathbb{H}_2$ is in the result. Then, $\mathcal{P}_n \cup \{x\}$ is equivalent in distribution to \mathcal{P}_n^x .

We indicate explicitly whenever we use the PPP instead of the normal hyperbolic random graph. In particular, section 4 does not require use of this technique.

4. Polylogarithmic upper bound. In this section, we show a polylogarithmic upper bound on the diameter of the hyperbolic random graph. The proof proceeds in two steps. First, we show that nodes close to the center form a connected component of diameter $\mathcal{O}(\log \log n)$. This covers all nodes that are at least b_O away from the boundary of D_R . We call this area $B_I := B_0(R - b_O)$ the *inner band*, where $b_O = \Theta(\log R)$ will be chosen suitably later. See Figure 1 for an illustration. Afterward, we prove that all remaining nodes in the *outer band* $B_O := D_R \setminus B_I$ form components of at most polylogarithmic diameter.

²Note that we write $\exists v \in S$ informally to mean whether $S \subseteq D_R$ contains a vertex $v \in V$. To be formally precise, we would have to write $V \cap S \neq \emptyset$. Since it is usually clear from the context that v refers to a node, we choose to keep notation concise.

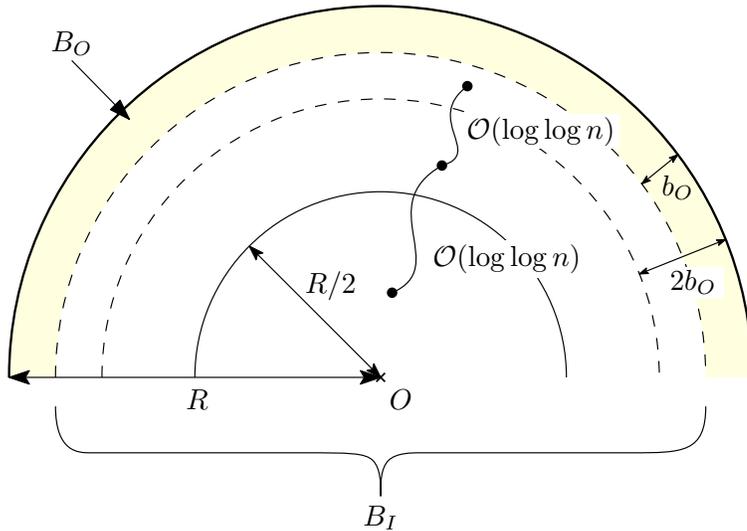


FIG. 1. The disk D_R is separated into an inner band $B_I = B_0(R - b_O)$ and an outer band $B_O = D_R \setminus B_I$ of thickness b_O . All nodes closer than $R/2$ to the center form a clique and thus have diameter 1. All nodes closer than $R - 2b_O$ to the center have a path of length $\mathcal{O}(\log \log n)$ to a node in $B_0(R/2)$. All nodes closer than $R - b_O$ have a path of length $\mathcal{O}(\log \log n)$ to a node in $B_0(R - 2b_O)$. Thus, all nodes in B_I are connected, and the diameter of the induced graph is $\mathcal{O}(\log \log n)$.

During the proof, it will sometimes be useful to use a discretization of the radial coordinates. To this end, we partition D_R into R layers of constant thickness 1, where the first layer contains all nodes furthest away from the origin. Thus, all nodes with radial coordinates in $(R - i, R - i + 1]$ are in layer i .³ We denote the layer i by $L_i := B_0(R - i + 1) \setminus B_0(R - i)$, where $i \geq 1$. The next lemma gives a bound on the maximal angle that two nodes in layers i, j may have while still being connected. Recall that for two nodes u, v with fixed radius r_u, r_v , the term $\theta(r_u, r_v)$ describes the maximum angle $\Delta\varphi_{u,v}$ such that u, v are still connected; see (2).

LEMMA 8. Let $1 \leq i, j \leq R/2$, and consider two nodes $u \in L_i, v \in L_j$. Then,

$$\frac{2}{e} e^{\frac{i+j-R}{2}} (1 + \Theta(e^{i+j-R})) \leq \theta(r_u, r_v) \leq 2e^{\frac{i+j-R}{2}} (1 + \Theta(e^{i+j-R})).$$

Proof. By Lemma 4, $\theta(r_v, r_w) = 2e^{\frac{R-r_v-r_w}{2}} (1 + \mathcal{O}(e^{R-r_v-r_w}))$. Since $v \in L_i$, we have that $R - i \leq r_v \leq R - i + 1$, and similarly $R - j \leq r_w \leq R - j + 1$. Thus, we obtain

$$2e^{\frac{i+j-2-R}{2}} (1 + \Theta(e^{i+j-R})) \leq \theta(r_u, r_v) \leq 2e^{\frac{i+j-R}{2}} (1 + \Theta(e^{i+j-R})). \quad \square$$

Furthermore, we require an estimate for the probability that a node u in layer L_i has a neighbor in layer L_j . To this end, the next lemma computes the probability mass of the area $B_u(R) \cap L_j$.

LEMMA 9. Let $1 \leq i, j \leq R/2$, and consider a node $u \in L_i$. Then, for all $\alpha > 0$,

$$\mu(L_j \cap B_u(R)) = \Theta\left(e^{-\alpha j + \frac{i+j-R}{2}}\right).$$

³Though we never need it explicitly, we note that the last layer only covers $[0, R - \lfloor R \rfloor]$.

If further $(i + j)/R < 1 - \varepsilon$ for some constant $\varepsilon > 0$ holds, we obtain the explicit bounds for large n

$$\frac{\alpha}{e\pi} \cdot e^{-\alpha j + \frac{i+j-R}{2}} \leq \mu(L_j \cap B_v(R)) \leq \frac{e^\alpha \alpha}{\pi} \cdot e^{-\alpha j + \frac{i+j-R}{2}}.$$

Proof. We have by Lemma 4 and (3)

$$\begin{aligned} \mu(L_j \cap B_v(R)) &\leq \mu(L_j)\theta(R - i, R - j) \\ &\leq f(R - j + 1) \cdot 2e^{\frac{i+j-R}{2}} (1 + \Theta(e^{i+j-R})) \\ &\leq \frac{\alpha}{\pi} e^{\frac{i+j-R}{2} + \alpha(R-j+1) - \alpha R} \left(1 + \Theta\left(e^{-\alpha R} - e^{-2\alpha(R-i)} + e^{i+j-R}\right)\right) \\ &\leq \frac{e^\alpha \alpha}{\pi} e^{\frac{i+j-R}{2} - \alpha j} (1 + \Theta(e^{-\alpha R} + e^{i+j-R})). \end{aligned}$$

