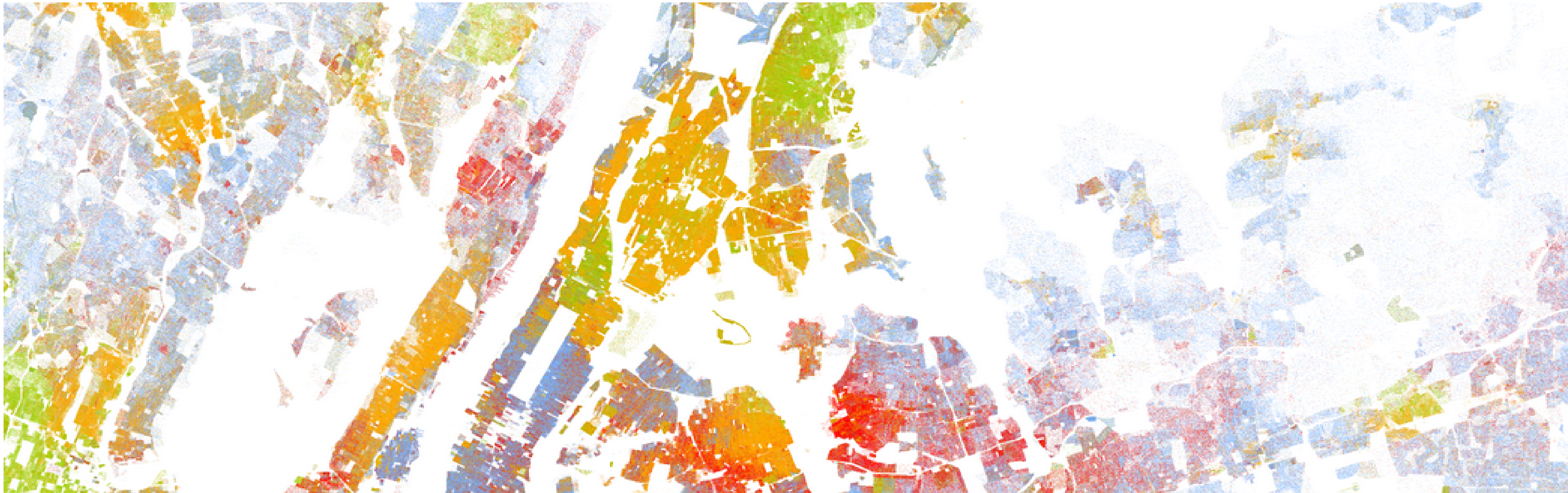
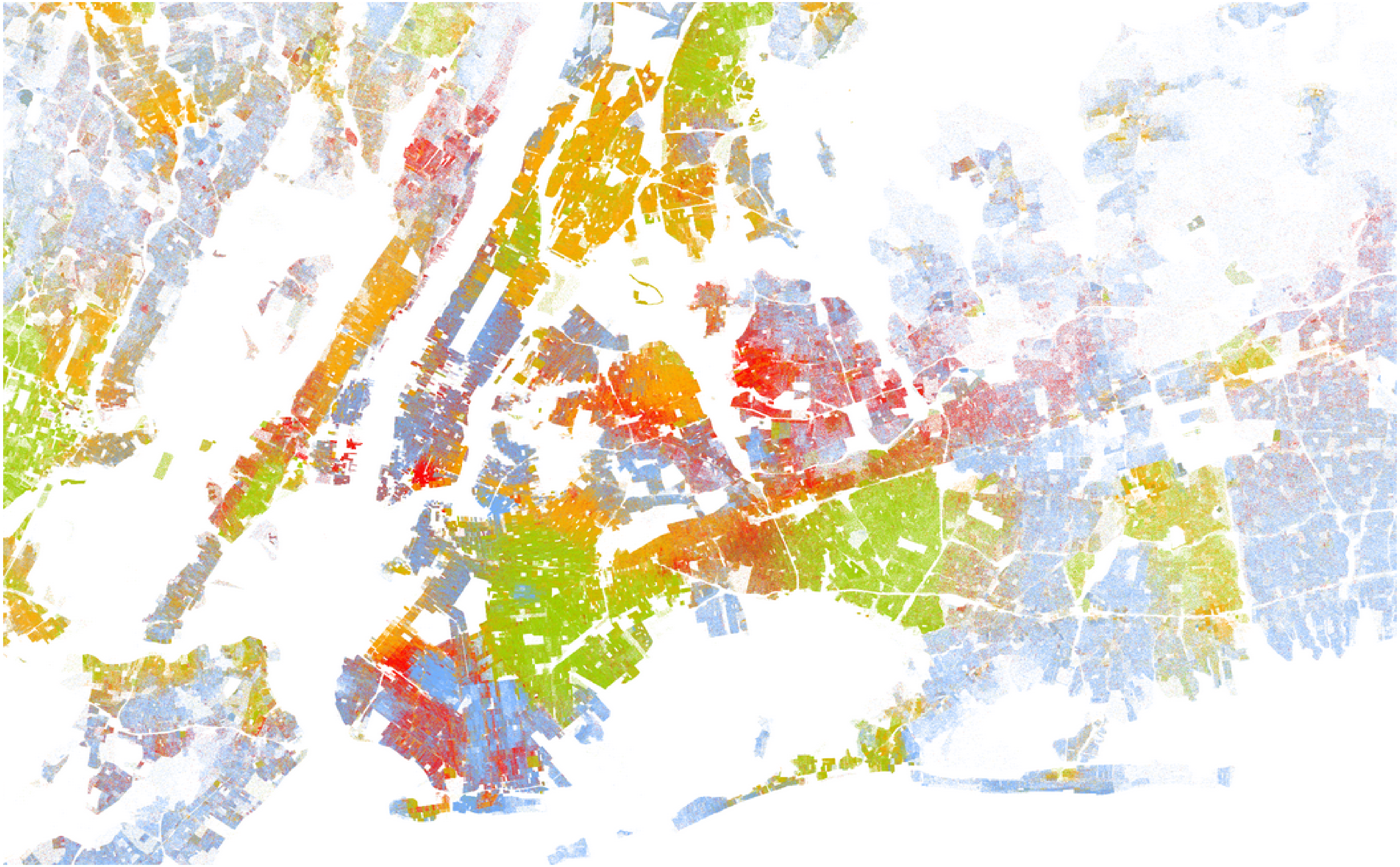


