Maps of Restrictions for Behaviourally Correct Learning

Vanja Doskoč♣ and Timo Kötzing♠

Hasso Plattner Institute, University of Potsdam, Potsdam, Germany
♣vanja.doskoc@hpi.de (corresponding) and ♠timo.koetzing@hpi.de

Abstract. In *language learning in the limit*, we study computable devices (learners) learning formal languages. We consider learning tasks paired with restrictions regarding, for example, the hypotheses made by the learners. We compare such restrictions with each other in order to study their impact and depict the results in overviews, the so-called *maps*. In the case of *explanatory* learning, the literature already provides various maps.
On the other hand, in the case of *behaviourally correct* learning, only partial results are known. In this work, we complete these results and provide full behaviourally correct maps for different types of data presentation. In particular, in all studied settings, we observe that monotone learning implies non-U-shaped learning and that cautiousness, semantic conservativeness and weak monotonicity are equally powerful.

Keywords: Language Learning in the Limit · Behaviourally Correct Learning · Learning Restrictions · Map.

1 Introduction

Motivation In his seminal work, Gold [10] introduced the *language learning in the limit* framework. Here, a learner (a computable function) successively receives positive information about a target language (a subset of the natural numbers). With each new datum, the learner produces a conjecture which language it believes to be presented. Once these guesses converge to a *single*, correct explanation of the target language, we say that the learner successfully learned the target language.

This is known as *explanatory* learning and denoted as[1] **TxtGEx**. We focus on the semantic version thereof, namely *behaviourally correct* learning [5,22], denoted as **TxtGBc**. Here, almost all conjectures of the learner have to be correct (but do not need to be syntactically identical). Naturally, each single language may be learned by a learner which always suggests (a conjecture for) this language. Thus, we focus on learning classes of languages learnable by a single learner.

[1] Particularly, a *text* (**Txt**) provides positive information about the target language, from which *Gold-style* (**G**) learners then infer their conjectures. Lastly, **Ex** for stands for explanatory learning.

These learning criteria are extended or altered to study the impact of certain restrictions. These may limit which hypotheses are allowed, for example requiring them to follow a monotone behaviour or by constraining when changes of conjectures are allowed, or the data representation, for example, by leaving out information on the order the data is presented. An overview of the studied restrictions can be found in Section 2. A particular branch of study focuses on the pairwise relation between such restrictions. The findings are then depicted in overviews, so-called *maps*. The literature already provides maps of explanatory learners with various modes of data representation [11,15,16]. However, for behaviourally correct learning only partial results on the pairwise interaction of different restrictions are known so far [2,7,8,9,12,17].

Our Contribution In this work, we provide the missing relations. This way, we obtain a full picture regarding the pairwise relation of the studied restrictions. We provide the collected findings in Figure 1. In particular, we observe in all studied settings that classes of languages that can be learned by learners which never discard correctly conjectured elements, that is, monotone learners, can also be learned by learners which never change their mind from a correct guess, that is, non-U-shaped learners. These results are presented in Lemma 1 and Theorems 3 and 5. Furthermore, we find that learners which base their guess solely on the set of elements presented change their mind only when witnessing inconsistent information, see Theorem 2. We note that analogous results hold in the explanatory setting [14,15,16]. However, one difference between these settings is that neither learners which may change their mind only when inconsistent, nor learners which never fall back to a proper subset of a previous guess depend on the order or amount of data presented, see Theorem 2.

Another contribution of this work pertains to normal forms of learners. Particularly interesting are *strongly* **Bc**-*locking* learners [15]. These learners have, on each text for a target language, a **Bc**-*locking sequence*, that is, a sequence which contains enough information for the learner to be correct and never change its mind any more regardless what information from the target language it receives [3,12]. With Theorem 4, we complete the literature by showing that for all considered restrictions the learners may be assumed strongly **Bc**-locking.

Future Work We leave studying one important restriction to future work: decisiveness [21]. Here, a learner may never get back to a previous, rejected hypothesis. In particular, it is open to resolve whether each class of languages a monotone learner learns can be learned decisively. We note that Theorem 2 shows that decisiveness is no restriction for learners which base their hypotheses solely on the set of elements, that is, set-driven learners.

The impact of the data presented during learning varies depending on the studied restriction [6,7,8,17]. For example, cautious or semantically conservative learners do not rely on the order or amount of data presented while monotone learners do. Future work may resolve how the provided data impacts decisive and non-U-shaped learners.

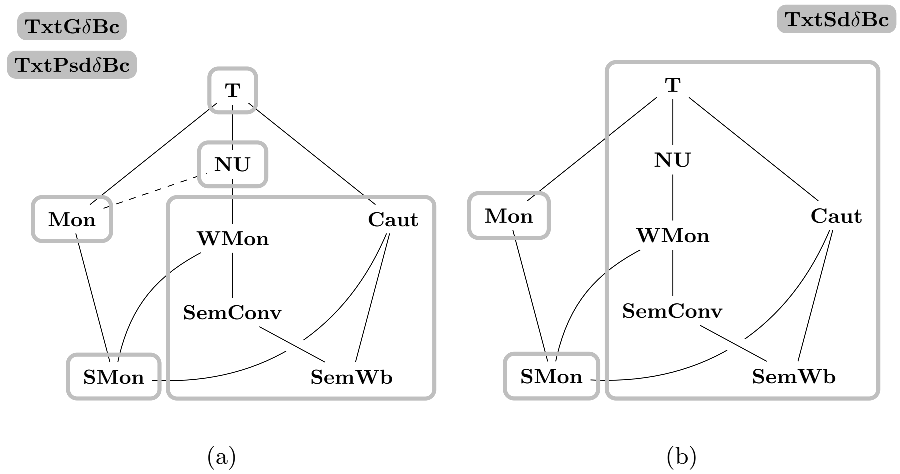

Fig. 1: A depiction of the relation between the studied restrictions (compare Section 2) for Gold-style and partially set-driven (see Figure 1(a)) as well as set-driven learning (see Figure 1(b)). Solid and dashed lines imply trivial and non-trivial inclusions (both bottom-to-top), respectively. Greyly edged areas illustrate a collapse of the enclosed learning criteria. There are no further collapses.

Structure of the Work This work is structured as follows. In Section 2, we shortly discuss important concepts for this work. In the remaining sections, we discuss the initial situation for the respective studied mode of data representation and provide our results to complete the particular map.

2 Preliminaries

In this section, we introduce the notation used as well as important concepts for language learning in the limit.

2.1 Mathematical Notation

We mainly follow the textbook by Rogers Jr. [23]. With $\mathbb{N} = \{0, 1, 2, \dots\}$ we denote the set of all natural numbers and with $\emptyset$ the empty set. We use $\subseteq$ ($\subsetneq$) to denote the (proper) inclusion relation between two sets as well as the (proper) extension relation between two finite sequences. Furthermore, to denote the concatenation of two sequences σ and τ, we write $\sigma^\frown\tau$ or simply $\sigma\tau$. Given a non-empty, finite sequence σ, we use σ^- to denote σ without its last element. We let $\mathcal{P}$ ($\mathcal{R}$) be the set of all (total) computable functions $p\colon \mathbb{N} \to \mathbb{N}$. We furthermore fix an effective numbering $\{\varphi_e\}_{e\in\mathbb{N}}$ of all partial computable functions and denote the e-th computable set as $W_e = \mathrm{dom}(\varphi_e)$, where dom denotes the domain of a function. We call e the *program*, *index* or *hypothesis* of W_e. For any time-step $t \in \mathbb{N}$, we let W_e^t be the set of all elements which the program e enumerates in at most t steps.

