Hasso-Plattner-Institute

Heuristic Optimization – Homework 2

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2 Exercise 2

2.1 Given

$$\forall x \in \mathbb{R} : e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

2.2 Task a

To show:

$$\forall x \geq -1, k \in \mathbb{N} : (1+x)^k \geq 1 + kx$$

Base case: k=0

$$(1+x)^0 \geq 1 \geq 1 + 0 \cdot x$$

True.

Induction hypothesis:

$(1+x)^k \geq 1 + k \cdot x$ is correct for some natural number k.

Induction-step:

We now have to prove, that $(1+x)^{k+1} \geq 1 + (k+1) \cdot x$.

$$(1+x)^k \geq 1 + k \cdot x \quad \text{(induction hypothesis)}$$

$$\Rightarrow (1+x)^k \cdot (1+x) \geq (1+x) + kx \cdot (1+x)$$

$$\Rightarrow (1+x)^{k+1} \geq 1 + x + kx + kx^2$$

Since kx^2 is greater 0, we can make the right side even smaller by subtracting it.

$$\Rightarrow (1+x)^{k+1} \geq 1 + kx + x$$

$$\Rightarrow (1+x)^{k+1} \geq 1 + (k+1)x$$

Hence by mathematical induction the inequality is correct for all natural numbers.

q.e.d.

2.3 Task b

To show:

$$\forall x>-1, x\neq 0: \ln(1+x)<x$$

Proof:

$$\boxed{\ln(1+x)<x}$$

$$\Leftrightarrow\ 1+x<e^{x}$$

$$\Leftrightarrow\ 1+x\ <\ \sum_{i=0}^{\infty}\frac{x^{i}}{i!}\qquad \Leftrightarrow\ 1+x\ <\ 1+x+\sum_{i=2}^{\infty}\frac{x^{i}}{i!}$$

$$\Leftrightarrow\ 0\ <\ \sum_{i=2}^{\infty}\frac{x^{i}}{i!}\ =\ \sum_{i=1}^{\infty}\frac{x^{2i}}{(2i)!}+\frac{x^{2i+1}}{(2i+1)!}$$

If both summands together are greater zero, the sum also will be greater than zero.

$$\Leftarrow\ 0\ <\ \frac{x^{2i}}{(2i)!}+\frac{x^{2i+1}}{(2i+1)!}\ =\ \frac{x^{2i}\cdot(2i+1)}{(2i)!\cdot(2i+1)}+\frac{x^{2i+1}}{(2i+1)!}\ =\ \frac{x^{2i}\cdot(2i+1)}{(2i+1)!}+x^{2i}\cdot\frac{x}{(2i+1)!}$$

$$\Leftrightarrow\ 0\ <\ \frac{x^{2i}\cdot(2i+1+x)}{(2i+1)!}$$

$$\Leftrightarrow\ 0\ <\ x^{2i}\cdot(2i+1+x)$$

$$\Leftarrow\ 0\ <\ 2i+1+x\quad , x^{2i}>0$$

$$\Leftarrow\ 0\ <\ 1+x\quad , 2i>0$$

$$\Leftrightarrow\ -1\ <\ x$$

This is true, since it is given in the task.

Now, that the last statement is true, you can read the proof from bottom up, and you will get $\ln(1+x)<x$ in the end.

2.4 Task c

To show:

$$\forall x>-1, x\neq 0\,\forall r>0: (1+x)^{r}<e^{rx}$$

Proof:

$$\boxed{(1+x)^{r}<e^{rx}}$$

$$\Leftrightarrow\ 1+x<e^{x}$$

$$\Leftrightarrow\ \ln(1+x)<x$$

Already proven in 2.3(task b)

q.e.d.

3 Exercise 3

3.1 Given

The probability that a certain bit flips is $P(\text{a certain bit flips})=\frac{1}{n}$ and the number of current zeros is k. k>0, since all ones would already be the optimum.

3.2 Task a

To show:

$$P(\text{create all 1s individually})=\Theta(n^{-k})$$

Proof:

$$P(\text{create all 1s individually})=P(K)$$

$$P(K)=P(\text{all zeros flip to ones})\cdot P(\text{all ones stay})$$

$$P(K)=P(\text{a certain zero flips to one})^k\cdot P(\text{a certain one stays})^{n-k}$$

$$P(K)=P(\text{a certain zero flips to one})^k\cdot(1-P(\text{a certain one flips to zero}))^{n-k}$$

$$P(K)=\left(\frac{1}{n}\right)^k\cdot\left(1-\frac{1}{n}\right)^{n-k}=n^{-k}\cdot\left(1-\frac{1}{n}\right)^{n-k}$$

For k>0, what is given, $\left(1-\frac{1}{n}\right)^{n-k}$ goes towards e^{-1} from above. *

$$\Rightarrow\left(1-\frac{1}{n}\right)^{n-k}>e^{-1}$$

Since $0\leq 1-\frac{1}{n}\leq 1$, and $n-k\geq 1$, $\Rightarrow\left(1-\frac{1}{n}\right)^{n-k}\leq 1$

$$\Rightarrow\exists c_{1,}c_{2,}n_0>0\,\forall n>n_0:\ c_1\leq\left(1-\frac{1}{n}\right)^{n-k}\leq c_2\quad,\quad\left(c_1=e^{-1}\wedge c_2=1\right)$$

Knowing, that n^{-k} is always positive, we can multiply it

$$\Leftrightarrow\exists c_{1,}c_{2,}n_0>0\,\forall n>n_0:\ n^{-k}\cdot c_1\leq n^{-k}\cdot\left(1-\frac{1}{n}\right)^{n-k}\leq n^{-k}\cdot c_2$$

$$\Leftrightarrow\exists c_{1,}c_{2,}n_0>0\,\forall n>n_0:\ n^{-k}\cdot c_1\leq P(K)\leq n^{-k}\cdot c_2$$

By definition of Θ :

$$\boxed{\Leftrightarrow P(K)=\Theta(n^{-k})}$$

q.e.d.