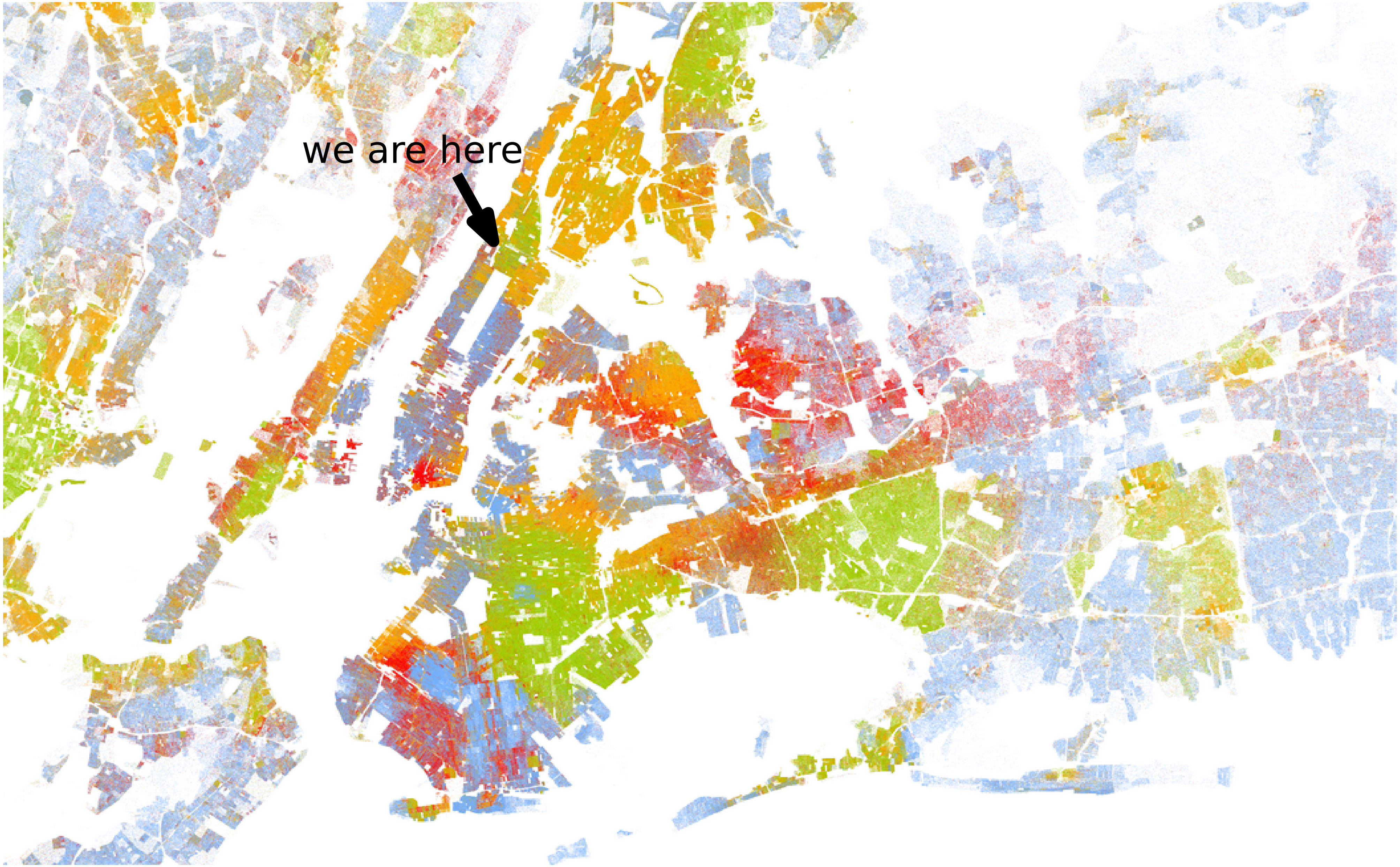
Convergence and Hardness of Strategic Schelling Segregation

WINE Conference 2019

Algorithm Engineering Research Group

H. Echzell, T. Friedrich, P. Lenzner, L. Molitor, **M. Pappik**, F. Schöne, F. Sommer, D. Stangl

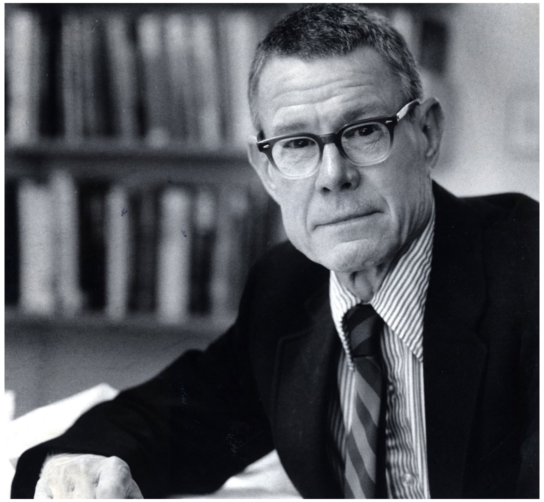



we are here

Schelling Segregation

Thomas Schelling (1921-2016)

- "economics Nobel prize" winner
- *Micromotives and Macrobehavior* (1978)



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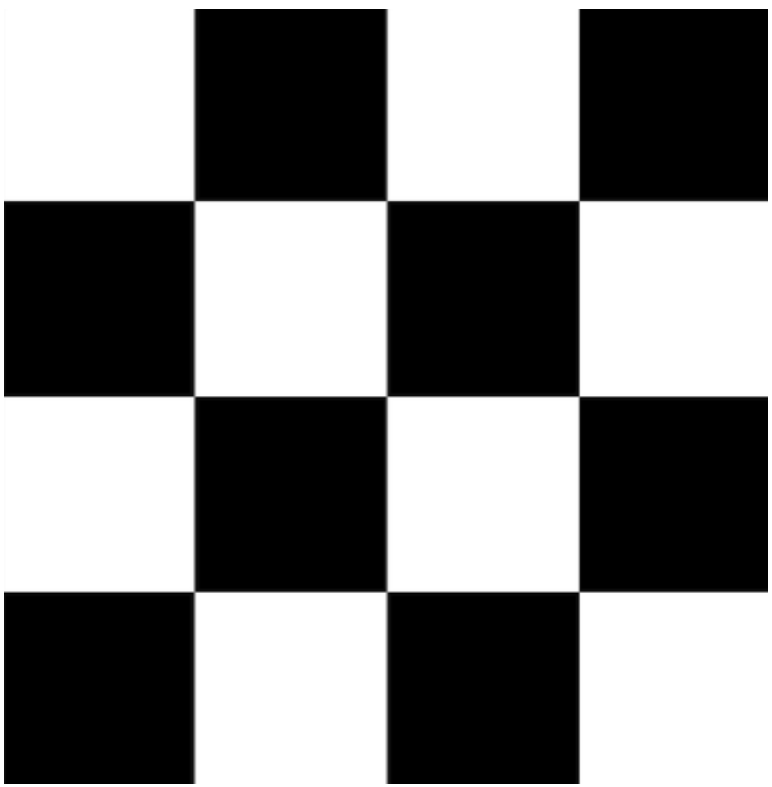

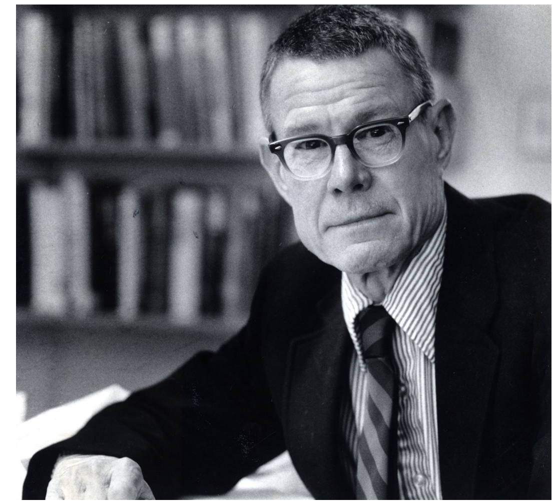

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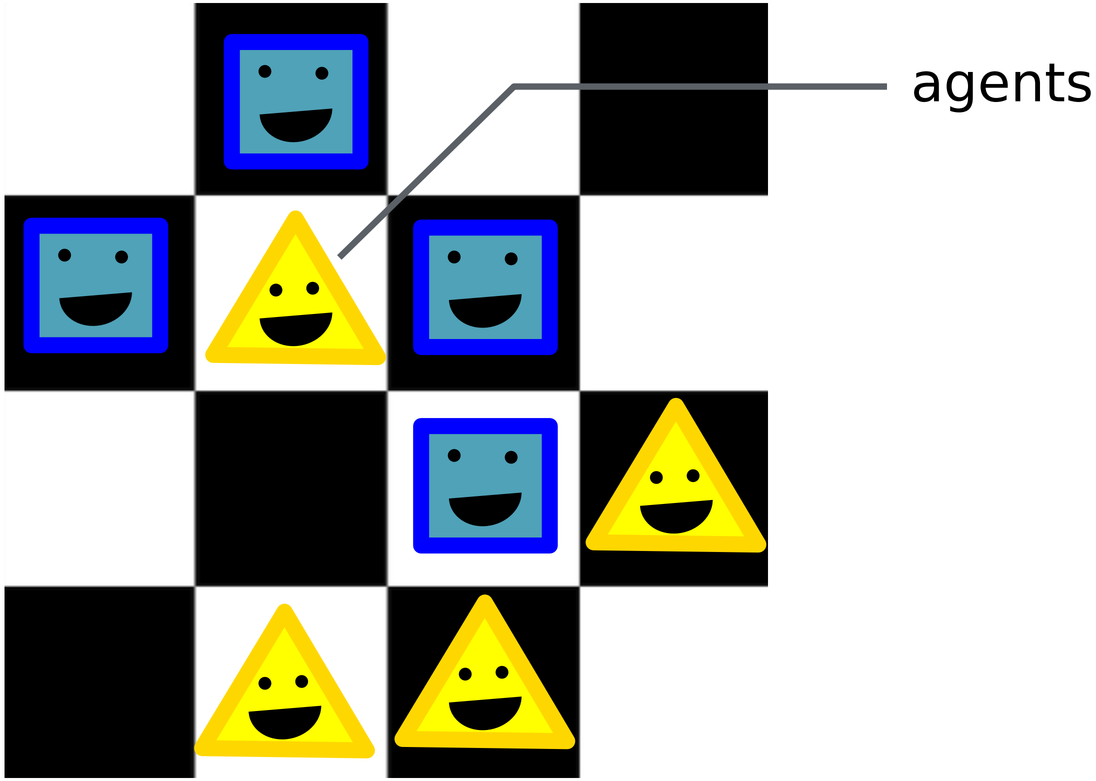

