Hasso-Plattner-Institute

Heuristic Optimization – Homework 4

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5 Exercise 5

5.1 Given

$$g(x)=\begin{cases} 1, & if \ |x|_1=n \\ 0, & if \ 3n/4 \leq |x|_1 < n \\ 1/2, & otherwise \end{cases}$$

5.2 Task a

Task:

What is the expected runtime of RandomSearch to maximize g?

Solution:

To maximize *g*, the number of ones in *x* has to be n.

Let T_{RS} be a random variable, describing the number of tries to get an all-one-string.

T_{RS} has a geometric distribution. Therefore, $E(T_{RS})=\frac{1}{p}$ where p is the probability to get the all-one-string in one try.

There are 2^n possibilities for *x*, where each has the same probability to occur after one try.

$$\Rightarrow \quad p=\frac{1}{2^n}$$

$$\boxed{\Rightarrow \quad E(T_{RS})=2^n}$$

5.3 Task b

Task:

Let Ɛ be the event that the initial solution of the (1+1) EA has at most n/4 zero bits. Derive an upper bound on Pr(Ɛ).

Solution:

Ɛ is the event to have at most n/4 zero bits.

Let *X* be the random variable, describing the number of zero bits in a randomly chosen x.

$$E(X)=\frac{n}{2}$$

To calculate *Pr(Ɛ),* we use the *Chernoff Bounds*:

$$Pr(X \leq (1-\delta)E(X)) \leq e^{\frac{-E(X)\delta^2}{2}}$$

For this formula, the event $X \leq \frac{n}{4}$ has to be equal to the event $X \leq (1-\delta)E(X)$.

$$\Rightarrow \quad \frac{n}{4} = (1-\delta)E(X)$$

$$\Rightarrow \quad \frac{n}{4} = (1-\delta)\frac{n}{2}$$

$$\Rightarrow \quad \frac{1}{2} = 1-\delta$$

$$\Rightarrow \quad \delta=\frac{1}{2}$$

$$\Rightarrow \quad Pr(\varepsilon) \leq e^{\frac{-E(X)\delta^2}{2}} = e^{\frac{-\frac{n}{2}\frac{1}{2}^2}{2}} = e^{-\frac{n}{16}}$$

$$\boxed{\Rightarrow \quad Pr(\varepsilon) \leq e^{-\frac{n}{16}}}$$

$e^{-\frac{n}{16}}$ is an upper bound for *Pr(Ɛ).*

5.4 Task c

Task:

Suppose Ɛ has not occurred. Give a lower bound on the expected time for the (1+1) EA to first generate the all-ones string under this condition.

Solution:

Let *X* be the random variable, describing the number of zero bits in a randomly chosen x.

When Ɛ has not occurred, then $X > \frac{n}{4}$.

The more ones the initial solution has, the closer g already is to the optimum and the lower is the expected runtime. Therefore, if we assume $X = \frac{n}{4}$ it ends in a shorter runtime than for $X > \frac{n}{4}$ and will definitely give us a lower bound for the runtime.

Because of $X = \frac{n}{4}$ the value of *g* is *0. g* only will give a higher result, if all ones are found. This event "*Z*" only occurs, if all missing ones are found at the same time, because all other events won't change the value and are thrown away. An exception is the case, where $X > \frac{n}{4}$, so that $g(x) = \frac{1}{2}$. But this only will increase the runtime. For the lower bound of the runtime, we ignore this case.

To find all ones at the same time, every zero has to flip in a single iteration.

The probability for a bit to flip is $p = \frac{1}{n}$.

Therefore, the probability to flip all zeros is $Pr(Z) = \left(\frac{1}{n}\right)^{\frac{n}{4}}$, since $X = \frac{n}{4}$.

Let *T* be a random variable, describing the number of runs, to randomly switch all zeros to ones. T is the runtime we are looking for, is geometrically distributed, and has an expected value of $E(T \mid X = \frac{n}{4}) = \frac{1}{Pr(Z)} = n^{\frac{n}{4}}$. Since the case $X = \frac{n}{4}$ gives a lower bound for the runtime, $\boxed{E(T \mid \neg\varepsilon) \geq n^{\frac{n}{4}}}$.

5.5 Task d

Task:

Conclude that the expected runtime to maximize g for the (1+1) EA is worse than the runtime of RandomSearch on g.

Solution:

The runtime T_{EA} of the *(1+1) EA* is:

$$E(T_{EA})=E(T_{EA}|\varepsilon)\cdot Pr(\varepsilon)+E(T_{EA}|\neg\varepsilon)\cdot Pr(\neg\varepsilon) \quad \text{, since} \quad \varepsilon\cap\neg\varepsilon=\emptyset$$

$$E(T_{EA}|\varepsilon)\geq 0 \ \wedge \ Pr(\varepsilon)\geq 0$$

$$\Rightarrow \ E(T_{EA}|\varepsilon)\cdot Pr(\varepsilon)\geq 0$$

$$\Rightarrow E(T_{EA})\geq E(T|\neg\varepsilon)\cdot Pr(\neg\varepsilon)$$

$$\Rightarrow E(T_{EA})\geq E(T_{EA}|\neg\varepsilon)\cdot(1-Pr(\varepsilon))$$

From the tasks before, we know that $E(T_{EA}|\neg\varepsilon) \geq n^{\frac{n}{4}}$ and $Pr(\varepsilon) \leq e^{-\frac{n}{16}}$ and therefore

$1-Pr(\varepsilon) \geq 1-e^{-\frac{n}{16}}$.

$$\Rightarrow E(T_{EA})\geq n^{\frac{1}{4}n}\cdot(1-e^{-\frac{n}{16}})$$

$$\Rightarrow E(T_{EA})\geq n^{\frac{n}{4}} \quad (\quad 1-e^{-\frac{n}{16}}\geq 0 \quad)$$

Now, we want to show, that *(1+1) EA* is slower than *RandomSearch* for *g*. To show that, we prove, that $E(T_{EA})\neq O(E(T_{RS}))$ (*(1+1) EA* is not faster or as fast as *RandomSearch*).

We assume, that $E(T_{EA})=O(E(T_{RS}))$

$$\Leftrightarrow \exists c>0\,\exists n_0>0\,\forall n>n_0: \quad E(T_{EA})\leq c\cdot E(T_{RS})$$

$$\Rightarrow \exists c>0\,\exists n_0>0\,\forall n>n_0: \quad n^{\frac{1}{4}n}\leq c\cdot 2^n$$

$$\Rightarrow \exists c>0\,\exists n_0>0\,\forall n>n_0: \quad n^n\leq c\cdot 2^n \quad \text{, because} \quad n^{\frac{1}{4}}\geq 1$$

$$\Leftrightarrow \exists c>0\,\exists n_0>0\,\forall n>n_0: \quad n\leq 2\cdot c^{\frac{1}{n}}$$

false, since $2\cdot c^{\frac{1}{n}}$ goes towards 2 and n towards infinity.

We concluded this false statement from the assumption that $E(T_{EA})=O(E(T_{RS}))$.

Therefore $\boxed{E(T_{EA})\neq O(E(T_{RS}))}$.

This proves that *(1+1) EA* is slower than *RandomSearch* for *g*.