Heuristic Optimization

Today: Linear Programming

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Linear programming

- Let’s first define it formally:
„A **linear program** is an optimization problem in finitely many variables having a linear objective function and a constraint region determined by a finite number of linear equality and/or inequality constraints.“

Linear programming

- A linear function of the variables $x_1, x_2, \ldots, x_n$ is any function f of the form

$$f(x) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

for fixed $c_i \in \mathbb{R}$ $i = 1, \ldots, n$.

- A linear equality constraint is any equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = \alpha,$$

where $\alpha, a_1, a_2, \ldots, a_n \in \mathbb{R}$.

- A linear inequality constraint is any inequality of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq \alpha,$$

or

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq \alpha,$$

where $\alpha, a_1, a_2, \ldots, a_n \in \mathbb{R}$.

Linear programming

- The standard compact representation is

$$
\begin{array}{lll}
\text{minimize} & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n & \\
\text{subject to} & a_{i1} x_i + a_{i2} x_2 + \cdots + a_{in} x_n \leq \alpha_i & i = 1, \ldots, s \\
& b_{i1} x_i + b_{i2} x_2 + \cdots + b_{in} x_n = \beta_i & i = 1, \ldots, r.
\end{array}
$$

Linear programming

- The standard matrix representation is

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq a \text{ and } Bx = b \end{array}$$

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \quad a = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_s \end{bmatrix}, \quad b = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix}.$$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ & \ddots & \\ a_{s1} & \dots & a_{sn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ & \ddots & \\ b_{r1} & \dots & b_{rn} \end{bmatrix}$$

Introductory Example #1: Optimization in Politics

- Assume you are a politician, who wants to get 50% of the votes
  - in metropolitan region #1 with 100k inhabitants
  - in suburban region #2 with 200k inhabitants
  - in rural region #3 with 50k inhabitants
- You want to buy advertisements; each commercial block costs 1000€

Introductory Example #1: Optimization in Politics

- The commercial blocks have the following effect:

	Metropolitan region #1	Suburban region #2	Rural region #3
Build more streets	-2	5	3
Disempower secret service	8	2	-5
More agriculture subvention	0	0	10
Increase petrol taxes	10	0	-2

Introductory Example #1: Optimization in Politics

- Some sample spending:

Total Spending	33.000€
Build more streets	20.000€
Disempower secret service	0€
More agriculture subvention	4.000€
Increase petrol taxes	9.000€

Introductory Example #1: Optimization in Politics

- Effect of our sample spending:

	Build more streets	Disempower secret service	More agriculture subvention	Increase petrol taxes	**Resultiung Votes**
Metropolitan region #1	20*(-2)	+0*8	+4*0	+9*10	=50.000
Suburban region #2	20*5	+0*2	+4*0	+9*0	=100.000
Rural region #3	20*3	+0*(-5)	+4*10	+9*(-2)	=82.000

- Is there a cheaper way to win the election?

Introductory Example #1: Optimization in Politics

- Let's formulate this mathematically:
  - x_1 = #commercial blocks for building more streets
  - x_2 = #commercial blocks for disempower secret service
  - x_3 = #commercial blocks for more agriculture subvention
  - x_4 = #commercial blocks for increasing petrol taxes

Introductory Example #1: Optimization in Politics

- Mathematically speaking, our target is:

  Minimize $x_1+x_2+x_3+x_4$ such that

  - $-2\ x_1 + 8\ x_2 + 0\ x_3 + 10\ x_4 \geq 50$
  - $5\ x_1 + 2\ x_2 + 0\ x_3 + 0\ x_4 \geq 100$
  - $3\ x_1 - 5\ x_2 + 10\ x_3 - 2\ x_4 \geq 25$
  - $x_1, x_2, x_3, x_4 \geq 0$

Introductory Example #2: Optimization in Economics

- Let's look at another problem:

  Maximize profit

  $$G(x_1,x_2) = 300\ x_1 + 500\ x_2$$

  such that

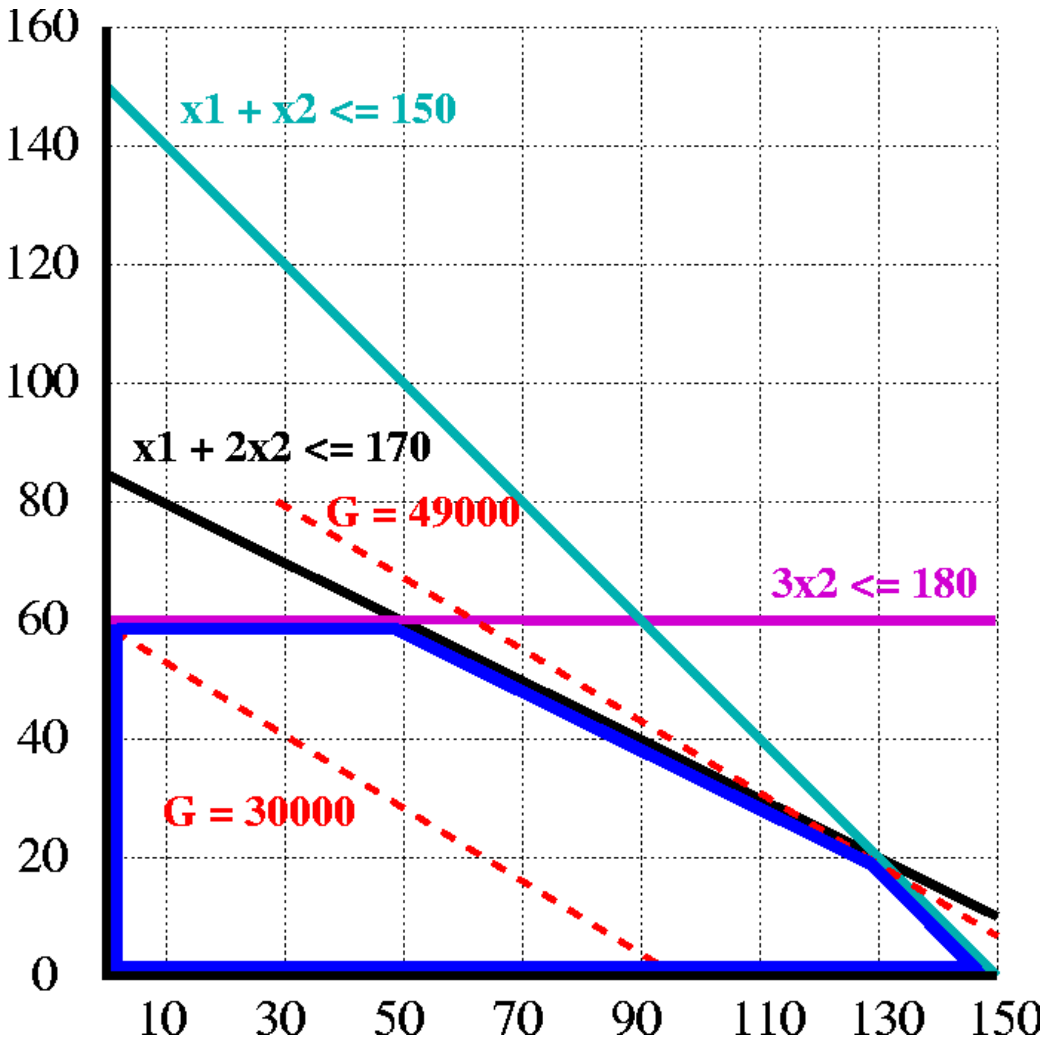
  (1) $x_1 + 2\ x_2 \leq 170$

  (2) $x_1 + x_2 \leq 150$

  (3) $3\ x_2 \leq 180$

  (4) $x_1\ ,\ x_2 \geq 0$

Introductory Example #2: Optimization in Economics



Introductory Example #2: Optimization in Economics

- Conclusion: Look for $y \geq 0$, such that $y^T A$ is an upper bound for $y^T A \geq c^T$

Primal vs. Dual

	$\mathcal{P}$			$\mathcal{D}$
Primal:	max	$c^T x$	**Dual:**	min	$b^T y$
	s.t.	$Ax \leq b$		s.t.	$A^T y \geq c$
		$0 \leq x$			$0 \leq y$

Weak Duality Theorem

- Weak Duality Theorem:
  If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then
  $$c^T x \leq y^T A x \leq b^T y.$$
- Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible.