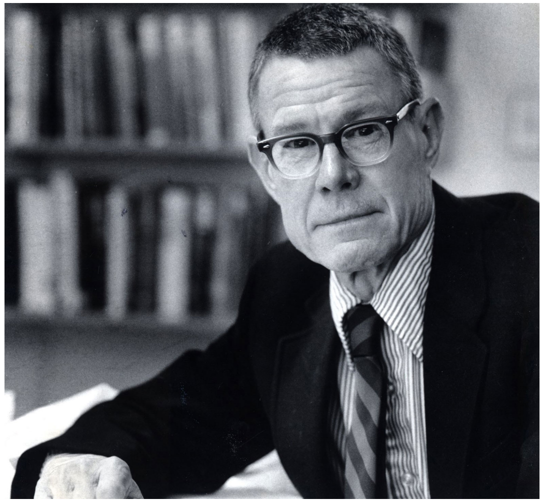

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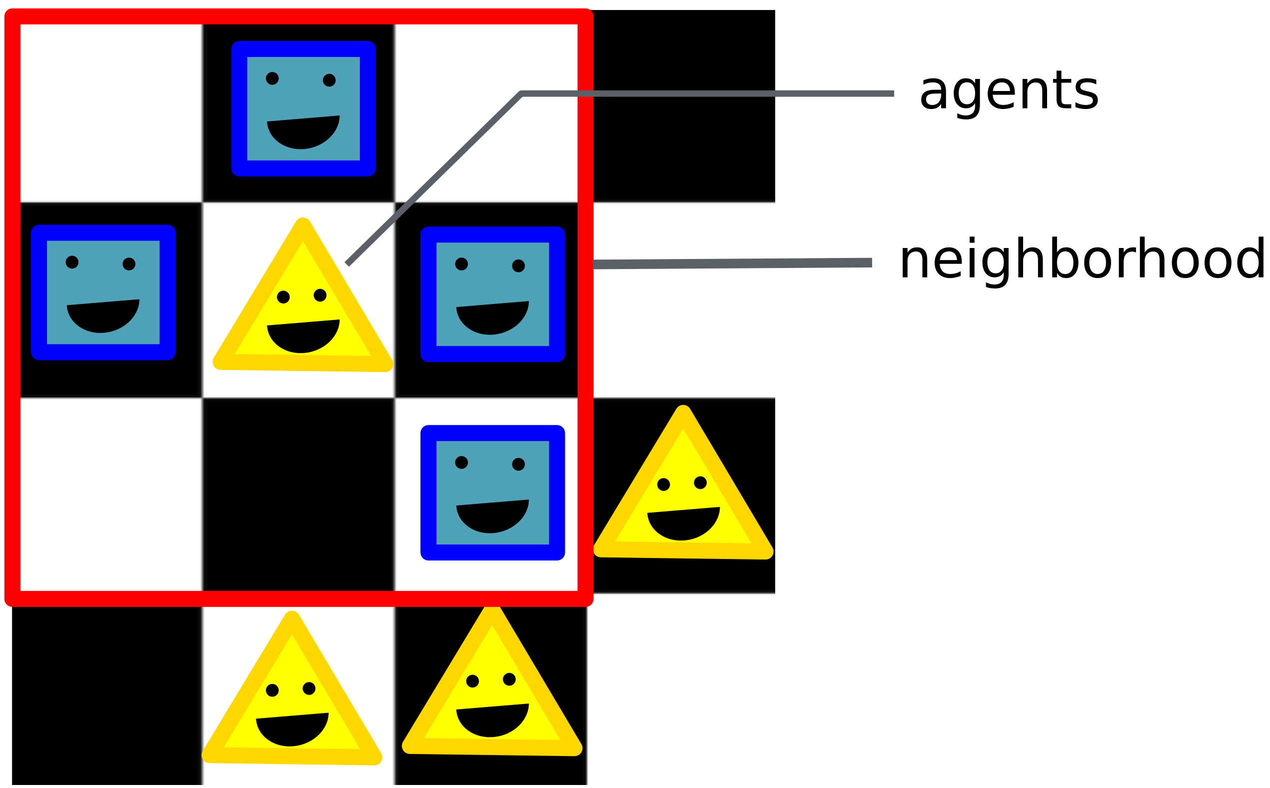

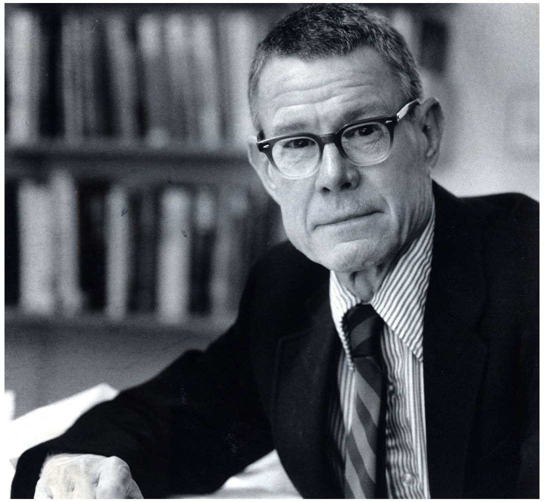

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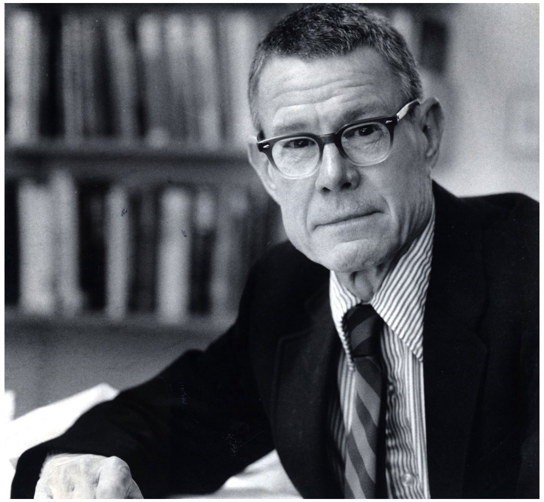