We learn languages (recursively enumerable sets) $L \subseteq \mathbb{N}$ using learners (partial computable functions). With $\#$ we denote the *pause symbol* and, for any set $S \subseteq \mathbb{N}$, we denote $S_\# \coloneqq S \cup \{\#\}$. Then, a *text* is a total function $T\colon \mathbb{N} \to \mathbb{N}_\#$ and $\mathbf{Txt}$ is the set of all texts. For any text (or sequence) T we define the *content* of T as $\mathrm{content}(T) \coloneqq \mathrm{range}(T) \setminus \{\#\}$. Here, range denotes the image of a function. Now, a text of a language L is such that $\mathrm{content}(T) = L$ and the set of all texts of L is denoted as $\mathbf{Txt}(L)$. Furthermore, for $n \in \mathbb{N}$ we let $T[n]$ be the initial sequence of T of length n, that is, $T[0] \coloneqq \epsilon$ (the empty string) and, if $n > 0$, $T[n] \coloneqq (T(0), T(1), \dots, T(n-1))$. Lastly, for $t, t' \in \mathbb{N}$ and finite sets $D, D' \subseteq \mathbb{N}$, we define $(D, t) \preceq (D', t')$ if $t \leq t'$ and there exists a text T such that $D = \mathrm{content}(T[t])$ and $D' = \mathrm{content}(T[t'])$.

2.2 Language Learning in the Limit

We discuss the formalization of learning criteria [18]. An *interaction operator* β provides the learner with the information to make its guess from. Formally, β takes as input a learner $h \in \mathcal{P}$ and a text $T \in \mathbf{Txt}$ and outputs a possibly partial function p. In this work, we consider the interaction operators $\mathbf{G}$ for Gold-style learning [10], which provides the learner with the full information on the input, $\mathbf{Psd}$ for partially set-driven learning [3,24], where the learner receives the set of input elements and a counter, and $\mathbf{Sd}$ for set-driven learning [25], where the learner only receives the set of elements. Formally, we have

$$\begin{aligned} \mathbf{G}(h,T)(i) &= h(T[i]); \\ \mathbf{Psd}(h,T)(i) &= h(\mathrm{content}(T[i]), i); \\ \mathbf{Sd}(h,T)(i) &= h(\mathrm{content}(T[i])). \end{aligned}$$

Learning criteria are formalized as follows. Initially, for *explanatory* learning ($\mathbf{Ex}$, [10]) the learner is expected to converge to a single, correct hypothesis when presented a target language. In our work, we focus on a relaxation thereof: We expect the learner to converge to a semantically correct hypothesis, while it may change its mind syntactically. This is known as *behaviourally correct* learning ($\mathbf{Bc}$, [5,22]). Formally, a *learning restriction* δ is a predicate on a total function p (the sequence of hypotheses) and a text $T \in \mathbf{Txt}$. For the mentioned criteria we have

$$\begin{aligned} &\mathbf{Ex}(p,T) \Leftrightarrow \exists n_0 \, \forall n \geq n_0 \colon p(n) = p(n_0) \wedge W_{p(n_0)} = \mathrm{content}(T), \\ &\mathbf{Bc}(p,T) \Leftrightarrow \exists n_0 \, \forall n \geq n_0 \colon W_{p(n)} = \mathrm{content}(T). \end{aligned}$$

We consider further restrictions. In *non-U-shaped* learning ($\mathbf{NU}$, [2]) the learner may never discard a correct guess. For *consistent* learning ($\mathbf{Cons}$, [1]) each hypothesis must include the information it is based on. We also consider various monotonic restrictions [13,19,26]. For *strongly monotone* learning ($\mathbf{SMon}$)

the learner may not discard elements present in any previous guess, while for *monotone* learning (**Mon**) this applies only to correctly guesses elements. On the other hand, for *weakly monotone* learning (**WMon**) the learner must not discard any elements while its hypothesis is consistent with the information seen. Similarly, in *cautious* learning (**Caut**, [21]) the hypotheses may never fall back to the proper subset of a previous hypothesis, and, as a relaxation thereof, in *target-cautious* learning ($\mathbf{Caut_{Tar}}$, [15]) the hypotheses may not be a proper superset of the target language. For *semantically conservative* learning (**SemConv**, [17]) the learner may not change a hypothesis while it is consistent with the data given. Lastly, in *semantically witness-based* learning (**SemWb**, [17]) the learner must justify each mind change. Formally, we have for a total function p and a text $T \in \mathbf{Txt}$

$$\begin{aligned}
\mathbf{NU}(p,T) &\Leftrightarrow \forall i,j,k\colon i \le j \le k \wedge W_{p(i)} = W_{p(k)} = \mathrm{content}(T) \Rightarrow \\
&\qquad\qquad \Rightarrow W_{p(i)} = W_{p(j)}, \\
\mathbf{Cons}(p,T) &\Leftrightarrow \forall i\colon \mathrm{content}(T[i]) \subseteq W_{p(i)}, \\
\mathbf{SMon}(p,T) &\Leftrightarrow \forall i,j\colon i < j \Rightarrow W_{p(i)} \subseteq W_{p(j)}, \\
\mathbf{Mon}(p,T) &\Leftrightarrow \forall i,j\colon i < j \Rightarrow \mathrm{content}(T) \cap W_{p(i)} \subseteq \mathrm{content}(T) \cap W_{p(j)}, \\
\mathbf{WMon}(p,T) &\Leftrightarrow \forall i,j\colon i < j \wedge \mathrm{content}(T[j]) \subseteq W_{p(i)} \Rightarrow W_{p(i)} \subseteq W_{p(j)}, \\
\mathbf{Caut}(p,T) &\Leftrightarrow \forall i,j\colon W_{p(i)} \subsetneq W_{p(j)} \Rightarrow i \le j, \\
\mathbf{Caut_{Tar}}(p,T) &\Leftrightarrow \forall i\colon \neg(\mathrm{content}(T) \subsetneq W_{p(i)}), \\
\mathbf{SemConv}(p,T) &\Leftrightarrow \forall i,j\colon (i \le j \wedge \mathrm{content}(T[j]) \subseteq W_{p(i)}) \Rightarrow W_{p(i)} = W_{p(j)}, \\
\mathbf{SemWb}(p,T) &\Leftrightarrow \forall i,j\colon \big(\exists k\colon i < k \le j \wedge W_{p(i)} \neq W_{p(k)}\big) \Rightarrow \\
&\qquad\qquad \Rightarrow \big(\mathrm{content}(T[j]) \cap W_{p(j)}\big) \setminus W_{p(i)} \neq \emptyset.
\end{aligned}$$

We combine any two restrictions δ and δ' by intersecting them and denote this as $\delta\delta'$. With $\mathbf{T}$ we denote the predicate which is always true and interpret it as absence of a learning restriction.

Finally, a *learning criterion* $(\alpha, \mathcal{C}, \beta, \delta)$ consists of learning restrictions α and δ, a set of admissible learners $\mathcal{C}$, usually $\mathcal{P}$ or $\mathcal{R}$, and an interaction operator β. We denote the learning criterion as $\tau(\alpha)\mathcal{C}\mathbf{Txt}\beta\delta$ and omit $\mathcal{C}$ if it equals $\mathcal{P}$ and the learning restrictions in case they equal $\mathbf{T}$. We say that an admissible learner $h \in \mathcal{C}$ $\tau(\alpha)\mathcal{C}\mathbf{Txt}\beta\delta$-learns a language L if on any text T the restriction α is met, that is, we have $\alpha(\beta(h,T),T)$, and for all $T \in \mathbf{Txt}(L)$ the restriction δ is fulfilled, that is, we have $\delta(\beta(h,T),T)$. We write $\tau(\alpha)\mathcal{C}\mathbf{Txt}\beta\delta(h)$ for the set of all languages h $\tau(\alpha)\mathcal{C}\mathbf{Txt}\beta\delta$-learns and $[\tau(\alpha)\mathcal{C}\mathbf{Txt}\beta\delta]$ for the set containing, for all $h' \in \mathcal{C}$, $\tau(\alpha)\mathcal{C}\mathbf{Txt}\beta\delta(h')$. The latter set is referred to as the *learning power* of $\tau(\alpha)\mathcal{C}\mathbf{Txt}\beta\delta$-learners.