For the other direction, a similar computation yields

$$\begin{aligned} \mu(L_j \cap B_v(R)) &\geq \mu(L_j)\theta(R - i + 1, R - j + 1) \\ &\geq f(R - j) \cdot \frac{2}{e} e^{\frac{i+j-R}{2}} (1 + \Theta(e^{i+j-R})) \\ &\geq \frac{\alpha}{e\pi} e^{\frac{i+j-R}{2} + \alpha(R-j) - \alpha R} \left(1 + \Theta\left(e^{-\alpha R} - e^{-2\alpha(R-i)} + e^{i+j-R}\right)\right) \\ &\geq \frac{\alpha}{e\pi} e^{\frac{i+j-R}{2} - \alpha j} (1 + \Theta(e^{-\alpha R} + e^{i+j-R})). \quad \square \end{aligned}$$

Using Lemmas 8 and 9, we can now prove that every node $v \in B_I$ has a path of length $\mathcal{O}(\log \log n)$ that leads to a node in $B_0(R/2)$. Recall that the inner band was defined as $B_I := B_0(R - b_O)$.

LEMMA 10. Consider a node $v \in L_i \subset B_I$. If $\alpha < 1$, it holds with probability $1 - \mathcal{O}(n^{-3})$ that

1. if $i \in [b_O, 2b_O]$, then v has a neighbor in layer L_{i+1} , and
2. if $i \in [2b_O, R/2]$, then v has a neighbor in layer L_j for $j = \frac{\alpha}{2\alpha-1}i$.

Proof. We begin by proving the first claim. By combining (6) and Lemma 9, the probability that node $v \in L_i$ does not contain a neighbor in $L_j = L_{i+1}$ is at most

$$\begin{aligned} (7) \quad \exp(-n\mu(L_j \cap B_v(R))) &\leq \exp\left(-\Theta(1) \cdot e^{R/2} \cdot e^{-\alpha j + \frac{i+j-R}{2}}\right) \\ &= \exp\left(-\Theta(1) \cdot e^{-\alpha j + \frac{i+j}{2}}\right) \\ &= \exp\left(-\Theta(1) \cdot e^{-\alpha(i+1) + i + \frac{1}{2}}\right) \\ &= \exp\left(-\Theta(1) \cdot e^{(1-\alpha)i}\right). \end{aligned}$$

We now choose b_O appropriately. Since in the first case we have $i \geq b_O$, our goal is to set b_O to a value such that above term is at most $\mathcal{O}(n^{-3})$. This is achieved by $b_O := \frac{\log R}{1-\alpha} + c$ for some large enough constant c . Then, we have

$$\exp(-n\mu(L_j \cap B_v(R))) \leq \exp(-\Theta(1) \cdot e^{\log R + (1-\alpha)c}) \leq \exp(-3 \log n).$$

This proves part 1 of the claim. For part 2, we set $j = \frac{\alpha}{2\alpha-1}i$ and $i \geq 2b_O$ in (7). It is then upper bounded by

$$\begin{aligned} \exp\left(-\Theta(1) \cdot e^{-\frac{\alpha^2}{2\alpha-1}i + \frac{i}{2} + \frac{\alpha}{4\alpha-2}i}\right) &= \exp\left(-\Theta(1) \cdot e^{(3\alpha-1-2\alpha^2)\frac{i}{4\alpha-2}}\right) \\ &= \exp\left(-\Theta(1) \cdot e^{(1-\alpha)\frac{i}{2}}\right) \\ &\leq \exp\left(-\Theta(1) \cdot e^{\log R + \frac{1-\alpha}{2}c}\right) \\ &\leq \exp(-3 \log n), \end{aligned}$$

which again holds if the constant c in $b_O = \frac{\log R}{1-\alpha} + c$ was chosen large enough. \square

Plugging everything together, we obtain that the diameter of the inner band is at most $\mathcal{O}(\log \log n)$.

COROLLARY 11. *Let $\frac{1}{2} < \alpha < 1$. With probability $1 - \mathcal{O}(n^{-2})$, each pair of nodes $u, v \in B_I$ in the hyperbolic random graph is connected by a path of length $\mathcal{O}(\log \log n)$.*

Proof. By a union bound over at most n nodes in B_I , the statement in Lemma 10 holds for every node in B_I with probability $1 - \mathcal{O}(n^{-2})$. Consider thus a node in a layer $[b_O, 2b_O]$. Since by Lemma 10, every such node has a neighbor in the subsequent layer, we need at most $\mathcal{O}(\log \log n)$ hops to reach a node in layer $i \in [2b_O, \frac{R}{2}]$. Similarly, every such node has a neighbor in layer $j = \frac{\alpha}{2\alpha-1}i = (1 + \varepsilon)i$ for some constant $\varepsilon > 0$. Thus, we need at most $\mathcal{O}(\log R) = \mathcal{O}(\log \log n)$ hops to reach some node in $B_0(R/2)$. Since all nodes in $B_0(R/2)$ form a clique by the triangle inequality, we therefore obtain that all nodes in B_I form a connected component with diameter $\mathcal{O}(\log \log n)$. \square

4.1. Outer band. By Corollary 11, we obtain that the diameter of the graph induced by nodes in the inner band B_I is at most $\mathcal{O}(\log \log n)$. In particular, since all nodes in $B_0(R/2)$ belong to the giant component [7], the nodes in the inner band all belong to the giant component as well. In this section, we prove that each component in the outer band B_O has a polylogarithmic diameter. One can then conclude that the overall diameter of the giant component is at most polylogarithmic, since the diameter is then dominated by the components in the outer band.

To argue over sequences of nodes on a path, we introduce the concept of *betweenness*: We say that a node v is *between* two nodes u, w if $\Delta\varphi_{u,v} + \Delta\varphi_{v,w} = \Delta\varphi_{u,w}$. As an example, consider the nodes $u = (r_1, 0), v = (r_2, \frac{\pi}{2})$ and $w = (r_3, \pi)$. Then, v lies between u and w , but w does not lie between u and v as $\Delta\varphi_{u,v} = \pi/2$ but $\Delta\varphi_{u,w} + \Delta\varphi_{w,v} = \frac{3}{4}\pi$.

If a node v is between two connected nodes u, w and has a small radial coordinate, it is also connected to u, w as shown by the following lemma. Figure 2 contains an illustration.