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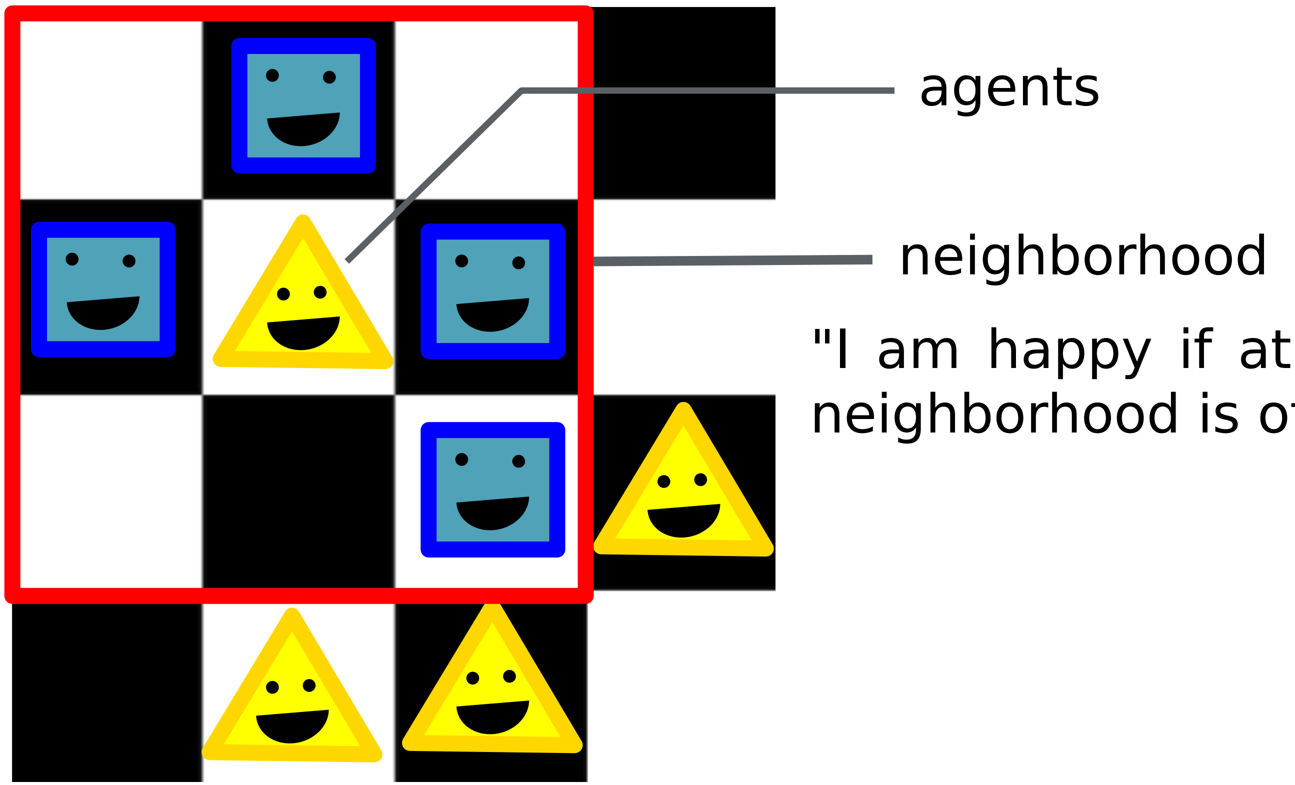

"I am happy if at least a fraction τ of my neighborhood is of my type."

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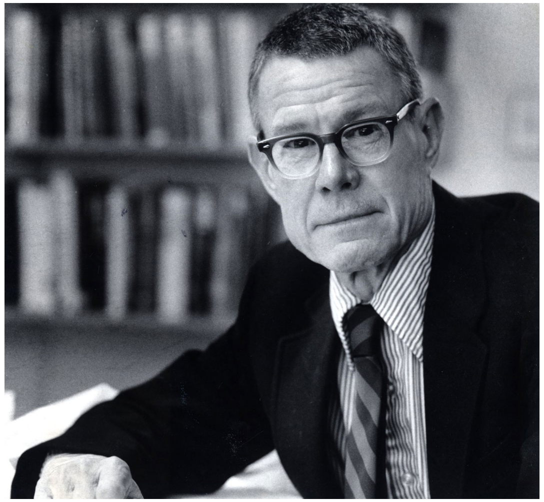

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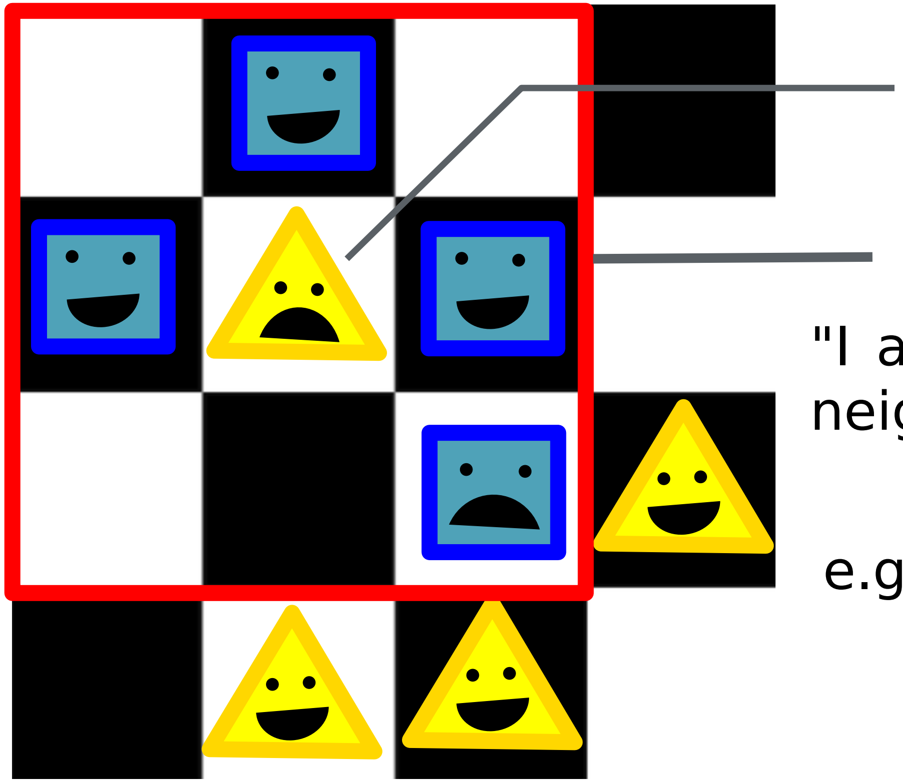

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e.g. $\tau = \frac{1}{4}$

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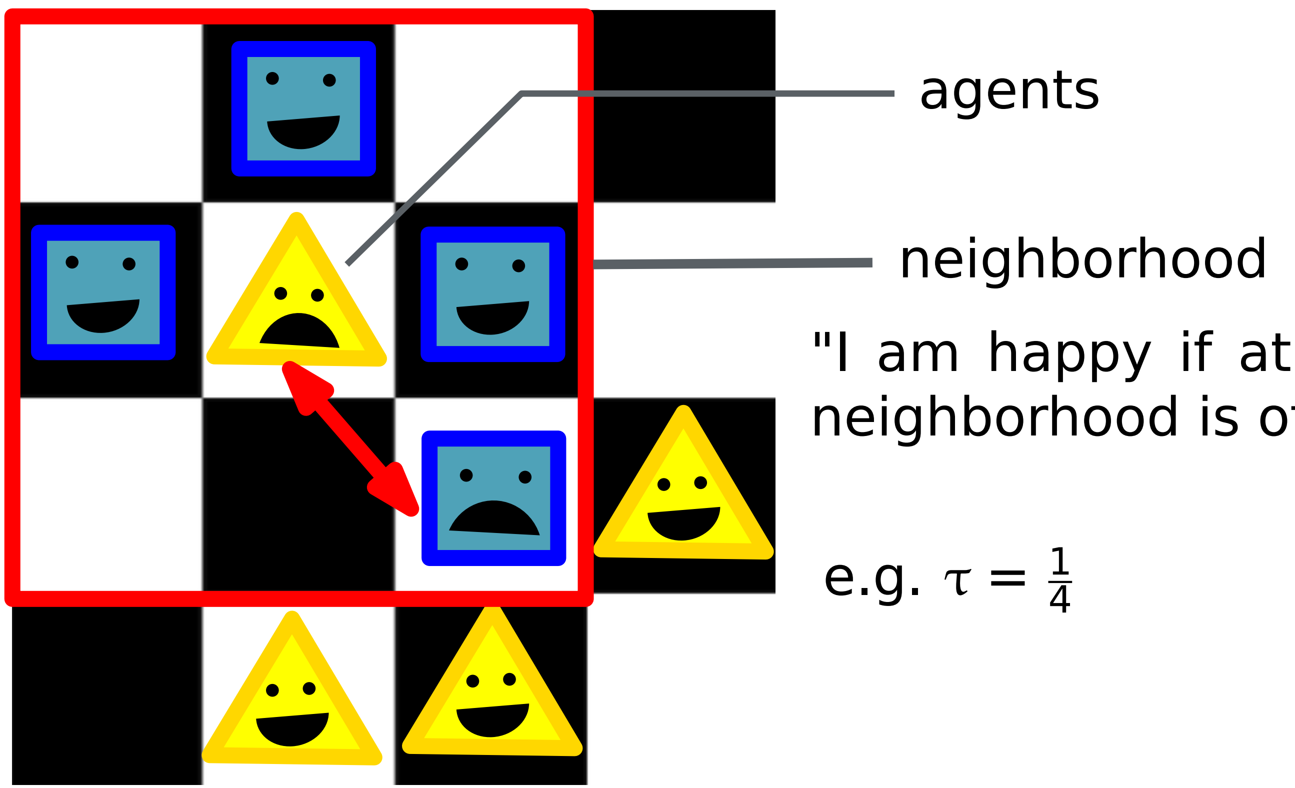