2.3 Locking Sequences

Certain sequences may contain especially valuable information for learners. *Locking sequences* contain sufficient information on the target language so that the learner makes a correct guess and does not change its mind any more, regardless what information from the target language it witnesses. Formally, a sequence σ is a *locking sequence* for a learner h on a language L if, for all $\tau \in L'_{\#}$, we have $W_{h(\sigma)} = L$ and $h(\sigma) = h(\sigma\tau)$ [3]. Analogously, for behaviourally correct learning, a sequence σ is a **Bc**-*locking sequence* for a learner h on a language L if, for all $\tau \in L'_{\#}$, we have $W_{h(\sigma\tau)} = L$ [12]. It is known that *every* **Bc**-learner has a **Bc**-locking sequence [3], however, there are learners and texts where no initial sequence of the text is a **Bc**-locking sequence [3]. We call a learner h strongly **Bc**-locking on some language L if for each text $T \in \mathbf{Txt}(L)$ there exists an n such that $T[n]$ is a **Bc**-locking sequence for h on L. If h is strongly **Bc**-locking on each language it learns, we call h *strongly* **Bc**-*locking* [15,16,17]. The transition to **Bc**-*locking information* for partially set-driven learner and **Bc**-*locking sets* for set-driven learner is immediate and, thus, omitted.

3 Set-Driven Map

In set-driven learning, unrestricted learners may be assumed cautious and consistent at the same time [8]. Furthermore, semantically conservative and semantically witness-based learners may be assumed consistent and equally powerful [17]. As a matter of fact, they are even as powerful as their Gold-style counterpart [7]. In this section we show that all of these learners are equal regarding their learning power. In particular, we exploit two known concepts. First, we use the same approach as when showing that set-driven learners may be assumed target-cautious [8] to show that without loss of generality they are also non-U-shaped, see Lemma 1. Using this, we can obtain semantically conservative learners (see Theorem 2) in a similar fashion as when making semantically conservative learners so *everywhere* [7]. In particular, we can override wrong hypotheses of the learners using witnessing elements (as, by target-cautious learning, incorrect guesses cannot overgeneralize the target language) and right hypotheses are never discarded (by non-U-shaped learning).

Lemma 1. *Every* **TxtSdBc**-*learner h may be assumed to be target-cautious* ($\mathbf{Caut_{Tar}}$)*, non-U-shaped* (**NU**) *and consistent* (**Cons**) *simultaneously.*

Proof. Let h be a learner and let $\mathcal{L} = \mathbf{TxtSdBc}(h)$. Applying a *weak forward search* algorithm [8] we may assume h to be target-cautious and consistent. We show that, by applying the same algorithm again, we get a learner which also **NU**-learns $\mathcal{L}$.

Let h' be given as in Algorithm 1 with parameter h. Then h' **TxtSdBc**-learns $\mathcal{L}$ [8]. We further remark that h' remains target-cautious and consistent [8] as h is, in particular, a **Sd**-learner. It remains to be shown that h' is also non-U-shaped. To that end, let $L \in \mathcal{L}$ and assume there exists a finite $D \subseteq L$ with $W_{h'(D)} = L$. We show that for all finite D'', with $D \subseteq D'' \subseteq L$, we have $W_{h'(D'')} = L$.

Algorithm 1: Weak Forward Verification [8]

Parameter: **Sd**-learner h, function enum$(.,.)$ such that $\forall e\colon W_e = \text{range}(\text{enum}(e,.))$.
Input: Finite set $D \subseteq \mathbb{N}$.
Semantic Output: $W_{h'(D)} = \bigcup_{i \in \mathbb{N}} E_i$.
Initialization: $E_0 \leftarrow D$.

1 **for** $i = 0$ **to** ∞ **do**
2     $x_i \leftarrow \text{enum}(h(D), i)$
3   **if** $x_i \notin E_i$ **then**
4   **for** $D', D \subseteq D' \subseteq E_i \cup \{x_i\}$ **do**
5   search for t such that $E_i \cup \{x_i\} \subseteq W^t_{h(D')}$
6     $E_{i+1} \leftarrow E_i \cup \{x_i\}$

We first show that D is a **Bc**-locking set for h on L. For finite D' with $D \subseteq D' \subseteq L$ and $x \in L$, let i be the step[2] in Algorithm 1 such that x is enumerated into $W_{h'(D)}$, that is, $D' \cup \{x\} \not\subseteq E_i$ and $D' \cup \{x\} \subseteq E_{i+1}$. Then, as $D \subseteq D' \subseteq E_{i+1}$ we have by lines 4 and 5

$$x \in E_{i+1} \subseteq W_{h(D')}.$$

Thus, for each finite D' with $D \subseteq D' \subseteq L$ we get for all $x \in L$ that $x \in W_{h(D')}$. So we have $L \subseteq W_{h(D')}$ and, since h is target cautious, even $L = W_{h(D')}$. Altogether, D is a **Bc**-locking set for h on L.

Now we show that for finite D'', with $D \subseteq D'' \subseteq L$, the algorithm runs through every step i successfully. This way, we obtain $W_{h'(D'')} = W_{h(D'')} = L$. Let $E_0 = D''$ and let i be the next step in Algorithm 1. If $x_i \in E_i$, step i is completed successfully. Otherwise, the algorithm checks whether for each finite D', with $D'' \subseteq D' \subseteq L$, we have some t such that $E_i \cup \{x_i\} \subseteq W^t_{h(D')}$. As $W_{h(D')} = L$ and as $E_i \cup \{x_i\}$ is a finite subset of L, such a t will eventually be found. Thus, x_i will be enumerated into E_{i+1} and, hence, into $W_{h'(D'')}$. This concludes the proof. □

Theorem 2. *We have that*

$$[\textbf{TxtSdBc}] = [\textbf{TxtSdCautBc}] = [\textbf{TxtGCautBc}] = \\ = [\textbf{TxtSdSemWbBc}] = [\textbf{TxtGSemWbBc}].$$

Proof. We show that [**TxtGSemConvBc**] = [**TxtSdBc**]. This suffices, since we have that [**TxtSdBc**] = [**TxtSdCautBc**] = [**TxtGCautBc**] [8] as well as [**TxtGSemConvBc**] = [**TxtGSemWbBc**] = [**TxtSdSemWbBc**] [7]. By the latter equality, we immediately get the inclusion [**TxtGSemConvBc**] $\subseteq$ [**TxtSdBc**]. For the other direction, let h **TxtSdBc**-learn $\mathcal{L}$. By Lemma 1, we may assume h to be target-cautious, non-U-shaped and consistent. We now

[2] Note that x and x_i may differ.

provide a learner h' such that $\mathcal{L} \subseteq \mathbf{TxtGSemConvBc}(h')$. To that end, we use the learner h' as described in Algorithm 2.

Algorithm 2: The **TxtGSemConvBc**-learner h'.

Parameter: A **Sd**-learner h.
Input: A finite sequence σ.
Semantic Output: $W_{h'(\sigma)} = \bigcup_{t \in \mathbb{N}} E_t$.
Initialization: $t' \leftarrow 0$, $E_0 \leftarrow \mathrm{content}(\sigma)$ and, for all $t > 0$, $E_t \leftarrow \emptyset$.

1 **if** $\sigma = \varepsilon$ *or* $\mathrm{content}(\sigma^-) \subsetneq \mathrm{content}(\sigma)$ **then**
2   **for** $t = 0$ **to** ∞ **do**
3   **if** $\exists \sigma' \subsetneq \sigma \colon \mathrm{content}(\sigma) \subseteq W^t_{h'(\sigma')}$ **then**
4             $\Sigma'_t \leftarrow \{\sigma' \subsetneq \sigma \mid \mathrm{content}(\sigma) \subseteq W^t_{h'(\sigma')}\}$
5             $E_{t+1} \leftarrow E_t \cup \bigcup_{\sigma' \in \Sigma'_t} W^t_{h'(\sigma')}$
6   **else**
7             $C_\sigma \leftarrow \mathrm{content}(\sigma)$
8             $F_{\sigma,t'} \leftarrow W^{t'}_{h(\mathrm{content}(\sigma))}$
9   **if** $\forall D \subseteq F_{\sigma,t'} \colon \bigcup_{D' \subseteq F_{\sigma,t'}} W^{t'}_{h(C_\sigma \cup D')} \subseteq W^t_{h(C_\sigma \cup D)}$ **then**
10                 $E_{t+1} \leftarrow E_t \cup F_{\sigma,t'}$
11                 $t' \leftarrow t' + 1$
12 **else**
13     $W_{h'(\sigma)} \leftarrow W_{h'(\sigma^-)}$

We discuss the learner h' obtained from Algorithm 2 with parameter h and a finite sequence σ as input. First note that the outer if-clause checks whether the current information σ contains a new datum or is empty. If not, then the learner outputs just the same as when given σ^-. This way, the learner only may change its mind when a new datum occurs. Otherwise, h' checks whether, on any previous sequence $\sigma' \subseteq \sigma$, it is consistent with the currently given information content(σ). If so, the learner only enumerates the same as such hypotheses (lines 3 to 5). While no such hypothesis is found, h' does a forward search with regard to h (lines 6 to 11). Then, h' only enumerates elements which are witnessed by *all* visible future hypotheses. This is possible to check, as the learner h is set-driven.