LEMMA 12. *Let $u, v, w \in V$ be nodes such that v lies between u and w , and let $\{u, w\} \in E$. If $r_v \leq r_u$, then v is connected to w .*

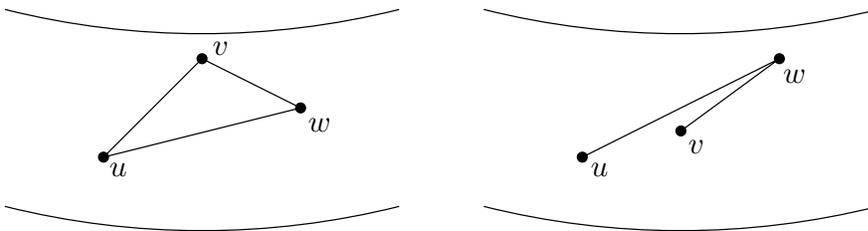


FIG. 2. Illustration of the statement in Lemma 12. By definition, the edge $\{u, w\}$ passes under v in both cases.

Proof. By Lemma 6, we know that if two nodes $(r_1, \varphi_1), (r_2, \varphi_2)$ are connected, then so are $(r'_1, \varphi_1), (r'_2, \varphi_2)$, where $r'_1 \leq r_1$ and $r'_2 \leq r_2$. Observe that $\Delta\varphi_{v,w} \leq \Delta\varphi_{u,w}$, since v lies between u, w . As the hyperbolic distance is monotone on $\Delta\varphi \leq \pi$, we therefore have $d(v, w) \leq d(u, w) \leq R$. \square

Note that by symmetry, if $r_v \leq r_u$ and $r_v \leq r_w$ both hold, then v is connected to both u, w .

We say that an edge $\{u, w\}$ passes under v if the requirement of Lemma 12 is fulfilled. Using this, we are ready to show Theorem 1. In this argument, we investigate the angular distance a path can at most traverse until it passes under a node in B_I . By Lemma 12, we then have with high probability (w.h.p.) a short path to the center $B_0(R/2)$ of the graph.

THEOREM 1. *Let $2 < \beta < 3$. The diameter of the giant component in the hyperbolic random graph is $\mathcal{O}((\log n)^{\frac{2}{3-\beta}})$ with probability $1 - \mathcal{O}(n^{-2})$.*

Proof. Partition the hyperbolic disc into n disjoint sectors of equal angle $\Theta(1/n)$. Recall that $b_O = \frac{\log R}{1-\alpha} + c$ for a large enough constant c . By (4) and (6), the probability that k consecutive sectors contain no node in B_I is

$$\begin{aligned} (1 - \Theta(k/n) \cdot \mu(B_0(R - b_O)))^n &\leq \exp\left(-\Theta(1) \cdot k \cdot e^{-\alpha \log R/(1-\alpha)}\right) \\ &= \exp\left(-\Theta(1) \cdot k \cdot (\log n)^{-\frac{\alpha}{1-\alpha}}\right). \end{aligned}$$

By choosing $k := \Theta((\log n)^{\frac{1}{1-\alpha}})$ large enough, we obtain that with probability $1 - \mathcal{O}(n^{-3})$, there are no k such consecutive sectors. By a Chernoff bound, the number of nodes in k such consecutive sectors is $\Theta((\log n)^{\frac{1}{1-\alpha}})$ with probability $1 - \mathcal{O}(n^{-3})$. Applying a union bound, we get that with probability $1 - \mathcal{O}(n^{-2})$, every sequence of k consecutive sectors contains at least one node in B_I and at most $c'k = \Theta(k)$ nodes in total, for some constant c' .

Consider now a node $v \in B_O$ that belongs to the giant component. Any path (without loops) from v that is longer than $c'k$ must thus span more than k sectors. This holds since by the argument above there are no k consecutive sectors containing more than $c'k$ nodes.

In particular, this path then either uses a node in B_I or passes under a node in B_I . By Lemma 12, there must thus exist a path from v to some node $u \in B_I$ of length at most $\mathcal{O}(k)$. From u , there is a path of length $\mathcal{O}(\log \log n)$ to the center $B_0(R/2)$ of the hyperbolic disc by Corollary 11. Since this holds for all nodes and the center forms a clique, the diameter is therefore $\mathcal{O}((\log n)^{\frac{1}{1-\alpha}}) = \mathcal{O}((\log n)^{\frac{2}{3-\beta}})$. \square

From the proof it follows that every component inhabiting $\Omega((\log n)^{\frac{2}{3-\beta}})$ sectors is connected to the center. We derive the following corollary.

COROLLARY 13. *Let $2 < \beta < 3$. The second largest component of the hyperbolic random graph is of size at most $\mathcal{O}((\log n)^{\frac{2}{3-\beta}})$ with probability $1 - \mathcal{O}(n^{-3/2})$.*

Proof. The second largest component may not be connected to a node in B_I . Otherwise, as shown above, it belongs (w.h.p.) to the giant component. By the same argument as in Theorem 1, the largest such component can contain at most $\mathcal{O}((\log n)^{\frac{2}{3-\beta}})$ nodes. \square

These bounds improve upon the results in [24], who show an upper bound on the diameter of $\mathcal{O}((\log n)^{\frac{32}{(3-\beta)(5-\beta)}+1})$ and an upper bound on the second largest

component of $\mathcal{O}((\log n)^{\frac{64}{(3-\beta)(5-\beta)}+1})$. As we will see in Theorem 3, however, the lower bound on the diameter is only $\Omega(\log n)$. It is an open problem to show a tight result for $\frac{1}{2} < \alpha < 1$. For the case $\alpha > 1$, we bridge this gap in the next section.

5. Logarithmic upper bound. In this section, we show that the diameter of the hyperbolic random graph is $\mathcal{O}(\log n)$, if $\beta > 3$, or, equivalently, $\alpha > 1$. Note that we use a different intuition for our analysis in this section than before. Instead of showing the existence of short paths to the center of the graph from all nodes, we show that all shortest paths terminate after $\mathcal{O}(\log n)$ steps since they reach the boundary of D_R . This holds because for each node v , its largest degree neighbor has (in expectation) a degree smaller than v itself. Thus, a shortest path visits successively nodes of a smaller and smaller degree, until it cannot continue.

In this section, we prove all intermediate results using the PPP; see subsection 3.1. Recall that we defined a layer L_i as $B_0(R - i + 1) \setminus B_0(R - i)$. We begin by showing that each node's largest degree neighbor is of small degree or, equivalently, is in a small layer. Here, we have to deal with an additional technicality. When sampling a shortest path, we already have uncovered a neighbor of the current node. To resolve this issue, recall that conditioning a PPP on a point is equivalent to adding this point to the distribution, as established in subsection 3.1.

Given a node v and a *forbidden* neighbor f , we denote by the random variable $Y(v, f)$ the largest layer in which v has a neighbor that is not f . If v has no other neighbors than f , we set $Y(v, f) = 0$. We show the following.