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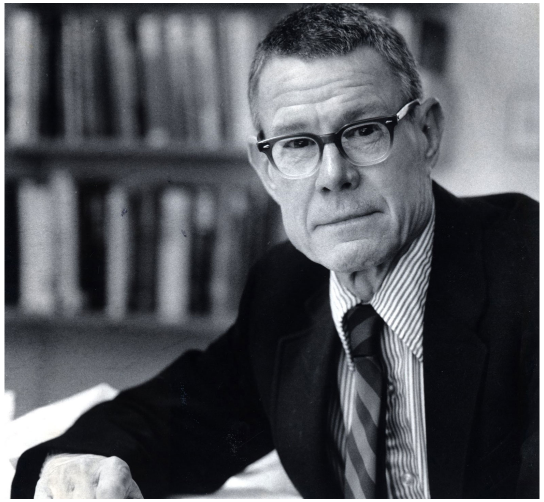

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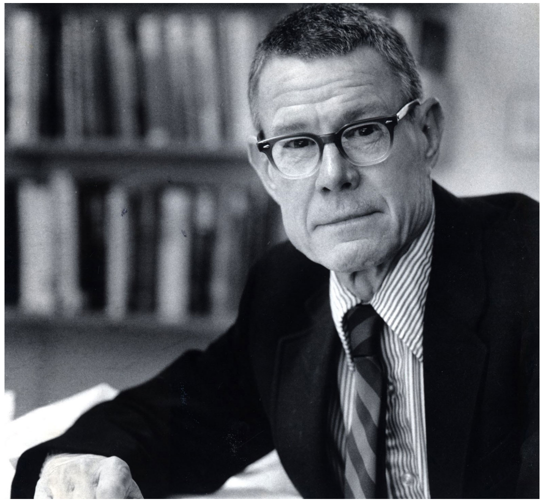

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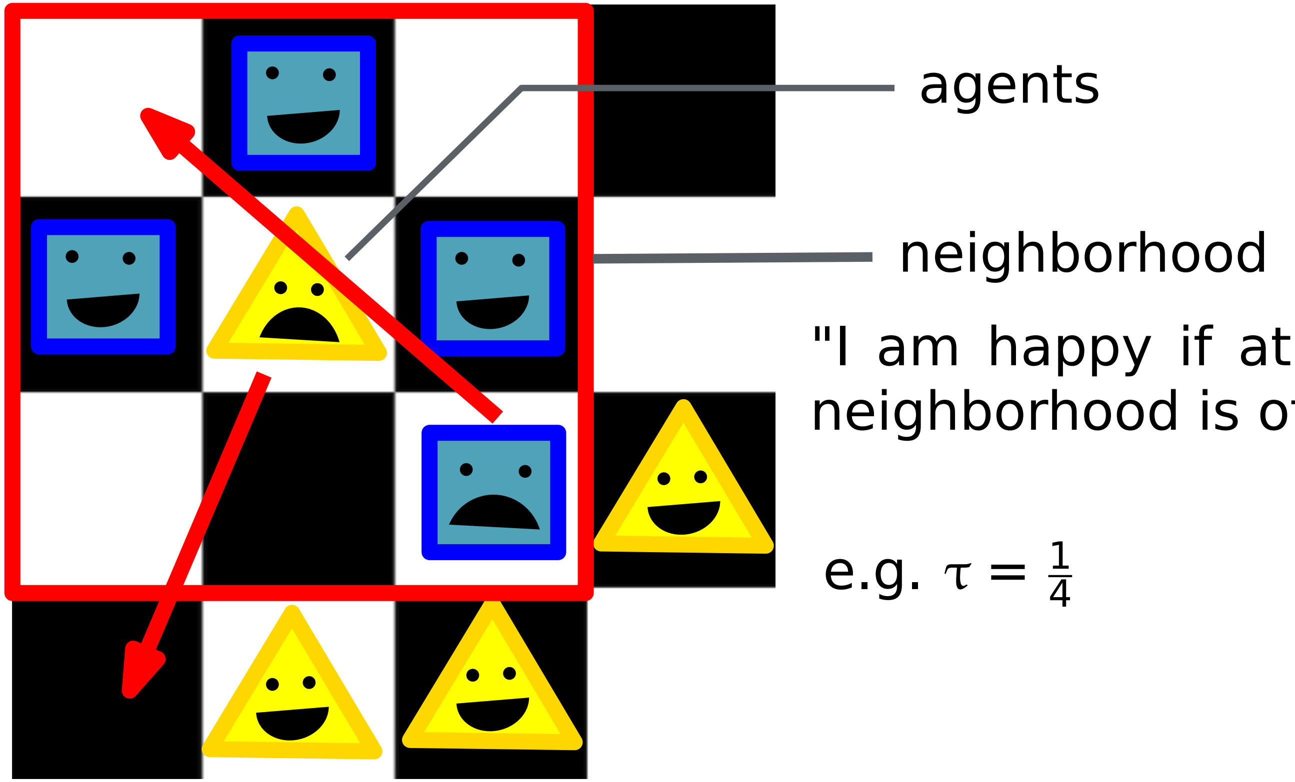

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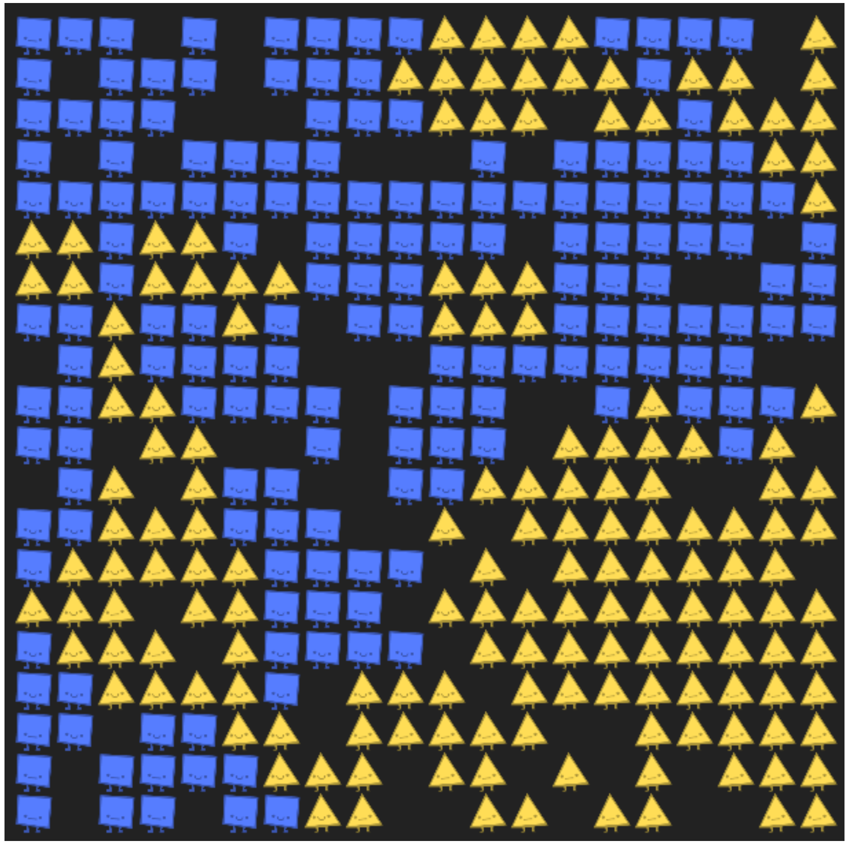