Note that this is a similar approach as when making **TxtGSemConvBc**-learner *everywhere* semantically conservative [7]. We maintain the monitoring of the time of enumeration for each element (lines 6 to 11) and checking for previous consistent hypotheses (lines 3 to 5) to prevent non-conservative behaviour. A main observation is that for the learner h' to converge correctly, the initial learner h need not be semantically conservative. It suffices that h is target-cautious (so that wrong hypotheses lack elements from the target language which then can be used for mind-changes) and non-U-shaped (so that we do not "unintentionally" output a correct guess prematurely).

We first show that h' is **SemConv** on arbitrary text $T \in \mathbf{Txt}$. The problem is that when a previous hypothesis becomes consistent with information currently given, the learner may have already enumerated incomparable data in its current hypothesis. This is prevented by closely monitoring the time of enumeration, namely by waiting until the enumerated data will certainly not cause such problems. We prove that h' is $\tau(\mathbf{SemConv})$ formally. Let $n < n'$ be (the lexicographically first pair) such that $\mathrm{content}(T[n']) \subseteq W_{h'(T[n])}$. We show that $W_{h'(T[n])} = W_{h'(T[n'])}$ by separately looking at each inclusion.

$\subseteq$: The inclusion $W_{h'(T[n])} \subseteq W_{h'(T[n'])}$ follows immediately since by assumption $\mathrm{content}(T[n']) \subseteq W_{h'(T[n])}$, meaning that at some point during the enumeration of $W_{h'(T[n'])}$ the first if-clause (lines 3 and 5) will find $T[n]$ as a candidate and then $W_{h'(T[n'])}$ will enumerate $W_{h'(T[n])}$.

$\supseteq$: Assume there exists $x \in W_{h'(T[n'])} \setminus W_{h'(T[n])}$. Let x be the first such element enumerated and let t_x be the step of enumeration with respect to $h(\mathrm{content}(T[n']))$, that is, $x \in W^{t_x}_{h(\mathrm{content}(T[n']))}$ but $x \notin W^{t_x-1}_{h(\mathrm{content}(T[n']))}$. Similarly, let t_{content} be the step where $\mathrm{content}(T[n']) \subseteq W_{h'(T[n])}$ is witnessed for the first time. Now, by the definition of h', we have

$$W_{h'(T[n'])} \subseteq W^{t_{\mathrm{content}}-1}_{h(\mathrm{content}(T[n']))} \cup W_{h'(T[n])},$$

as $W_{h'(T[n'])}$ enumerates at most $W^{t_{\mathrm{content}}-1}_{h(\mathrm{content}(T[n']))}$ until it sees the consistent prior hypothesis, namely $h'(T[n])$. As this happens exactly at step t_{content}, $W_{h'(T[n'])}$ enumerates (at most) elements from $W^{t_{\mathrm{content}}-1}_{h(\mathrm{content}(T[n']))}$ before it continues to follow $W_{h'(T[n])}$. Now, we have $x \in W_{h'(T[n'])}$ but $x \notin W_{h'(T[n])}$ and, therefore, $x \in W^{t_{\mathrm{content}}-1}_{h(\mathrm{content}(T[n']))}$. Thus, $t_x < t_{\mathrm{content}}$. In particular, x is enumerated via the second if-clause (lines 6 to 11). Furthermore, since $W_{h'(T[n])}$ also enumerates $\mathrm{content}(T[n'])$ via the second if-clause (lines 6 to 11), we have that

$$\bigcup_{D' \subseteq W^{t_{\mathrm{content}}}_{h(\mathrm{content}(T[n]))}} W^{t_{\mathrm{content}}}_{h(\mathrm{content}(T[n]) \cup D')} \subseteq W_{h'(T[n])}.$$

As $D' = \mathrm{content}(T[n'])$ is a candidate in the big union, we get that

$$x \in W^{t_{\mathrm{content}}-1}_{h(\mathrm{content}(T[n']))} \subseteq \bigcup_{D' \subseteq W^{t_{\mathrm{content}}}_{h(\mathrm{content}(T[n]))}} W^{t_{\mathrm{content}}}_{h(\mathrm{content}(T[n]) \cup D')} \subseteq W_{h'(T[n])},$$

contradicting $x \notin W_{h'(T[n])}$. This concludes this part of the proof.

Now that h' is shown to be semantically conservative, we show that for any $L \in \mathcal{L}$ and any $T \in \mathbf{Txt}(L)$ we have, for $n \in \mathbb{N}$,

$$W_{h'(T[n])} \subseteq W_{h(\mathrm{content}(T[n]))}. \tag{1}$$

We show Equation (1) by induction on n. The case $n = 0$ follows immediately. Assume Equation (1) holds up to n. Note that, by definition of h', we have

$$W_{h'(T[n+1])} \subseteq \bigcup_{\substack{n' \leq n, \\ \mathrm{content}(T[n+1]) \subseteq W_{h'(T[n'])}}} W_{h'(T[n'])} \cup W_{h(\mathrm{content}(T[n+1]))}. \tag{2}$$

Let n_{m} be the minimal n' such that $\mathrm{content}(T[n+1]) \subseteq W_{h'(T[n'])}$ (if such n_{m} exists). By **SemConv** of h', for all n'' with $n_{\mathrm{m}} \leq n'' \leq n$, we have

$$W_{h'(T[n_{\mathrm{m}}])} = W_{h'(T[n''])}.$$

Furthermore, for $n'' < n_{\mathrm{m}}$, no previous guess $W_{h'(T[n''])}$ contains $\mathrm{content}(T[n_{\mathrm{m}}])$, as otherwise, by **SemConv** of h', if $\mathrm{content}(T[n_{\mathrm{m}}]) \subseteq W_{h'(T[n''])}$ we obtain $W_{h'(T[n''])} = W_{h'(T[n_{\mathrm{m}}])} \supseteq \mathrm{content}(T[n+1])$, a contradiction to the minimality of n_{m}. Hence,

$$\bigcup_{\substack{n' \leq n, \\ \mathrm{content}(T[n+1]) \subseteq W_{h'(T[n'])}}} W_{h'(T[n'])} = W_{h'(T[n_{\mathrm{m}}])}. \tag{3}$$

In particular, h' does only a forward search on input $T[n_{\mathrm{m}}]$ (lines 6 to 11). As, by doing so, it eventually witnesses $\mathrm{content}(T[n+1])$, we get by definition of h' that

$$W_{h'(T[n_{\mathrm{m}}])} \subseteq W_{h(\mathrm{content}(T[n+1]))}. \tag{4}$$

Combining Equations (3) and (4) with Equation (2), we have that Equation (1) holds for the induction step and, therefore, for all $n \in \mathbb{N}$.

We close the proof by showing that h' **TxtGBc**-learns $\mathcal{L}$. To that end, let $L \in \mathcal{L}$ and $T \in \mathbf{Txt}(L)$. As h' is semantically conservative, it suffices to show that there exists a n such that $W_{h'(T[n])} = L$. We provide such n by case distinction.

1.C.: L is finite. Then there exists n_0 with $\mathrm{content}(T[n_0]) = L$. As h is consistent and target-cautious, we have $L = W_{h(\mathrm{content}(T[n_0]))}$. By Equation (1), we have $L = W_{h(\mathrm{content}(T[n_0]))} \supseteq W_{h'(T[n_0])}$ and, by consistency of h', $W_{h'(T[n_0])} \supseteq \mathrm{content}(T[n]) = L$. Altogether we have $W_{h'(T[n_0])} = L$ as required.