LEMMA 14. *Let $i \geq \varepsilon, j$ be such that $v \in L_i$ and $f \in L_j$. Then, there exist constants $\varepsilon, \delta > 0$ such that*

$$\mathbb{E}_{\mathcal{P}_n^{v,f}}[Y(v, f)] \leq (1 - \delta)i.$$

Proof. Recall that $\mathcal{P}_n^{v,f}$ is distributionally equivalent to $\mathcal{P}_n \cup \{v, f\}$. We may therefore consider the standard PPP with the points v, f added for this proof.

We first compute the probability that $v \in L_i$ has no neighbors in layer x . This happens when no nodes are sampled in the area $L_x \cap B_v(R)$. Recall that in the PPP, it holds that $\Pr[|\mathcal{P}_n^{v,f} \cap S| > 0] = 1 - \exp(-n\mu(S))$; see subsection 3.1. Thus, by Lemma 9,

$$\begin{aligned} \Pr[|\Gamma(v) \cap L_x| = 0] &= \exp(-n\mu(L_x \cap B_v(R))) \\ (8) \qquad \qquad \qquad &= \exp(-\Theta(1) \cdot e^{\frac{i}{2} - (\alpha - \frac{1}{2})x}). \end{aligned}$$

We now compute the probability that all neighbors $\Gamma(v) \setminus f$ are below layer m .

$$\begin{aligned} \Pr_{\mathcal{P}_n^{v,f}}[Y(v, f) < m] &= \Pr[\forall x \geq m: |\Gamma(v) \cap L_x| = 0] \\ &= \prod_{x \geq m} \exp\left(-\Theta(1) \cdot e^{\frac{i}{2} - (\alpha - \frac{1}{2})x}\right) \\ &= \exp\left(-\Theta(1) \cdot \sum_{x \geq m} e^{\frac{i}{2} - (\alpha - \frac{1}{2})x}\right) \\ &= \exp\left(-\Theta(1) \cdot e^{\frac{i}{2} - (\alpha - \frac{1}{2})m}\right), \end{aligned}$$

since the sum is geometric.

Finally, to compute the expectation of $Y(v, f)$, we sum over the complementary cumulative distribution function. This yields

$$\begin{aligned} \mathbb{E}_{\mathcal{P}_n^{v,f}}[Y(v, f)] &= \sum_{m=1}^{\infty} \Pr_{\mathcal{P}_n^{v,f}}[Y(v, f) \geq m] \\ &= \sum_{m=1}^{\infty} \left(1 - \Pr_{\mathcal{P}_n^{v,f}}[Y(v, f) < m]\right) \\ &\leq \sum_{m=1}^{\infty} \left(1 - \exp\left(-\Theta(1) \cdot e^{\frac{i}{2} - (\alpha - \frac{1}{2})m}\right)\right). \end{aligned}$$

Since the first $\frac{1}{2\alpha-1}i$ terms of the sum are close to 1, we simply overestimate them with 1. For the remaining part of the sum, we again apply the inequality $1 - e^{-x} \leq x$ and obtain

$$\begin{aligned} \mathbb{E}_{\mathcal{P}_n^{v,f}}[Y(v, f)] &\leq \frac{i}{2\alpha - 1} + \sum_{m=\frac{1}{2\alpha-1}i}^{\infty} \left(1 - \exp\left(-\Theta(1) \cdot e^{\frac{i}{2} - (\alpha - \frac{1}{2})m}\right)\right) \\ &\leq \frac{i}{2\alpha - 1} + \sum_{m=\frac{1}{2\alpha-1}i}^{\infty} \Theta(1) \cdot e^{\frac{i}{2} - (\alpha - \frac{1}{2})m} \\ &\leq \frac{i}{2\alpha - 1} + \Theta(1). \end{aligned}$$

To prove the claim, we choose $\varepsilon > 0$ as a large enough constant and $\delta > 0$ as a small enough constant. Then, since $i \geq \varepsilon$ by assumption and $\alpha > 1$, it holds that $\frac{1}{2\alpha-1} + \Theta(\frac{1}{i}) \leq 1 - \delta$. □

We note that the same result can be achieved when there is no forbidden node f as the PPP distribution does not change. In this case, we simply write $Y(v)$ instead of $Y(v, f)$.

Assume we now fix some vertex u and sample an arbitrary shortest path $\pi = [u = V_0, V_1, V_2, \dots]$. We want to obtain a bound on the length $|\pi|$ that holds w.h.p. Unfortunately, this process is hard to analyze exactly since it is governed by many dependencies. For example, V_2 may not be connected to V_0 , as otherwise, π is not a shortest path. We may, however, analyze an alternative process that is closely related. To this end, consider the following sequence of random variables, also called a *random walk*:

$$(Y_i)_{i \geq 1}, \quad Y_1 := Y(V_0), \quad Y_i := Y(V_{i-1}, V_{i-2}) \text{ if } i \geq 2.$$

Recall that $Y(V_{i-1}, V_{i-2})$ denotes the largest layer in which V_{i-1} has a neighbor that is not V_{i-2} . It is therefore immediate that V_i is always in a layer smaller than or equal to Y_i , since Y_i denotes the highest layer in which the shortest path can continue. Further, if $Y_i = 0$, then $|\pi| \leq i$ as V_{i-1} has no further neighbors apart from V_{i-2} . Recall now that a node's neighborhood is monotone in its radial coordinate (i.e., the smaller the r_v , the more neighbors v has; see Lemma 6). Thus, we may overestimate the radial coordinates of the nodes V_0, V_1, \dots and obtain an upper bound on $|\pi|$, since each node in π may only gain additional neighbors.

A natural candidate for this overestimation is to use the upper bounds given by $Y(\cdot)$. For example, we know that V_0 has no neighbors in layers above $Y(V_0)$, thus we may overestimate that V_1 has radial coordinate $R - Y(V_0)$. The next lemma formalizes this intuition by giving a random walk $(X_i)_{i \geq 1}$ that *dominates* $(Y_i)_{i \geq 1}$, that is, it holds $\Pr[X_i \geq x] \geq \Pr[Y_i \geq x]$ for all i and x .

LEMMA 15. Consider the random walk $(X_i)_{i \geq 1}$ with $X_i \in \mathbb{N}$, $X_1 := Y_1$ and distribution

$$(9) \quad \begin{aligned} \Pr[X_{i+1} \geq j \mid X_i = \ell] &= 1 - \exp(-ce^{\frac{\ell}{2} - (\alpha - \frac{1}{2})j}) && \text{if } \ell > 0, \\ \Pr[X_{i+1} = 0 \mid X_i = 0] &= 1 && \text{otherwise.} \end{aligned}$$

If c is a large enough constant, this random walk dominates $(Y_i)_{i \geq 1}$.

Proof. We need to give a coupling between the two random walks on which X_i is always greater than or equal to Y_i . By definition, this holds for X_1 and Y_1 . We now assume inductively that such a coupling exists for $1, \dots, i$ and show how to produce it for $i + 1$.