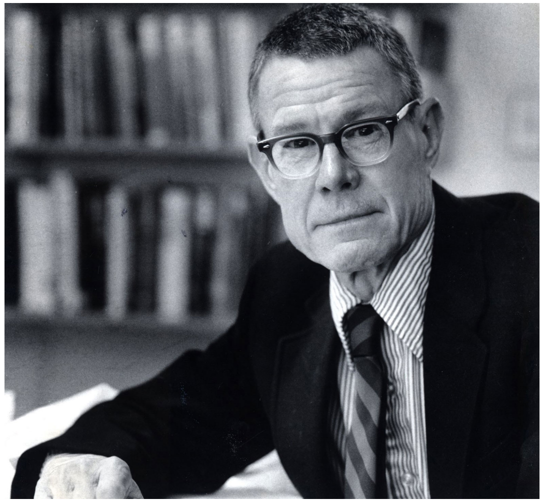

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- Brandt et al. (STOC 2012)
- Bhakta et al. (SODA 2014)
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many more...

Theoretical Approaches

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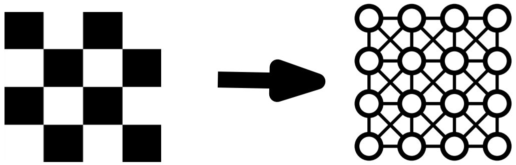
undirected (simple) graph $G = (V, E)$



Strategic Schelling Segregation

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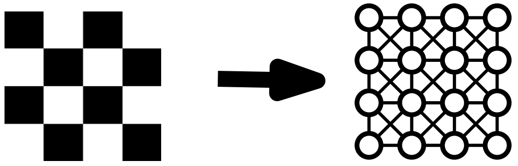
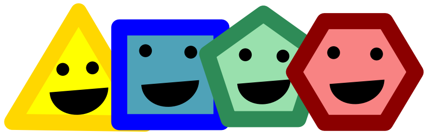


set of agents A with partitioning $P(A)$



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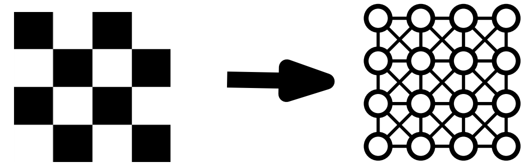

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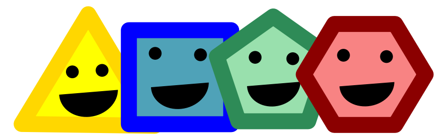
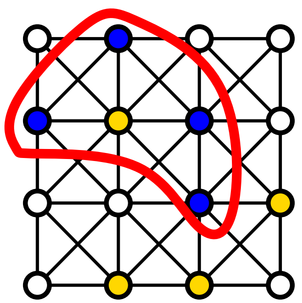


placement $p_G : A \to V$ (injective)

neighborhood $N_{p_G}(a)$:= adjacent agents



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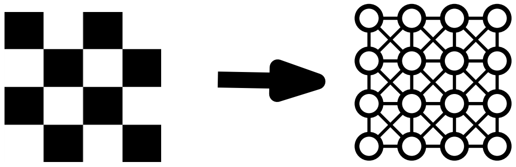

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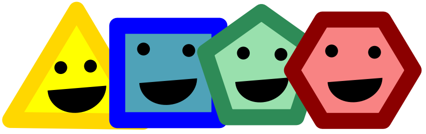

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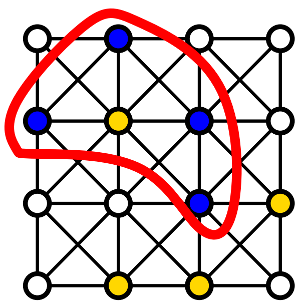
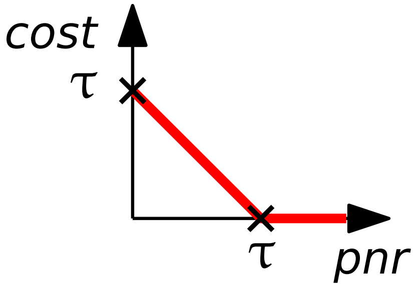


intolerance threshold $\tau \in [0, 1]$

$$N^+_{p_G}(a), N^-_{p_G}(a) \subseteq N_{p_G}(a)$$

$$cost_{p_G}(a) \begin{cases} \max(0, \tau - \frac{|N^+_{p_G}(a)|}{|N^+_{p_G}(a)|+|N^-_{p_G}(a)|}) \text{ if } N_{p_G}(a) \neq \emptyset \\ \tau \text{ else} \end{cases}$$



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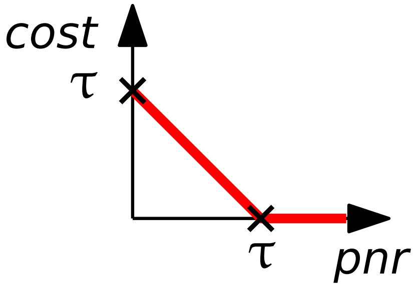

always: $N^+_{p_G}(a)$:= neighbors with same type as a

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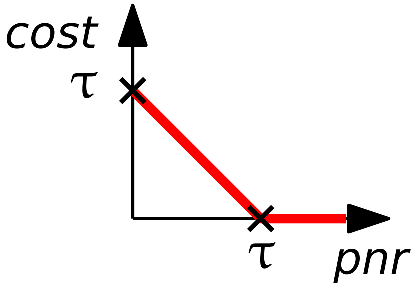

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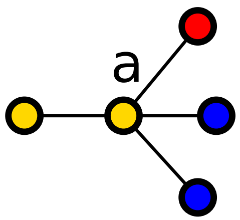
1 vs. all Schelling Game (1-k-SG)



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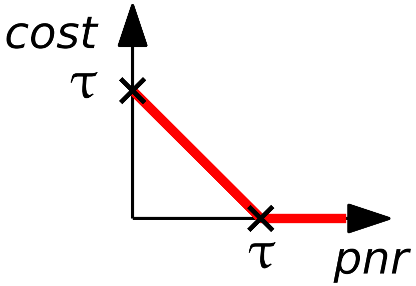

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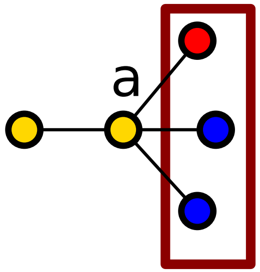
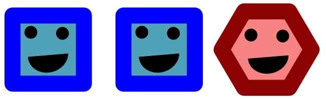


$N^+_{p_G}(a) =$ 

$N^-_{p_G}(a) =$ 

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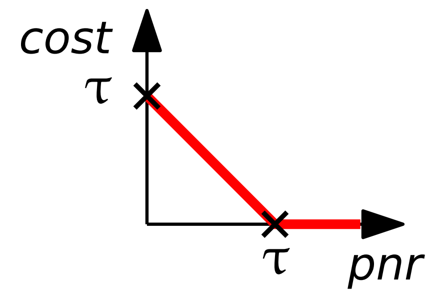

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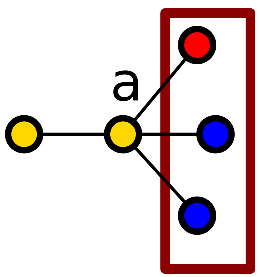