2.C.: L is infinite. Let n_0 be minimal such that $W_{h(\mathrm{content}(T[n_0]))} = L$. Then, as h is non-U-shaped, $\mathrm{content}(T[n_0])$ is a **Bc**-locking set for h on L. Let $n_1 \geq n_0$ be minimal such that

$$\forall i < n_0 \colon \mathrm{content}(T[n_1]) \not\subseteq W_{h'(T[i])}. \tag{5}$$

Such n_1 exists due to Equation (1) and h being target-cautious, that is, wrong hypotheses prior to $h(\mathrm{content}(T[n_0]))$ are either proper subsets of the target language or incomparable to it. In either case, some elements of the target language are not contained in these guesses.

Now, we show that Condition (5) actually holds for all $i < n_1$. If $n_0 = n_1$, this is immediately given. Otherwise, note that there exists some $i_0 < n_0$

such that $\text{content}(T[n_1 - 1]) \subseteq W_{h'(T[i_0])}$ (by the minimal choice of n_1 for Condition (5)) and thus, for all n with $i_0 \leq n \leq n_1 - 1$, $W_{h'(T[i_0])} = W_{h'(T[n])}$ (as h' is **SemConv**). In particular, we have

$$\exists i_0 < n_0 \, \forall n, n_0 \leq n < n_1 : W_{h'(T[i_0])} = W_{h'(T[n])}.$$

As $\text{content}(T[n_1]) \nsubseteq W_{h'(T[i_0])}$, we have

$$\forall i < n_1 \colon \text{content}(T[n_1]) \nsubseteq W_{h'(T[i])}. \tag{5'}$$

Hence, elements enumerated by $W_{h'(T[n_1])}$ cannot be enumerated by the first if-clause (lines 3 to 5) but only by the second one (lines 6 to 11). Next, we show $W_{h'(T[n_1])} = L$. As $W_{h'(T[n_1])} \subseteq W_{h(\text{content}(T[n_1]))}$ (see Equation (1)) and $\text{content}(T[n_1])$ is a **Bc**-locking set for h on L, we get $W_{h'(T[n_1])} \subseteq L$. For the other direction, let t' be the current step of enumeration. Observe that Condition (5′) implies that $W_{h'(T[n_1])}$ enumerates elements only via the second if-clause (see lines 6 to 11). As $\text{content}(T[n_1])$ is a **Bc**-locking set for h on L, we have, for all $D \subseteq W^{t'}_{h(\text{content}(T[n_1]))}$,

$$\bigcup_{D' \subseteq W^{t'}_{h(\text{content}(T[n_1]))}} W^{t'}_{h(\text{content}(T[n_1]) \cup D')} \subseteq W_{h(\text{content}(T[n_1]) \cup D)} = L.$$

Thus, at some step t, $E_{t+1} \leftarrow W^{t'}_{h(\text{content}(T[n_1]))}$ and then the enumeration continues with $t' \leftarrow t'+1$. In the end we have $L \subseteq W_{h'(T[n_1])}$ and, altogether, $L = W_{h'(T[n_1])}$. This concludes the proof. □

Now, the set-driven map is completed, as monotone learning is a restriction, but strictly more powerful than strongly monotone learning [12].

4 Partially Set-Driven Map

Theorem 2 already shows that semantically conservative, semantically witness-based, cautious and weakly monotone learning coincide in the partially set-driven setting. However, these restrictions are known to be restrictive [12,17]. Furthermore, they are known to be incomparable to monotone learning [12], while both are more powerful than strongly monotone learning [12]. Non-U-shaped learning separates from the mentioned restrictions [12,17]. However, non-U-shaped learning is a restriction for Gold-style learners [2,9] and, equivalently, for partially set-driven learners [8]. Note that for explanatory learners this is not the case [4].

We complete this map by showing that monotone learning implies non-U-shaped learning, see Theorem 3. In particular, we create a new hypothesis by adding all information obtainable by some future hypothesis generated from all seen elements. If this generates a correct hypothesis, no future hypotheses may be wrong, as otherwise the current hypothesis must contain further elements.

Theorem 3. *We have that* $[\mathbf{TxtPsdMonBc}] \subseteq [\mathbf{TxtPsdNUBc}]$.

Proof. Let h be a **TxtPsdMonBc**-learner. Note that h is, without loss of generality, strongly **Bc**-locking [17]. Let furthermore $\mathcal{L} = \mathbf{TxtPsdMonBc}(h)$. We provide a **TxtPsdNUBc**-learner h' which learns $\mathcal{L}$. For all finite sets $D \subseteq \mathbb{N}$, all $t < \infty$ and all $s \in \mathbb{N}_{\geq 1}$, define

$$W^0_{h'(D,t)} = D,$$
$$W^s_{h'(D,t)} = \bigcup_{\substack{(D',t') \text{ with} \\ (D,t) \preceq (D',t') \preceq (W^{s-1}_{h'(D,t)}, t+s)}} W^{s-1}_{h(D',t')}.$$

Finally, $W_{h'(D,t)} = \bigcup_{s \in \mathbb{N}} W^s_{h'(D,t)}$. Intuitively, the learner h' produces its hypothesis on (D,t) iteratively. At stage s, $W^s_{h'(D,t)}$ enumerates all elements witnessed by the learner h on some hypothesis extending (D,t) using elements witnessed so far, that is, elements in $W^{s-1}_{h'(D,t)}$.

We show that h' **TxtPsdNUBc**-learns $\mathcal{L}$. Let $L \in \mathcal{L}$ and $T \in \mathbf{Txt}(L)$. We provide a proof in two steps.

1. We first show that there exists an n_0 such that $W_{h'(\mathrm{content}(T[n_0]),n_0)} = L$.
2. Afterwards, we show that, for all n, whenever $W_{h'(\mathrm{content}(T[n]),n)} = L$ we have, for all $n' > n$, also $W_{h'(\mathrm{content}(T[n']),n')} = L$.

For the first, let n_0 be such that $(D,t) \coloneqq (\mathrm{content}(T[n_0]), n_0)$ is a **Bc**-locking information for h on L. Then, by definition of h', we have $W_{h'(D,t)} \supseteq W_{h(D,t)} = L$. For the other direction, we show that for all $s \in \mathbb{N}$ we have $W^s_{h'(D,t)} \subseteq L$ by induction on s. We get the statement for $s = 0$ immediately. Assuming it holds for $s \in \mathbb{N}$, we show it for $s+1$. Since $W^s_{h'(D,t)} \subseteq L$, we have that (D',t') with $(D,t) \preceq (D',t') \preceq (W^s_{h'(D,t)}, t+s+1)$ is also a **Bc**-locking information for h on L. In particular, we have $W_{h(D',t')} = L$. This results in

$$W^{s+1}_{h'(D,t)} = \bigcup_{\substack{(D',t') \text{ with} \\ (D,t) \preceq (D',t') \preceq (W^{s}_{h'(D,t)}, t+s+1)}} W^{s}_{h(D',t')} \subseteq L.$$

For the second claim, let $n \in \mathbb{N}$ and $(D,t) \coloneqq (\mathrm{content}(T[n]), n)$ be such that $W_{h'(D,t)} = L$, let $n' \geq n$ and $(D'',t'') \coloneqq (\mathrm{content}(T[n']), n')$. Note that $D \subseteq D'' \subseteq L$ and $t'' \geq t$. We show that $W_{h'(D'',t'')} = L$. First, note that (D'',t'') will eventually be considered when enumerating $W_{h'(D,t)}$, that is, there exists an $s \in \mathbb{N}$ such that $(D'',t'') \preceq (W^{s-1}_{h'(D,t)}, t+s)$. Hence,

$$\begin{aligned} W_{h'(D'',t'')} &= \bigcup_{s \in \mathbb{N}} \bigcup_{\substack{(D',t') \text{ with} \\ (D'',t'') \preceq (D',t') \preceq (W^{s-1}_{h'(D'',t'')}, t+s)}} W^{s-1}_{h(D',t')} \subseteq \\ &\subseteq \bigcup_{s \in \mathbb{N}} \bigcup_{\substack{(D',t') \text{ with} \\ (D,t) \preceq (D',t') \preceq (W^{s-1}_{h'(D,t)}, t+s)}} W^{s-1}_{h(D',t')} = W_{h'(D,t)} = L. \end{aligned}$$

Secondly, we show that for each $x \in L = W_{h'(D,t)}$ we also have $x \in W_{h'(D'',t'')}$. We show (by induction on s) that $W^s_{h'(D,t)} \subseteq W_{h'(D'',t'')}$. For $s = 0$ we have $W^s_{h'(D,t)} = D \subseteq D'' = W^0_{h'(D'',t'')} \subseteq W_{h'(D'',t'')}$. Let the statement be fulfilled until s. At step $s+1$, we distinguish the following cases.