We first reveal $\Gamma(V_i)$, i.e., all neighbors of V_i . Obviously, the shortest path formed by the vertices V_1, \dots, V_i will continue on a node from $\Gamma(V_i) \setminus V_{i-1}$ or end at V_i . Consider now a fresh instance of a hyperbolic random graph in which no nodes have been sampled yet. Assume that in this new instance, we place a node V'_i at position $(R - X_i, \varphi_{V_i})$. Observe that this node has a radius smaller than or equal to V_i , since by induction we know that $X_i \geq Y_i$. Further, we have that by Lemma 6, $B_{V'_i}(R) \cap B_0(R)$ is a superset of $B_{V_i}(R) \cap B_0(R)$. Since we are in the PPP, vertices in the additional area $B_{V'_i}(R) \cap B_0(R) \setminus B_{V_i}(R)$ may be sampled independently from $B_{V_i}(R) \cap B_0(R)$ since these regions are disjoint.

Thus, we may couple the neighborhood of V'_i to contain copies of all nodes $\Gamma(V_i)$ and possibly more. Therefore, the largest layer containing a neighbor of V'_i that is not the copy of V_{i-1} satisfies $Y(V'_i, V_{i-1}) \geq Y(V_i, V_{i-1})$. And as derived in Lemma 14, there is a constant c such that $Y(V'_i, V_{i-1})$ is distributed as

$$\Pr[Y(V'_i, V_{i-1}) \geq j \mid X_i = \ell] \leq 1 - \exp(-ce^{\frac{\ell}{2} - (\alpha - \frac{1}{2})j}).$$

This agrees with (9), and since $Y(V'_i, V_{i-1}) \geq Y(V_i, V_{i-1})$, so is $X_{i+1} \geq Y_{i+1}$. □

Observe that Lemma 14 shows that by definition of $(X_i)_{i \geq 1}$, it also holds that

$$\mathbb{E}[X_{i+1} \mid X_i] \leq (1 - \delta)X_i$$

if X_i is at least a large enough constant ε . In other words, $(X_i)_{i \geq 1}$ has a so-called multiplicative drift toward 0 while it is above some constant layer ε .

We now finally turn to analyzing the length of the random walk $(X_i)_{i \geq 1}$ until it reaches 0 and thus, by our explanations above, the length $|\pi|$ of a shortest path. Let $T := \min\{i \mid X_i = 0\}$ be the random variable describing the number of iterations until X_i hits 0. We bound T by a multiplicative drift theorem as presented by Lehre and Witt [27, Theorem 7] and originally developed by Doerr and Goldberg [14, Theorem 1] for the analysis of evolutionary algorithms. For the sake of completeness, we restate their result.

THEOREM 16 (from [14, 27]). Let $(X_i)_{i \geq 1}$ be a stochastic process over some state space $\{0\} \cup [x_{\min}, x_{\max}]$, where $x_{\min} > 0$. Suppose that there exists some $0 < \delta < 1$ such that $\mathbb{E}[X_{i+1} \mid X_0, \dots, X_i] \leq (1 - \delta)X_i$. Then, for the hitting time $T := \min\{i \mid X_i = 0\}$ it holds that

$$\Pr[T \geq \frac{1}{\delta}(\ln(X_0/x_{\min}) + r) \mid X_0] \leq e^{-r} \quad \forall r > 0.$$

Unfortunately, in our case, the multiplicative drift vanishes once $X_i < \varepsilon$. We therefore split the random walk $(X_i)_{i \geq 1}$ into several stages, where a certain stage

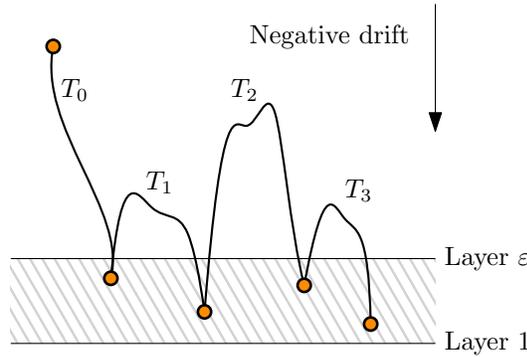


FIG. 3. A sketch of the diameter proof. Until layer ε , there is a negative drift on the next node of the random walk. Between layer ε and layer 1, there is a constant probability for the walk to end. Thus, the overall walk visits the marked area $\mathcal{O}(\log n)$ times. Naively, the random walk is thus of length $\mathcal{O}(\log^2 n)$; however, Lemma 17 shows that it is w.h.p. of length $\mathcal{O}(\log n)$.

ends when the random walk falls below ε ; see Figure 3. Thus, we map all points $x < \varepsilon$ to 0 and set $X_0 \leq R$ and $x_{\min} = \varepsilon$. Using Lemma 14 this shows that

$$(10) \quad \Pr[T_0 \geq \frac{1}{\delta} \cdot (\log \log n - \log \varepsilon + r)] \leq e^{-r},$$

where T_0 refers to the hitting time of the first stage of the random walk. Hence, by setting $r = 4 \log n$ we obtain that with probability $1 - \mathcal{O}(n^{-4})$, the random walk $(X_i)_{i \geq 1}$ ends after $\mathcal{O}(\log n)$ steps below ε .

Once X_i crosses ε , we consider two possibilities: either the random walk ends, or it continues at ε . For the latter case, observe that $\Pr[X_{i+1} \geq j \mid X_i]$ is monotonously increasing in X_i ; see (9). Thus, increasing X_i to ε results in a dominating random walk. Again, by Theorem 16, its stopping time $T_j, j > 0$, is distributed as

$$(11) \quad \Pr[T_j \geq \frac{1}{\delta} \cdot r] \leq e^{-r}.$$

As we prove later, the probability for X_i to end is constant if it is below ε . By the Chernoff bound, the random walk therefore visits w.h.p. at most $\mathcal{O}(\log n)$ nodes below layer ε before stopping. A naive application of (10) and (11) thus yields that $(X_i)_{i \geq 1}$ is w.h.p. of length $\mathcal{O}(\log^2 n)$; see Figure 3. It is, however, possible to improve this result. The reason is that when adding together $\mathcal{O}(\log n)$ random variables that are exponentially distributed, most of them will be of constant size. Thus, intuitively, the main contribution to the sum comes from just one variable achieving a value of $\Omega(\log n)$, whereas all others are small. In the following, we prove this intuition and thereby show that if $(X_i)_{i \geq 1}$ drops below layer ε not more than $\mathcal{O}(\log n)$ times, then the total length of the random walk is still w.h.p. $\mathcal{O}(\log n)$.