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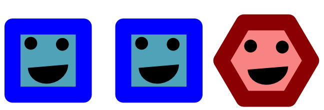
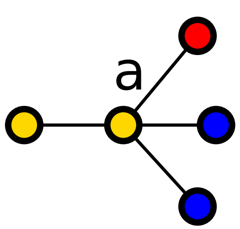
$N^-_{p_G}(a) =$ 

1 vs. 1 Schelling Game (1-1-SG)



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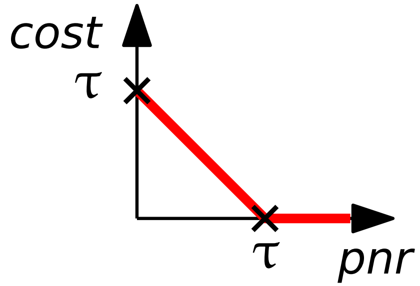

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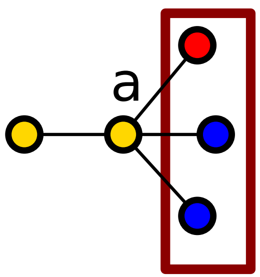

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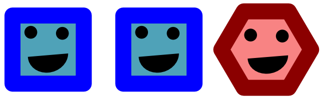
$N^-_{p_G}(a) =$ 

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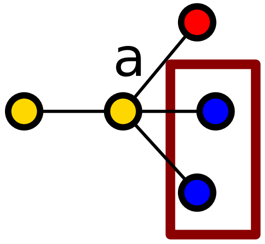

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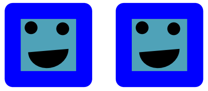
$N^-_{p_G}(a) =$ 

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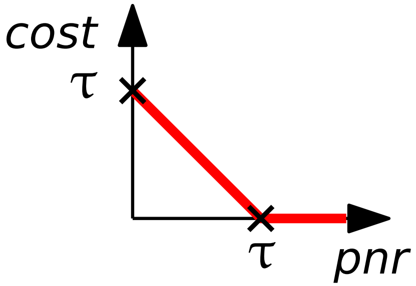
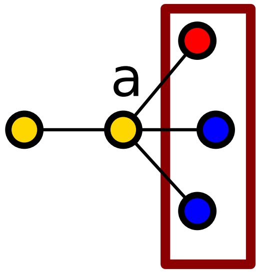


always: $N^+_{p_G}(a) :=$ neighbors with same type as a

1 vs. all Schelling Game (1-k-SG) $\tau = \frac{1}{3}$



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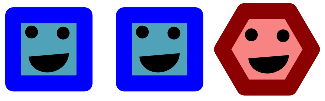
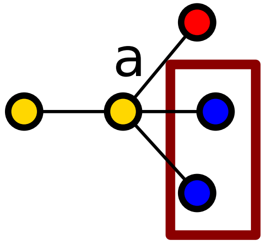
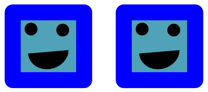
$N^-_{p_G}(a) =$ 

$cost_{p_G}(a) = \frac{1}{12}$

1 vs. 1 Schelling Game (1-1-SG)



$N^+_{p_G}(a) =$ 

$N^-_{p_G}(a) =$ 

$cost_{p_G}(a) = 0$

$$N^+_{p_G}(a), N^-_{p_G}(a) \subseteq N_{p_G}(a)$$

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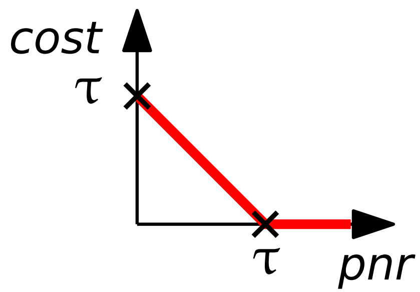

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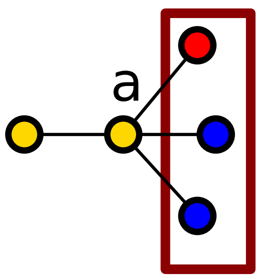
$\tau = \frac{1}{3}$



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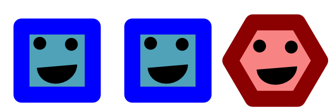
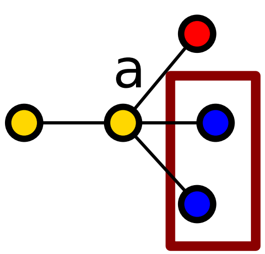
$N^-_{p_G}(a) =$ 

$cost_{p_G}(a) = \frac{1}{12}$ 

1 vs. 1 Schelling Game (1-1-SG)



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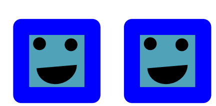
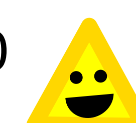
$N^-_{p_G}(a) =$ 

$cost_{p_G}(a) = 0$ 

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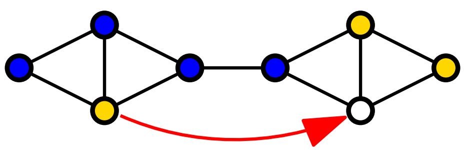

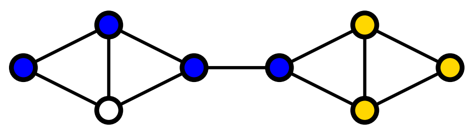

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1-k-JSG

1-1-JSG

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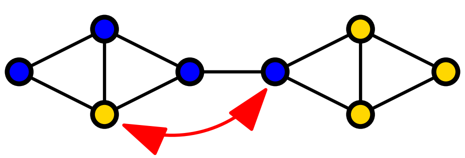

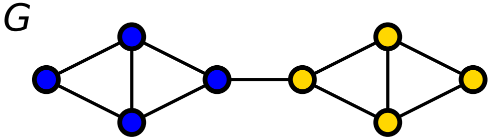

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- sequence of placements $p_G^1, ..., p_G^k$
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not weakly acyclic:
there is an unstable placement p_G from which no stable placement p'_G can be reached

Previous results by Chauhan et al (SAGT 2018):

	$1-k-$SSG	$1-1-$SSG	$1-k-$JSG	$1-1-$JSG
$\Delta-$regular	$\checkmark \lvert P(A)\rvert = 2$		$\checkmark \lvert P(A)\rvert = 2, \Delta = 2$	
arbitrary	$\checkmark \lvert P(A)\rvert = 2, \tau \leq \frac{1}{2}$			

✓guaranteed convergence

Convergence

Our results:

	$1-k-$SSG	$1-1-$SSG	$1-k-$JSG	$1-1-$JSG
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		o $\tau > \frac{6}{\Delta}$	o $\tau > \frac{2}{\Delta}$	o $\tau > \frac{2}{\Delta}$
arbitrary	$\checkmark \lvert P(A)\rvert = 2, \tau \leq \frac{1}{2}$			
	$\times$ else	$\times$ else	$\times$	$\times$

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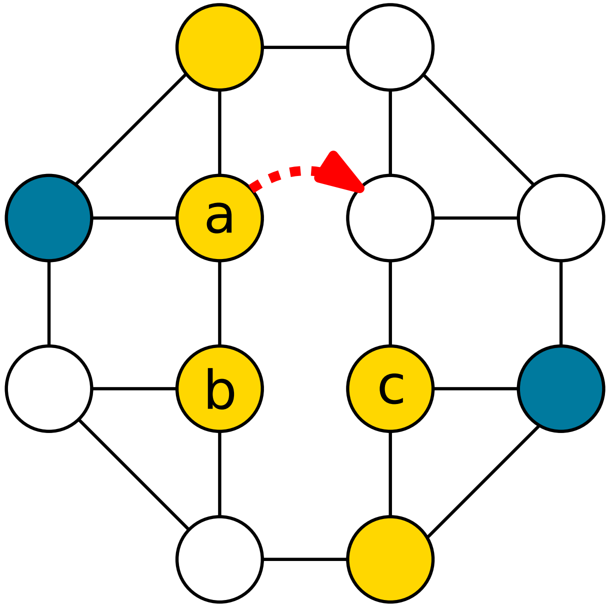