1. Case: If $W^s_{h'(D,t)} = W^{s+1}_{h'(D,t)}$, that is, no new element is enumerated, the statement of the induction step is true immediately.
2. Case: If $W^s_{h'(D,t)} \subsetneq W^{s+1}_{h'(D,t)}$, let $x \in W^{s+1}_{h'(D,t)} \setminus W^s_{h'(D,t)}$. Note that $x \in L$. Let $(\tilde{D}, \tilde{t})$, with $(D,t) \preceq (\tilde{D}, \tilde{t}) \preceq (W^s_{h'(D,t)}, t+s)$, be the information on which x was witnessed, that is, $x \in W_{h(\tilde{D},\tilde{t})}$. By $W^s_{h'(D,t)} \subseteq W_{h'(D'',t'')}$ (the induction assumption), there exists s'' such that $(\tilde{D}, \tilde{t}) \preceq (W^{s''}_{h'(D'',t'')}, t'' + s'')$. Since h is monotone and $x \in L$ we have

$$x \in W_{h(W^{s''}_{h'(D'',t'')}, t''+s'')} \overset{\text{def. of } h'}{\subseteq} W_{h'(D'',t'')}.$$

Altogether, we get the desired result. □

5 Gold-Style Learning Map

The overall situation for Gold-style learning is basically analogous to the initial situation for partially set-driven learning as discussed in Section 4, compare the literature [2,9,12,17] and Theorem 2. We complete the map by showing that monotone learning implies non-U-shaped learning, see 5.

We aim to employ a similar approach as for the partially set-driven case. To that end, we have to overcome two obstacles. Firstly, we show that monotone Gold-style learners are strongly **Bc**-locking, see Theorem 4. In particular, this shows that *all* restrictions studied in this paper allow for strongly **Bc**-locking learning [4,17]. Secondly, Gold-style learners infer from sequences, meaning that extensions considered at a certain step do not necessarily have to be considered in later steps (as opposed to partially set-driven learning). We circumvent this by also enumerating elements from previous guesses on which the learner shows a monotone behaviour, as they are likely part of the target language.

Theorem 4. *Any* **TxtGMonBc***-learner may be assumed strongly* **Bc***-locking.*

Proof. This proof is inspired by the proof based on private communication with Sanjay Jain where, for certain restrictions δ, [**TxtPsd**δ**Bc**] = [**TxtG**δ**Bc**] is shown [8, Thm. 10]. Let h be a learner and let $\mathcal{L} = \mathbf{TxtGMonBc}(h)$. We provide a strongly **Bc**-locking **TxtGMonBc**-learner h' for $\mathcal{L}$ as follows. For two finite sequences σ, σ', define the auxiliary function g as

$$W_{g(\sigma',\sigma)} = \bigcap_{\tau \in \mathrm{content}(\sigma)^{\leq |\sigma|}_{\#}} W_{h(\sigma'\tau)} \cap \bigcap_{\substack{\sigma'' \leq \sigma', \\ \sigma'' \in \mathrm{content}(\sigma')^*_{\#}}} \bigcup_{\tau'' \in \mathrm{content}(\sigma')^*_{\#}} W_{h(\sigma''\tau'')}.$$

Then, define the learner h' on finite sequences σ as

$$W_{h'(\sigma)} = \bigcup_{\sigma' \subseteq \sigma} W_{g(\sigma',\sigma)}.$$

The intuition is the following. With the function g, we search for minimal **Bc**-locking sequences [8]. To ensure that g eventually only contains elements from the target language, we extend the left hand intersection to be based on σ. However, as σ contains more and more information, additional sequences are also considered in the right hand intersection. This may lead to already enumerated elements being discarded (even if they belong to a target language). To prevent this, we take the union over all possible $W_{g(\sigma',\sigma)}$.

We formally show that h' has the desired properties. First, we show that h' is **Mon**. Let $L \in \mathcal{L}$ and $\sigma_1, \sigma_2 \in L^*_\#$ with $\sigma_1 \subseteq \sigma_2$. We show that for all $x \in \mathbb{N}$

$$x \in W_{h'(\sigma_1)} \cap L \Rightarrow x \in W_{h'(\sigma_2)} \cap L.$$

As $x \in W_{h'(\sigma_1)}$, there exists $\sigma_1' \subseteq \sigma_1$ such that $x \in W_{g(\sigma_1',\sigma_1)}$, that is,

$$x \in \bigcap_{\tau \in \mathrm{content}(\sigma_1)^{\leq |\sigma_1|}_\#} W_{h(\sigma_1'\tau)} \cap \bigcap_{\substack{\sigma'' \leq \sigma_1', \\ \sigma'' \in \mathrm{content}(\sigma_1')^*_\#}} \bigcup_{\tau'' \in \mathrm{content}(\sigma_1')^*_\#} W_{h(\sigma''\tau'')}. \quad (6)$$

In particular, $x \in W_{h(\sigma_1')}$. We show that $x \in W_{g(\sigma_1',\sigma_2)}$. By monotonicity of h, we have that

$$x \in \bigcap_{\tau \in \mathrm{content}(\sigma_2)^{\leq |\sigma_2|}_\#} W_{h(\sigma_1'\tau)}.$$

As the right hand intersection in Equation (6) (of which x is an element) does not depend on σ_1, we have that

$$x \in \bigcap_{\tau \in \mathrm{content}(\sigma_2)^{\leq |\sigma_2|}_\#} W_{h(\sigma_1'\tau)} \cap \bigcap_{\substack{\sigma'' \leq \sigma_1', \\ \sigma'' \in \mathrm{content}(\sigma_1')^*_\#}} \bigcup_{\tau'' \in \mathrm{content}(\sigma_1')^*_\#} W_{h(\sigma''\tau'')} =$$
$$= W_{g(\sigma_1',\sigma_2)}.$$

By definition of h' and since $\sigma_1' \subseteq \sigma_1 \subseteq \sigma_2$, we have

$$W_{g(\sigma_1',\sigma_2)} \subseteq \bigcup_{\sigma' \subseteq \sigma_2} W_{g(\sigma',\sigma_2)} = W_{h'(\sigma_2)}.$$

Thus, $x \in W_{h'(\sigma_2)} \cap L$.

We now show that h' is strongly **Bc**-locking (and thus also **Bc**-learns $\mathcal{L}$). Let $L \in \mathcal{L}$ and $T \in \mathbf{Txt}(L)$. Let $\sigma_0 \in L^*_\#$ be the $\leq$-minimal **Bc**-locking sequence for h on L [3]. For each $\sigma' < \sigma_0$ with $\mathrm{content}(\sigma') \subseteq L$, let $\tau_{\sigma'} \in L^*_\#$ be such that $\sigma'\tau_{\sigma'}$ is a **Bc**-locking sequence for h on L [20]. Let n_0 be such that h converges on $T[n_0]$, that is, for all $n' \geq n_0$, $W_{h(T[n])} = L$. Let $n_1 \geq n_0$ be such that

– $\sigma_0 \leq T[n_1]$,
– $\sigma_0 \in \mathrm{content}(T[n_1])^*_\#$, and
– for all $\sigma' < \sigma_0$ such that $\mathrm{content}(\sigma') \subseteq L$, we have that $\mathrm{content}(\sigma'\tau_{\sigma'}) \subseteq \mathrm{content}(T[n_1])$ and $|\tau_{\sigma'}| \leq n_1$.