3.3 Task b

To show:

$$\exists c>0:\forall n:\ \ P(\text{k is bigger after the next step})\geq c\cdot\frac{k}{n}$$

Proof:

$$P(\text{k is bigger after the next step})=P(B)$$

$$P(B)=P(\text{exactly one zero flips to one})+P(\text{exactly two zeros flip to ones})+\ldots$$

Since there is no negative probability:

$$\Rightarrow P(B)\geq P(\text{exactly one zero flips to one})$$

$$\Rightarrow P(B)\geq P(\text{exact the first zero flips to one or the second zero flips to one or ...})$$

Since all these events are disjoint, the probability of their sum is just the sum of their probabilities.

$$\Rightarrow P(B)\geq P(\text{exact the first zero flips})+P(\text{exact the second zero flips})+\ldots$$

All these events have the same probability

$$\Rightarrow P(B)\geq k\cdot P(\text{exact a certain zero flips})$$

$$\Rightarrow P(B)\ \geq\ k\cdot P(\text{a certain zero flips})\cdot P(\text{all other bits stay})$$

$$\Rightarrow P(B)\geq k\cdot\frac{1}{n}\cdot\left(1-\frac{1}{n}\right)^{n-1}$$

$\left(1-\frac{1}{n}\right)^{n-1}$ goes towards e^{-1} from above *. Therefore: $\left(1-\frac{1}{n}\right)^{n-1}>e^{-1}$

$$\Rightarrow P(B)\geq\frac{k}{n}\cdot\left(1-\frac{1}{n}\right)^{n-1}\ >\ \frac{k}{n}\cdot e^{-1}=e^{-1}\cdot\frac{k}{n}$$

$$\Rightarrow P(B)\geq e^{-1}\cdot\frac{k}{n}$$

$$\boxed{\Rightarrow\exists c>0:\forall n:\ \ P(B)\geq c\cdot\frac{k}{n}}$$

q.e.d.

* Proof that for n>k, k>0 $f(n)=\left(1-\frac{1}{n}\right)^{n-k}$ goes towards e^{-1} from above.

First we want to show, that $f(n)$ goes towards e^{-1}:

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n-k} = \lim_{n\to\infty}\frac{\left(1-\frac{1}{n}\right)^{n}}{\left(1-\frac{1}{n}\right)^{k}} = \frac{e^{-1}}{1^k} = e^{-1}$$

To prove, that all values n>k are greater than e^{-1}, it is enough to show, that the function is monotonically decreasing. Since f goes towards e^{-1}, they have to be greater than e^{-1}.

$$f(n) \text{ Monotonically decreasing}$$

$$\Leftrightarrow \forall n>k:\ f(n+1)\le f(n) \quad \text{, (only for } f(n)\ge 0 \text{)}$$

$$\Leftrightarrow \forall n>k:\ \frac{f(n+1)}{f(n)}\le 1$$

$$\Leftrightarrow \forall n>k:\ \frac{\left(1-\frac{1}{n+1}\right)^{n-k+1}}{\left(1-\frac{1}{n}\right)^{n-k}} = \frac{\left(\frac{n}{n+1}\right)^{n-k+1}}{\left(\frac{n-1}{n}\right)^{n-k}} = \frac{\left(\frac{n}{n+1}\right)^{n-k}}{\left(\frac{n-1}{n}\right)^{n-k}}\cdot\left(\frac{n}{n+1}\right) \le 1$$

$$\Leftrightarrow \forall n>k:\ \left(\frac{n}{n+1}\right)^{n-k}\cdot\left(\frac{n}{n-1}\right)^{n-k}\cdot\frac{n}{n+1} = \left(\frac{n^2}{n^2-1}\right)^{n-k}\cdot\frac{n}{n+1} \le 1$$

$$\Leftrightarrow \forall n>k:\ \frac{n^{2n-2k}}{n^{2n-2k}-1}\cdot\frac{n}{n+1} = \frac{n^{2n-2k+1}}{n^{2n-2k+1}+n^{2n-2k}-n-1} \le 1$$

$$\Leftrightarrow \forall n>k:\ n^{2n-2k+1} \le n^{2n-2k+1}+n^{2n-2k}-n-1$$

$$\Leftrightarrow \forall n>k:\ 0 \le n^{2n-2k}-n-1$$

$$\Leftrightarrow \forall n>k:\ n+1 \le n^{2n-2k}$$

$$\Leftrightarrow \forall n>k:\ 1+\frac{1}{n} \le n^{2(n-k)-1}$$

Since $1+\frac{1}{n}$ is always smaller or equal 2:

$$\Leftarrow \forall n>k:\ 2 \le n^{2(n-k)-1}$$

$$\Leftarrow \forall n>k:\ 2 \le n^{2-1} \wedge n^{2-1} \le n^{2(n-k)-1}$$

$$\Leftarrow \forall n>k:\ 2 \le n \wedge \mathit{true} \quad \text{, because n>k}$$

This is obviously true for n>1. n can not be <1, since n>k>0.

Therefore you can read the proof from the bottom up.

q.e.d.