LEMMA 17. Let $(T_j)_{j=1 \dots x}$ be $x = \lceil c \log n \rceil$ independent random variables, each with distribution as in (11). Then, with probability $1 - \mathcal{O}(n^{-5})$, $\sum_{j=1}^x T_j \leq \mathcal{O}(\log n)$.

Proof. As we only know an exponential tail bound but not the exact distribution of T_j , we instead investigate the random variables T'_j whose distribution is given by

$$\Pr[T'_j \geq r] = \exp(-\delta r).$$

Note that T'_j dominates T_j ; therefore it suffices to find a tail bound on $T_x^* := \sum_{j=0}^{x-1} T'_j$. Since T_x^* is a sum of x exponentially distributed variables with equal mean, the distribution of T_x^* is an Erlang(x, δ) distribution (a special case of the Gamma distribution; see [17]) and we have

$$(12) \quad \Pr[T_x^* \geq t] = \sum_{i=0}^{x-1} \frac{1}{i!} e^{-\delta t} (\delta t)^i.$$

To estimate this term, we observe that a random variable P that is Poisson distributed with mean δt has probability mass function $\Pr[P = i] = (\delta t)^i e^{-\delta t} \frac{1}{i!}$. This term equals the summands in (12), and we can therefore write

$$\Pr[T_x^* \geq t] = \sum_{i=0}^{x-1} \Pr[P = i] = \Pr[P < x].$$

By a Chernoff bound for Poisson variables [29], we can estimate this with

$$\Pr[P < x] \leq \frac{e^{-\delta t} (e\delta t)^x}{x^x},$$

as long as $x = \lceil c \log n \rceil \leq \delta t$. Choosing $t = c' \log n$ large enough, we obtain

$$\begin{aligned} \Pr[T_x^* \geq t] &\leq \frac{e^{-\delta c' \log n} (e\delta c' \log n)^{c \log n}}{(c \log n)^{c \log n}} \\ &= n^{-\delta c'} \left(\frac{e\delta c'}{c}\right)^{c \log n} \\ &= n^{-\delta c' + c \log(\frac{e\delta c'}{c})} \leq n^{-5}. \end{aligned} \quad \square$$

Using our auxiliary lemmas, we can prove that the diameter of the hyperbolic random graph is $\mathcal{O}(\log n)$ if $\beta > 3$ or, equivalently, $\alpha > 1$.

THEOREM 2. *Let $\beta > 3$. Then, the diameter of the hyperbolic random graph is $\mathcal{O}(\log n)$ with probability $1 - \mathcal{O}(n^{-2})$.*

Proof. We show that for each of the $\mathcal{O}(n^2)$ connected node pairs, there exists a shortest path of length $\mathcal{O}(\log n)$ w.h.p. It then follows from a union bound over all pairs of nodes that the diameter is upper bounded by $\mathcal{O}(\log n)$.

Consider any node v and a shortest path beginning in v . The length of the shortest path is dominated by the length of the random walk $(X_i)_{i \geq 1}$ as defined in (9). Let $s = c \log n$ for some large enough constant c , and let ε be a large enough constant. By Lemma 14 and (10), $X_s < \varepsilon$ with probability $1 - \mathcal{O}(n^{-4})$. The probability that $X_{s+1} = 0$ is then by (9)

$$\begin{aligned} \Pr[X_{s+1} = 0 \mid X_s < \varepsilon] &= \Pr[X_{s+1} < 1 \mid X_s < \varepsilon] \\ &\geq \exp\left(-ce^{\frac{\varepsilon}{2} - (\alpha - \frac{1}{2})}\right) \\ &= \Theta(1). \end{aligned}$$

Thus, for a large enough constant c' , the probability that the random walk $(X_i)_{i \geq 1}$ returns more than $c' \log n$ times to a value $< \varepsilon$ is at most n^{-5} . Consequently, we may apply Lemma 17 and obtain that with probability $1 - \mathcal{O}(n^{-5})$, the length of the walk $(X_i)_{i \geq 1}$ is $\mathcal{O}(\log n)$. Thereby, the length of a shortest path from any node v in the Poisson point model is at most $\mathcal{O}(\log n)$ with a probability of at least $1 - \mathcal{O}(n^{-5})$. By the union bound, it thus holds that all $\mathcal{O}(n^2)$ shortest paths in the graph have a length of at most $\mathcal{O}(\log n)$ with probability $1 - \mathcal{O}(n^{-3})$. Finally, since we have done the analysis in the Poisson point model, we have that the probability that the shortest path in the hyperbolic random graph is $\mathcal{O}(\log n)$ with probability at least $1 - \mathcal{O}(n^{-2})$. \square

Let us conclude this section by mentioning that the case $2 < \beta \leq 3$ remains an open problem. Even though we provided a better polylogarithmic bound on the

diameter, the lower bound in the next section only produces a path of length $\Omega(\log n)$. These bounds leave an open gap, and until now it remains unclear whether the diameter is truly logarithmic or in fact polylogarithmic.

The main problem in proving a better bound in this case lies within the outer band B_O . Consider a similar approach as in Theorem 2: if $\beta < 3$, then the random walk Y_i has a multiplicative drift toward the center of D_R . While this sounds promising, there is a significant problem. In our case, we may simply discard the dependencies of previously visited nodes. After all, they only may exclude potential areas of D_R to contain neighbors, and discarding them increases the (expected) layer of the next vertex.

This estimation does not hold the other way around. If the random walk is to reach the center of the graph, we may not discard these dependencies as doing so decreases the length of the random walk. Thus, one has to consider the dependencies of previously visited nodes. This is difficult, since the influence of an earlier node depends on its (angular) distance from the current node and its radial coordinate. Imagine, for instance, that $X_1 = R/2$, and $X_2 = 1$. We would now like to sample X_3 only depending on X_2 . We know, however, that X_3 cannot be in a large layer like $R/2$. Otherwise, the node represented by X_3 would likely have been connected to the node represented by X_1 , and thus X_2 cannot be the layer containing the largest neighbor of the node in X_1 .

What makes matters even worse is that one needs to consider the conditions of *all* preceding nodes, not only the last. This combination of factors makes an analysis technically challenging. Compared to other random graph models (see Table 1), however, it would seem highly surprising if the diameter is indeed polylogarithmic. Furthermore, the upper bound on the diameter $\mathcal{O}((\log n)^{\frac{2}{3-\beta}})$ in Theorem 1 increases as $\beta \rightarrow 3$; however, for $\beta > 3$, the diameter is $\mathcal{O}(\log n)$ by Theorem 2. While such an abrupt phase transition seems unnatural, however, it is not completely unreasonable. The largest component in hyperbolic random graphs for $2 < \beta < 3$ is of linear size, whereas for $\beta > 3$ it is only of polynomial, sublinear order. The lower connectivity might disconnect long paths and therefore decrease the diameter overall. Nevertheless, we believe this to be unlikely and conclude this section with the following conjecture.