To show that $\sigma_1 \coloneqq T[n_1]$ is a **Bc**-locking sequence for h' on L, we show that, for any $\rho \in L^*_\#$, $\sigma_1 {}^\frown \rho \eqqcolon \sigma_1\rho$ is a correct guess, that is, $W_{h'(\sigma_1\rho)} = L$. Let $\rho \in L^*_\#$. We prove each direction of $W_{h'(\sigma_1\rho)} = L$ separately.

1.C.: $W_{h'(\sigma_1\rho)} \subseteq L$: Let $x \in W_{h'(\sigma_1\rho)}$. Then there exists $\sigma' \subseteq \sigma_1\rho$ such that $x \in W_{g(\sigma',\sigma_1\rho)}$. In particular,

$$x \in \bigcap_{\tau \in \mathrm{content}(\sigma_1\rho)^{\leq|\sigma_1\rho|}_\#} W_{h(\sigma'\tau)} \cap \bigcap_{\substack{\sigma'' \leq \sigma', \\ \sigma'' \in \mathrm{content}(\sigma')^*_\#}} \bigcup_{\tau'' \in \mathrm{content}(\sigma')^*_\#} W_{h(\sigma''\tau'')}. \tag{7}$$

We distinguish based on the relation between σ' and σ_1.

1.1.C.: If $\sigma' \subseteq \sigma_1$, then there exists $\tau \in \mathrm{content}(\sigma_1\rho)^{\leq|\sigma_1\rho|}_\#$ such that $\sigma'\tau = \sigma_1$. As $h(\sigma_1)$ is a correct guess and $W_{h(\sigma_1)}$ is considered in the left hand intersection of Equation (7), we have that $x \in L$.

1.2.C.: If $\sigma' \supsetneq \sigma_1$, we have $\sigma_0 \leq \sigma_1 \subseteq \sigma'$ and $\sigma_0 \in \mathrm{content}(\sigma_1)^*_\# \subseteq \mathrm{content}(\sigma')^*_\#$. Thus, σ_0 is considered in the right hand intersection of Equation (7). Since, for any $\tau \in L^*_\#$, we have $W_{h(\sigma_0\tau)} = L$, we get $x \in W_{h(\sigma_0\tau)} = L$.

2.C.: $L \subseteq W_{h'(\sigma_1\rho)}$: Let $x \in L$. We show that $x \in W_{g(\sigma_1,\sigma_1\rho)}$. As h is monotone, $\sigma_1 \subseteq \sigma_1\rho$ and h converges on σ_1, we have

$$x \in \bigcap_{\tau \in \mathrm{content}(\sigma_1)^*_\#} W_{h(\sigma_1 {}^\frown \tau)}.$$

Moreover, by choice of n_1, we have, for all $\sigma'' \leq \sigma_1$ with $\sigma'' \in \mathrm{content}(\sigma_1)^*_\#$, that $\tau''_{\sigma''} \in \mathrm{content}(\sigma_1)^*_\#$. As $\sigma''\tau''_{\sigma''}$ is a **Bc**-locking sequence for h on L, we get $x \in W_{h(\sigma''\tau''_{\sigma''})}$. Hence,

$$x \in \bigcap_{\substack{\sigma'' \leq \sigma_1, \\ \sigma'' \in \mathrm{content}(\sigma_1)^*_\#}} \bigcup_{\tau'' \in \mathrm{content}(\sigma_1)^*_\#} W_{h(\sigma''\tau'')}.$$

Altogether, $x \in W_{g(\sigma_1,\sigma_1\rho)} \subseteq W_{h'(\sigma_1\rho)}$.

In the end, we have $W_{h'(\sigma_1\rho)} = L$, which concludes the proof. □

Theorem 5. *We have that* $[\mathbf{TxtGMonBc}] \subseteq [\mathbf{TxtGNUBc}]$.

Proof. Let h be a **TxtGMonBc**-learner. Without loss of generality, h may be assumed strongly **Bc**-locking, see Theorem 4. Let $\mathcal{L} = \mathbf{TxtGMonBc}(h)$. We provide a learner h' which **TxtGNUBc**-learns $\mathcal{L}$. To do so, we employ both a *forward enumeration strategy* (via sets $F_{\sigma,s}$) as well as a *backward search strategy* (via sets $B_{\sigma,s}$). For a finite sequence σ and computation step $s \in \mathbb{N}$ we define $F_{\sigma,s}$ (forward enumeration set) and $B_{\sigma,s}$ (backwards search set) as follows. Let $F_{\sigma,0} = B_{\sigma,0} = \mathrm{content}(\sigma)$. Furthermore, let

$$F_{\sigma,s+1} = F_{\sigma,s} \cup \bigcup_{\tau \in (F_{\sigma,s} \cup B_{\sigma,s})^{\leq s}_{\#}} W^s_{h(\sigma\tau)}.$$

Intuitively, $F_{\sigma,s+1}$ contains all elements enumerated by some possible future guess, that is, for $\tau \in (F_{\sigma,s} \cup B_{\sigma,s})^{\leq s}_{\#}$, $W_{h(\sigma\tau)}$. Note that this is a similar approach as in the **Psd**-case, see the proof of Theorem 3. However, as opposed to partially set-driven learning, this alone does not suffice. In particular, $F_{\sigma,s}$ may consider $\sigma^\frown\tau$ and $\sigma^\frown\tau'$, where $\tau \neq \tau'$, in its enumeration, but, for a later hypothesis σ', $F_{\sigma',s}$ cannot consider both, as σ' cannot extend both $\sigma^\frown\tau$ and $\sigma^\frown\tau'$. To circumvent this, we need the backwards search set $B_{\sigma,s}$.

To define $B_{\sigma,s}$, we introduce the following auxiliary predicate and function. Given a learner h (we omit using Gödel numbers in favour of readability), finite sequences σ and ρ, an element $x \in \mathbb{N}$ and a counter $s \in \mathbb{N}$, we define

$$\mathbf{MonBeh}(h, \rho, x, s, \sigma) \Leftrightarrow \forall \tau \in \mathrm{content}(\sigma)^{\leq s+|\sigma|} : x \in W_{h(\rho^\frown\tau)}.$$

Intuitively, $\mathbf{MonBeh}(h, \rho, x, s, \sigma)$ checks whether h, starting on information ρ, exhibits a monotonic behaviour regarding the element x. We further introduce a function which gives us the newly enumerated element by some hypothesis. In particular, let $\tilde{x} := \mathbf{nextEl}(h', \sigma', \sigma, s)$ be the element enumerated next by $F_{\sigma',s}$ which is not yet in $W_{h'(\sigma)}$. Furthermore, let $\tilde{\sigma} := \sigma'^\frown\tau$ be the (minimal) sequence on which $\tilde{x}$ has been seen for the first time inside $F_{\sigma',s}$. We define the backwards search set as, for finite sequences σ, σ'' and $s \in \mathbb{N}$,

$$\begin{aligned} B_{\sigma,0,\sigma''} &= \mathrm{content}(\sigma''), \\ B_{\sigma,s+1,\sigma''} &= B_{\sigma,s,\sigma''} \cup \begin{cases} \{\tilde{x}\}, & \text{for } \tilde{x} = \mathbf{nextEl}(h', \sigma'', \sigma, s) \text{ via } \tilde{\sigma} \text{ if} \\ & \mathbf{MonBeh}(h', \tilde{\sigma}, \tilde{x}, s, \sigma), \\ \emptyset, & \text{else.} \end{cases} \\ B_{\sigma,s+1} &= B_{\sigma,s} \cup \bigcup_{\sigma'' \subsetneq \sigma} B_{\sigma,s,\sigma''}. \end{aligned}$$