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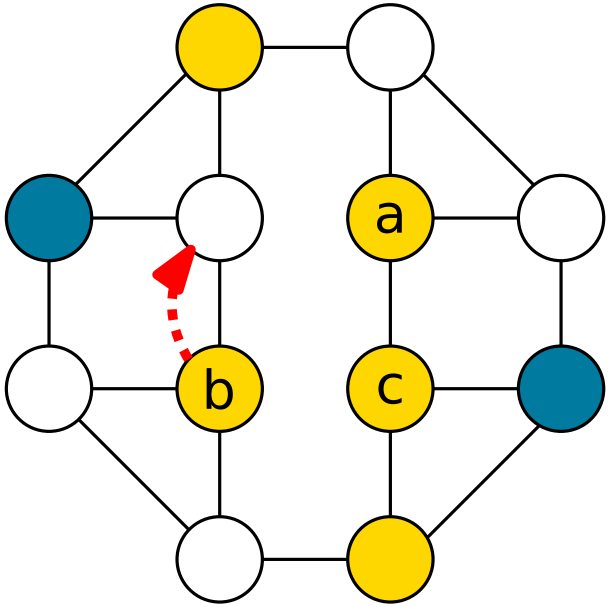

$cost_{p_G}(b) = \frac{5}{6}$

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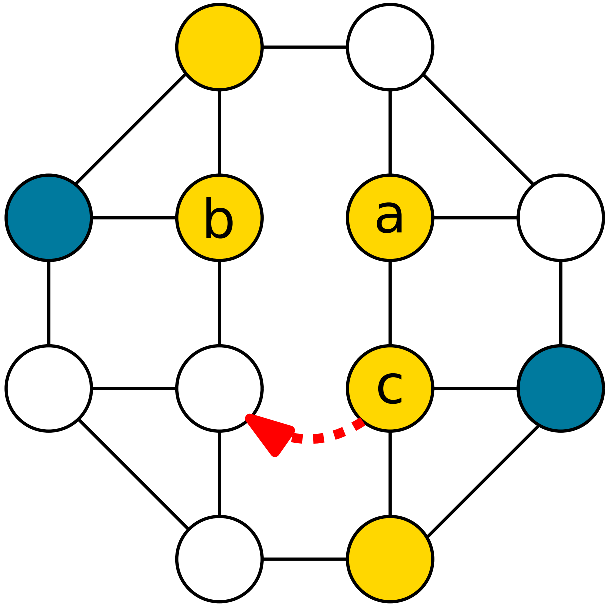

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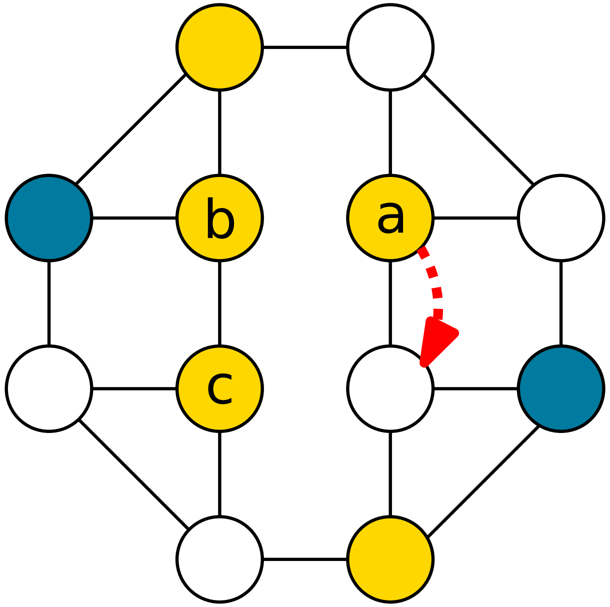

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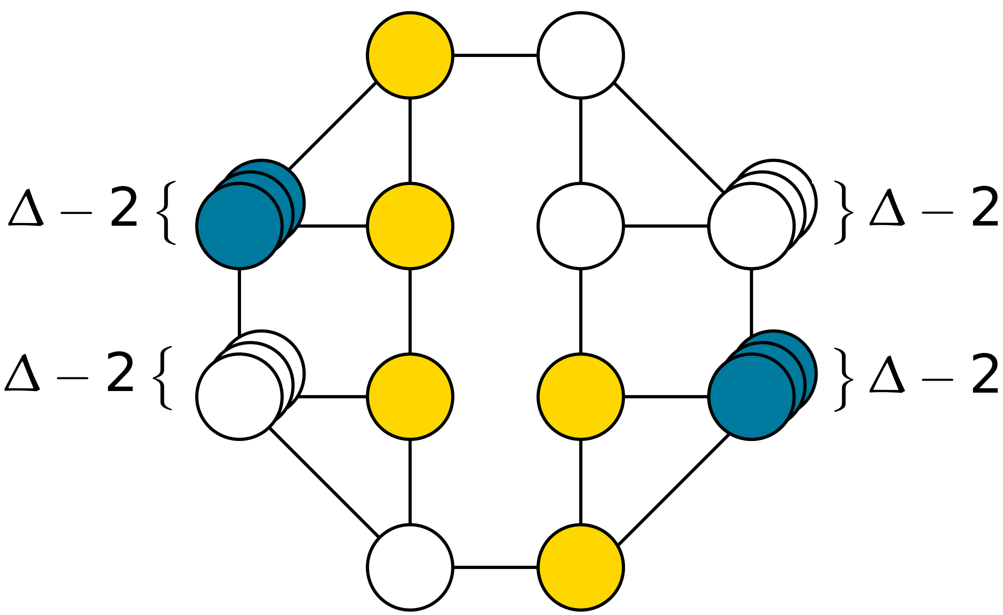
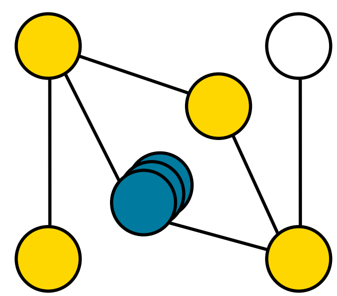


almost for free:
not weakly acyclic on arbitrary graphs



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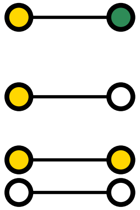
Proof (sketch): again search a potential function Φ

$$\Phi(p_G) = \sum_{(u,v)\in E} w_{p_G}(u,v) \qquad w_{p_G}(u,v) = \begin{cases} 1 \text{ if} \\ c \text{ if} \\ 0 \text{ ow.} \end{cases} \qquad \frac{1}{2} - \frac{1}{2\Delta} < c < \frac{1}{2}$$



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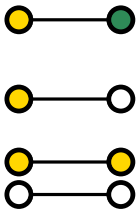

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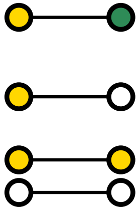

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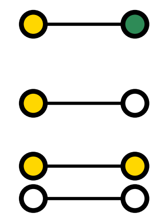

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2 cases: $|N^+_{p_G}(a)| < |N^+_{p'_G}(a)|$ and $|N^+_{p_G}(a)| = |N^+_{p'_G}(a)| = 1$ using regularity

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Theorem
It is NP-complete to decide the optimal placement problem for $1-k-$SSG and $1-1-$SSG on 2-regular graphs for $\tau > \frac{1}{2}$ and an arbitrary number of types.

Proof (sketch): reduction from 3-Partition

Summary & Future Work

Convergence highly depends on cost model, number of types, swap or jump, ...

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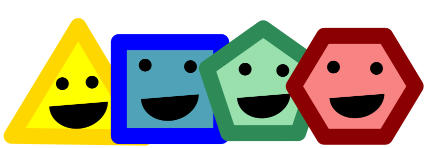
Thank you very much and let’s be happy polygons.



https://ncase.me/polygons/