CONJECTURE 18. *The diameter of the hyperbolic random graph with power law exponent $2 < \beta \leq 3$ is $\mathcal{O}(\log n)$ w.h.p.⁴*

6. Logarithmic lower bound. Kiwi and Mitsche [24] provide a proof for the existence of a path component of length $\Theta(\log n)$ w.h.p. In this section, we present a slightly simpler proof that there exists a component with diameter of $\Omega(\log n)$. We achieve this by considering $\Theta(\log n)$ subsequent sectors of angle $\Theta(\frac{1}{n})$, such that each sector contains exactly one node in layer L_1 and no further nodes. We can show that such a sequence of sectors occurs at least once in the graph w.h.p. and that it forms a path of length $\Omega(\log n)$ without shortcuts. In the case where $2 < \beta < 3$, we can further show that this path component is connected to B_I . This proves the intuition that the giant component has a diameter of at least $\Omega(\log n)$, which is not obvious a priori.

THEOREM 3. *Let $\beta > 2$. Then, there exists a component in the hyperbolic random graph with diameter $\Omega(\log n)$ with probability $1 - \mathcal{O}(n^{1-\frac{\beta}{2}})$. If $\beta < 3$, this is the giant component.*

⁴A proof for this conjecture appeared in [30] after submission of this paper.

Proof. Let $\varepsilon := (\frac{1}{2} - \frac{1}{4\alpha})$. Observe that for $\alpha > \frac{1}{2}$, we have $\varepsilon > 0$. Consider the hyperbolic random graph model (*not* the PPP). We first show that there are no nodes in $B_0(\varepsilon R)$ w.h.p. For this, we observe that $\mu(B_0(\varepsilon R)) = \Theta(1) \cdot \exp(-(\frac{\alpha}{2} + \frac{1}{4})R) = o(1)$ by (4). Thus, we may apply Lemmas 5 and 7 to obtain

$$\begin{aligned} \Pr[\text{there are no vertices in } B_0(\varepsilon R)] &= (1 - \mu(B_0(\varepsilon R)))^n \\ &\geq \exp\left(-\Theta(1) \cdot e^{R/2} \cdot e^{-(\frac{\alpha}{2} + \frac{1}{4})R}\right) \\ &\geq 1 - \Theta(1) \cdot e^{(\frac{1}{4} - \frac{\alpha}{2})R} \\ &= 1 - \Theta\left(n^{-(\alpha - \frac{1}{2})}\right). \end{aligned}$$

It is important to perform this computation in the hyperbolic random graph model, as the probability that there are no nodes in $B_0(\varepsilon R) \neq \emptyset$ is smaller than $1 - n^{-\frac{1}{2}}$ for some values of α . Thus, a direct application of the PPP will result in a useless tail bound. Instead, we condition in the PPP on the fact that there are no nodes in $B_0(\varepsilon R)$. Then, if the same holds in the hyperbolic random graph, we again recover the same distribution of nodes by simply applying the PPP to the area $D_R \setminus B_0(\varepsilon R)$ instead of D_R . The expected number of nodes in the PPP is then

$$\begin{aligned} \mathbb{E}[|\mathcal{P}_n \setminus B_0(\varepsilon R)|] &= n \cdot \mu(B_0(R) \setminus B_0(\varepsilon R)) \\ &= n \cdot \left(1 - \Theta\left(e^{\alpha(\varepsilon R - R)}\right)\right) \\ &= n \cdot \left(1 - \Theta\left(e^{-\alpha R(\frac{1}{2} + \frac{1}{4\alpha})}\right)\right) \\ &= n \cdot \left(1 - \Theta\left(e^{-\alpha \frac{R}{2} - \frac{R}{4}}\right)\right) \\ &= n \cdot \left(1 - \Theta\left(n^{-(\alpha + \frac{1}{2})}\right)\right) \\ &= n - o(1). \end{aligned}$$

Thus, the penalty term is still equal to $\Theta(n^{\frac{1}{2}})$:

$$\begin{aligned} \Pr[|\mathcal{P}_n \setminus B_0(\varepsilon R)| = n] &= (n - o(1))^n \exp(-n + o(1)) \frac{1}{n!} \\ &\geq \Theta(1) \cdot (n - o(1))^n \exp(-n + o(1)) n^{-n - \frac{1}{2}} e^n \\ &= \Theta(n^{-\frac{1}{2}}) \cdot (1 - o(\frac{1}{n}))^n \\ &\geq \Theta(n^{-\frac{1}{2}}). \end{aligned}$$

In the following, we therefore may condition on the fact that there are no nodes in $B_0(\varepsilon R)$ and switch to the PPP. Next, we compute the probability that a shortest path of length $\Omega(\log n)$ appears in a certain area. At the end, we amplify this probability by repeating the experiment independently multiple times to arrive at our desired result.

To this end, similarly to Theorem 1, we now partition the disk D_R into $\Theta(n)$ sectors of equal angle $\varphi := e^{-R/2} = \Theta(\frac{1}{n})$. Then, two nodes $u, v \in L_1$ in neighboring sectors have angular distance at most $2e^{-R/2}$ and are therefore by Lemma 4 connected. On the flip side, if two nodes are at least six sectors apart, they are not connected, since their angle is $6e^{-R/2} > 2e^{-R/2+1}(1 + \mathcal{O}(e^{-R}))$.

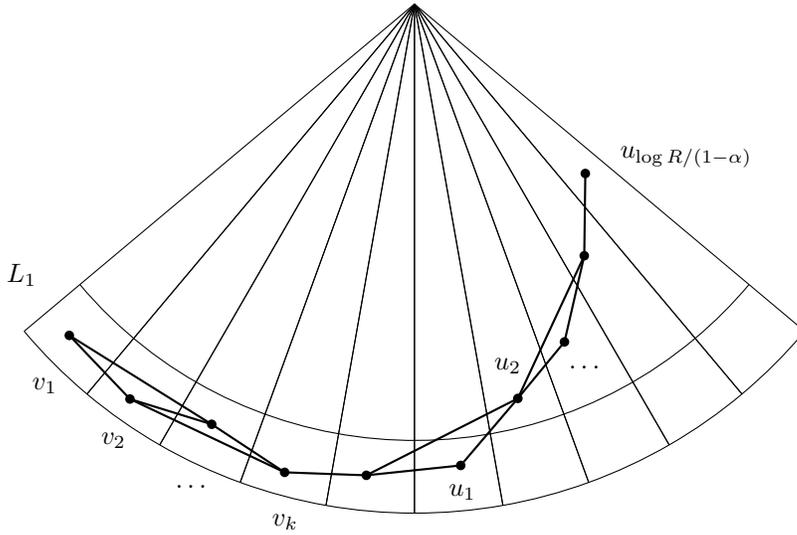


FIG. 4. Proof illustration for Theorem 3. The disk D_R is partitioned into sectors of angle $\frac{1}{n}$. Nodes v_1, \dots, v_k in neighboring sectors form a path component of length $\Theta(k)$. If $\alpha < 1$, the path is also connected via the nodes $u_1, \dots, u_{\frac{\log R}{1-\alpha} + c}$ to B_I .