Weak Duality Theorem

- Proof of Weak Duality Theorem:

$$
\begin{aligned}
c^T x &= \sum_{j=1}^{n} c_j x_j \\
&\le \sum_{j=1}^{n} \Big(\sum_{i=1}^{m} a_{ij} y_i\Big) x_j \qquad \Big[0 \le x_j,\ c_j \le \sum_{i=1}^{m} a_{ij} y_i \ \Rightarrow\ c_j x_j \le \Big(\sum_{i=1}^{m} a_{ij} y_i\Big) x_j\Big] \\
&= y^T A x \\
&= \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij} x_j\Big) y_i \\
&\le \sum_{i=1}^{m} b_i y_i \qquad \Big[0 \le y_i,\ \sum_{j=1}^{n} a_{ij} x_j \le b_i \ \Rightarrow\ \Big(\sum_{j=1}^{n} a_{ij} x_j\Big) y_i \le b_i y_i\Big] \\
&= b^T y
\end{aligned}
$$

Strong Duality Theorem

Theorem:
If either $\mathcal{P}$ or $\mathcal{D}$ has a finite optimal value, then so does the other, the optimal values coincide, and optimal solutions to both $\mathcal{P}$ and $\mathcal{D}$ exist.

Fundamental Theorem

Theorem:
Every LP has the following three properties:

(i) *If it has no optimal solution, then it is either infeasible or unbounded.*

(ii) *If it has a feasible solution, then it has a basic feasible solution.*

(iii) *If it is bounded, then it has an optimal basic feasible solution.*

Reduction to Standard Form

Any linear program can be brought into standard form:

- elimination of free (unbounded) variables x_j:

  replace x_j with $x_j^+, x_j^- \geq 0$: $\quad x_j = x_j^+ - x_j^-$

- elimination of non-positive variables x_j:

  replace $x_j \leq 0$ with $(-x_j) \geq 0$.

- elimination of inequality constraint ${a_i}^T \cdot x \leq b_i$:

  introduce slack variable $s \geq 0$ and rewrite: ${a_i}^T \cdot x + s = b_i$

- elimination of inequality constraint ${a_i}^T \cdot x \geq b_i$:

  introduce slack variable $s \geq 0$ and rewrite: ${a_i}^T \cdot x - s = b_i$

Simplex Algorithm

- Observations:
  - Feasible region ($Ax \leq b$, $x \geq 0$) is a (possibly unbounded) convex polytope
  - Maximum value on the feasible region occurs on one of the extreme points
  - Number of extreme points is finite, but exponentially large
  - It is enough to check corners of the polytope

Simplex Algorithm

- Walks along edges of the polytope to extreme points with greater and greater objective values
- Continues until the maximum value is reached or an unbounded edge is visited (i.e. problem has no solution)
- Runtime
  - Finite as the number of vertices in the polytope is finite
  - For every deterministic Pivot rule there is a worst-case which need exponential time
  - Still very efficient in practice
  - There are other algos for LPs with poly. time

Integer Linear Program

- Adding integrality constraints make things MUCH harder; example Weighted Vertex Cover Problem:

Given: undirected graph $G = (V, E)$, weight function $w : V \to \mathbb{R}_{\geq 0}$.

Task: find $U \subseteq V$ of minimum total weight such that every edge $e \in E$ has at least one endpoint in U.

Formulation as an integer linear program (IP):

Variables: $x_v \in \{0, 1\}$ for $v \in V$ with $x_v = 1$ if and only if $v \in U$.

$$\begin{aligned} \text{minimize} \quad & \sum_{v \in V} w(v) \cdot x_v \\ \text{subject to} \quad & x_v + x_{v'} \geq 1 && \text{for all } e = \{v, v'\} \in E, \\ & x_v \in \{0, 1\} && \text{for all } v \in V. \end{aligned}$$