Note that $\bigcup_{s \in \mathbb{N}} B_{\sigma,s,\sigma''} \subseteq \bigcup_{s \in \mathbb{N}} F_{\sigma'',s}$. The idea behind the backwards search is based on the following observation. Given two sequences $\sigma' \subseteq \sigma$, let x be the first element enumerated by $F_{\sigma',s}$ (which is not in $\mathrm{content}(\sigma')$). If x is an element of the target language, it will eventually be enumerated in $F_{\sigma,s}$ as well (as it has to appear in $W_{h(\sigma)}$ by monotonicity of h). However, further enumerations may not be similar, as $F_{\sigma',s}$ may build its further hypotheses on $\sigma'^\frown x$, which in general is no subsequence of σ. With the backwards search, we check for such elements and enumerate them in case the learner h shows a monotonic behaviour regarding them. In the end, we define the learner h' as

$$W_{h'(\sigma)} = \bigcup_{s \in \mathbb{N}} B_{\sigma,s} \cup F_{\sigma,s}.$$

We show that h' **TxtGNUBc**-learns $\mathcal{L}$. Let $L \in \mathcal{L}$ and $T \in \mathbf{Txt}(L)$. First, we show **Bc**-convergence and afterwards that h' is **NU**. To that end, let n_0 be such that $T[n_0]$ is a **Bc**-locking sequence for h on L (this exists by Theorem 4). For each $n < n_0$, let $\tilde{x}_n = \mathbf{nextEl}(h', T[n], T[n_0], s)$ (via $\tilde{\sigma}_n$) be the first newly enumerated element not in L (if such exists). Then, let Let $n_1 \geq n_0$ be such that, for $n < n_0$, for each $\tilde{\sigma}_n$ there exists a $\tau \in \text{content}(T[n_1])_{\#}^{\leq |T[n_1]|}$ such that $h(\tilde{\sigma}_n \frown \tau)$ is a correct guess. In particular, $\mathbf{MonBeh}(h, \tilde{\sigma}, \tilde{x}, s, T[n_1])$ fails and, therefore, no $B_{T[n_1],s,T[n]}$ contains elements which are not in L.

Also, for $n \geq n_0$, $B_{T[n_1],s,T[n]}$ only contains elements in L (as $T[n_0]$ is a **Bc**-locking sequence). Hence, for $n \geq n_1$, we have

$$\bigcup_{s \in \mathbb{N}} B_{T[n],s} \subseteq L.$$

In particular, as $T[n]$ is also a **Bc**-locking sequence, we get

$$\bigcup_{s \in \mathbb{N}} F_{T[n],s} = L.$$

Thus, $W_{h'(T[n])} = L$.

It remains to be shown that h' is **NU**. Let n be minimal such that $W_{h'(T[n])} = L$. We show that, for $n' \geq n$, we have $W_{h'(T[n'])} = L$ as well. Note that by definition of the backwards search sets, for $\tilde{n} \leq n$, we have

$$\bigcup_{s \in \mathbb{N}} B_{T[n],s,T[\tilde{n}]} \supseteq \bigcup_{s \in \mathbb{N}} B_{T[n'],s,T[\tilde{n}]}.$$

Furthermore,

$$\bigcup_{s \in \mathbb{N}} F_{T[n],s} \supseteq \bigcup_{s \in \mathbb{N}} F_{T[n'],s} \cup \bigcup_{\substack{\tilde{n} \in \mathbb{N}, \\ n \leq \tilde{n} \leq n'}} \bigcup_{s \in \mathbb{N}} B_{T[n'],s,T[\tilde{n}]},$$

as, firstly, $T[n']$ is a candidate within, from some s onwards, $F_{T[n],s}$ and, secondly, the backwards search set $\bigcup_{s \in \mathbb{N}} B_{T[n'],s,T[\tilde{n}]}$ can only enumerate as much as the forward enumeration set $\bigcup_{s \in \mathbb{N}} F_{T[\tilde{n}],s}$. Thus, $W_{h'(T[n'])} \subseteq W_{h'(T[n])} = L$. Next we show that each element $x \in W_{h'(T[n])}$ will be enumerated in $W_{h'(T[n'])}$. We show this by case distinction depending how x is enumerated in $W_{h'(T[n])}$.

1.C.: For some s', the element x is enumerated in $F_{T[n],s'}$. Then, we get $x \in \bigcup_{s \in \mathbb{N}} B_{T[n],s,T[n']}$ as the **MonBeh** check passes for elements in L.

2.C.: For some s' and $\tilde{n} \leq n$, we have $x \in B_{T[\tilde{n}],s',T[n]}$. Then, x is also enumerated in $\bigcup_{s \in \mathbb{N}} B_{T[\tilde{n}],s,T[n']}$ as the **MonBeh** check passes for elements in L.

Thus, $W_{h'(T[n'])} \supseteq L$ and, altogether, $W_{h'(T[n'])} = L$. □

References

1. Angluin, D.: Inductive inference of formal languages from positive data. Information and Control **45**, 117–135 (1980)

2. Baliga, G., Case, J., Merkle, W., Stephan, F., Wiehagen, R.: When unlearning helps. Information and Computation **206**, 694–709 (2008)
3. Blum, L., Blum, M.: Toward a mathematical theory of inductive inference. Information and Control **28**, 125–155 (1975)
4. Case, J., Kötzing, T.: Strongly non-U-shaped language learning results by general techniques. Information and Computation **251**, 1–15 (2016)
5. Case, J., Lynes, C.: Machine inductive inference and language identification. In: International Colloquium on Automata, Languages and Programming (ICALP). pp. 107–115 (1982)
6. Doskoč, V., Kötzing, T.: Mapping monotonic restrictions in inductive inference. In: Computability in Europe (CiE). pp. 12813:146–12813:157 (2021)
7. Doskoč, V., Kötzing, T.: Normal forms for semantically witness-based learners in inductive inference. In: Computability in Europe (CiE). pp. 12813:158–12813:168 (2021)
8. Doskoč, V., Kötzing, T.: Cautious limit learning. In: Algorithmic Learning Theory (ALT). pp. 117:251–117:276 (2020)
9. Fulk, M.A., Jain, S., Osherson, D.N.: Open problems in "systems that learn". J. Comput. Syst. Sci. **49**(3), 589–604 (1994)
10. Gold, E.M.: Language identification in the limit. Information and Control **10**, 447–474 (1967)
11. Jain, S., Kötzing, T., Ma, J., Stephan, F.: On the role of update constraints and text-types in iterative learning. Information and Computation **247**, 152–168 (2016)
12. Jain, S., Osherson, D., Royer, J.S., Sharma, A.: Systems that Learn: An Introduction to Learning Theory. MIT Press, Cambridge (MA), Second Edition (1999)
13. Jantke, K.: Monotonic and non-monotonic inductive inference. New Generation Computing **8**, 349–360 (1991)
14. Kinber, E.B., Stephan, F.: Language learning from texts: Mindchanges, limited memory, and monotonicity. Information and Computation **123**, 224–241 (1995)
15. Kötzing, T., Palenta, R.: A map of update constraints in inductive inference. Theoretical Computer Science **650**, 4–24 (2016)
16. Kötzing, T., Schirneck, M.: Towards an atlas of computational learning theory. In: Symposium on Theoretical Aspects of Computer Science (STACS). pp. 47:1–47:13 (2016)
17. Kötzing, T., Schirneck, M., Seidel, K.: Normal forms in semantic language identification. In: Algorithmic Learning Theory (ALT). pp. 76:493–76:516 (2017)
18. Kötzing, T.: Abstraction and Complexity in Computational Learning in the Limit. Ph.D. thesis, University of Delaware (2009)
19. Lange, S., Zeugmann, T.: Monotonic versus non-monotonic language learning. In: Nonmonotonic and Inductive Logic. pp. 254–269 (1993)
20. Osherson, D., Stob, M., Weinstein, S.: Systems that Learn: An Introduction to Learning Theory for Cognitive and Computer Scientists. MIT Press, Cambridge (MA) (1986)
21. Osherson, D.N., Stob, M., Weinstein, S.: Learning strategies. Information and Control **53**, 32–51 (1982)
22. Osherson, D.N., Weinstein, S.: Criteria of language learning. Information and Control **52**, 123–138 (1982)
23. Rogers Jr., H.: Theory of recursive functions and effective computability. Reprinted by MIT Press, Cambridge (MA) (1987)
24. Schäfer-Richter, G.: Über Eingabeabhängigkeit und Komplexität von Inferenzstrategien. Ph.D. thesis, RWTH Aachen University, Germany (1984)

25. Wexler, K., Culicover, P.W.: Formal principles of language acquisition. MIT Press, Cambridge (MA) (1980)
26. Wiehagen, R.: A thesis in inductive inference. In: Nonmonotonic and Inductive Logic. pp. 184–207 (1991)