Consider now k consecutive sectors, where k is to be fixed later. Let p_1 be the probability that a single sector contains exactly one node in L_1 . Since we are in the PPP, p_1 can be computed by using the Poisson distribution,

$$\begin{aligned} p_1 &= \exp(-n\varphi \cdot \mu(L_1)) \cdot n\varphi\mu(L_1) \\ &\geq \exp(-ne^{-R/2}f(R-1)) \cdot ne^{-R/2}f(R-1) \\ &= e^{-\Theta(1)}, \end{aligned}$$

meaning it is a constant bounded away from 0. Let p_2 be the probability that the node has no further neighbors than the ones in the k sectors. Since this node is in L_1 , it has a constant number of expected neighbors, and so $p_2 = e^{-\Theta(1)}$ is a constant probability bounded away from 0 as well.

We name the nodes in the k sectors v_1, \dots, v_k , respectively. As argued above, k such nodes form a shortest path of length $\Omega(k)$. We now argue that when $\beta < 3$, this path is also connected to the core of the hyperbolic random graph by exposing a path u_1, \dots, u_h to the inner band B_I , where $h = \mathcal{O}(\log \log n)$. Recall that by Corollary 11, a path that reaches B_I is connected to the giant component. Figure 4 contains an illustration of the proof.

The probability that sectors $k+1$ to $k+c+1$ also each contain exactly one node in L_1 is again $e^{-\Theta(1)}$ if c is constant. From here, we expose a path to the inner band B_I as follows. Assume we have a node $u_i \in L_i$ in sector $k+c+i$. Assume further u_i is to the right of all previous sectors. Then, we consider the probability that u_i has a neighbor to the right in layer L_{i+1} , while we still condition on the fact that all nodes v_1, \dots, v_k have no neighbors above L_1 as stated before. By Lemma 8, a node $u_{i+1} \in L_{i+1}$ is not connected to any of the nodes v_1, \dots, v_k if $\Delta\varphi_{u_{i+1}, v_k} > 6e^{\frac{i-R}{2}}$. Similarly, it is connected to u_i if $\Delta\varphi_{u_i, u_{i+1}} \leq \frac{2}{e}e^{\frac{2i-R}{2}}$. Since all nodes v_1, \dots, v_k are to

the left of u_i and there are c sectors between, u_{i+1} may fall into an angular range of

$$\frac{2}{e} e^{\frac{2i-R}{2}} - 6e^{\frac{i-R}{2}} + ce^{-R/2} \geq \Theta(1) \cdot e^{i-\frac{R}{2}}$$

if c is large enough. Therefore, the probability that node u_i has a neighbor in layer L_{i+1} that is not connected to v_1, \dots, v_k is at least

$$1 - \exp(-\Theta(1) \cdot n \cdot e^{-\alpha i} e^{i-\frac{R}{2}}) = 1 - \exp(-\Theta(1) \cdot e^{(1-\alpha)i}) = \Theta(1).$$

Such a path to B_I is of length $\frac{\log R}{1-\alpha} + c = \mathcal{O}(\log \log n)$ at most. In total, the probability that v_1, \dots, v_k exist as described above, and that they are connected to B_I , is thereby $e^{-\Theta(k+\log \log n)}$, or just $e^{-\Theta(k)}$ in the case where $\beta \geq 3$.

It remains to compute how often we can repeat this experiment independently. Consider a node in the outermost layer $v \in L_1$. Since we assumed that v_1, \dots, v_k have no neighbors in layers above 1, we have exposed the area $B_0(R) \cap (\bigcup_{i=1}^k B_{v_i}(R))$. The largest angular distance such a node v can have to one of its neighbors is by Lemma 4,

$$(13) \quad \Delta\varphi \leq 2e^{-\frac{\varepsilon R}{2}} (1 \pm \mathcal{O}(e^{-\varepsilon R})) \leq \mathcal{O}(n^{-\varepsilon}),$$

where $\varepsilon = (\frac{1}{2} - \frac{1}{4\alpha})$ as chosen in the beginning. This holds since we condition on the fact that there are no nodes in $B_0(\varepsilon R)$.

We thus expose at most an angle of $\mathcal{O}(\frac{k}{n} + n^{-\varepsilon} + \log \log n \cdot \frac{1}{n} (\log n)^{1/(1-\alpha)})$ of the graph. Therefore, if $\frac{k}{n} < n^{-\varepsilon}$, we can repeat this experiment independently $\Omega(n^\varepsilon)$ times. The probability that all of them fail is at most

$$\left(1 - e^{-\Theta(k+\log \log n)}\right)^{\Omega(n^\varepsilon)} \leq \exp\left(-e^{-\Theta(k)} \Omega(n^\varepsilon)\right) = \exp\left(-n^{\Omega(1)}\right)$$

if $k = \Theta(\log n)$ is chosen small enough. This proves the claim. □

7. Conclusion. We derive a new polylogarithmic upper bound on the diameter of hyperbolic random graphs for the case $2 < \beta < 3$ and show that it is $\mathcal{O}(\log n)$ if $\beta > 3$. We further prove a logarithmic lower bound. This immediately yields lower bounds for any broadcasting protocol that has to reach all nodes. Processes such as bootstrap percolation or rumor spreading therefore must run at least $\Omega(\log n)$ steps until they inform all nodes in the giant component. In particular, this result stands in contrast to the average distance of two nodes in the hyperbolic random graph, which is of order $\Theta(\log \log n)$ [1, 11]. This implies the existence of a path that is exponentially longer than the average path.

It remains an open problem to find a matching upper bound on the diameter in the case $2 < \beta \leq 3$, but we conjecture that it is of order $\mathcal{O}(\log n)$ as well. A natural direction to expand this research is to investigate rumor spreading on hyperbolic random graphs. Even though there exists a significant body of research on rumor spreading in other social network models, hyperbolic random graphs are largely unexplored in this context. The only work known to us in this direction is by Candellero and Fountoulakis [12], who study bootstrap percolation in this model. An interesting question in this context is whether the constant clustering of this model affects rumor spreading protocols in a positive or negative way. Previously inspected scale-free graph models have subconstant clustering.

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