Reductions to Sets of Low Information Content

(Extended Abstract)

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Abstract. In this paper we study the complexity of sets that reduce to sparse sets (and tally sets), and the complexity of the simplest sparse sets to which such sets reduce. We show even with respect to very flexible reductions that NP cannot have sparse hard sets unless P = NP; an immediate consequence of our results is: If any NP-complete set conjunctively reduces to a sparse set, then P = NP. We also show that any set A that reduces to some sparse set (via various types of reductions) in fact reduces by the same type of reduction to a sparse set that is simple relative to A. We give a complete characterization of the sets of low instance complexity in terms of reductions to tally sets; it follows that if P $\neq$ NP, then no set of low instance complexity can be complete for NP with respect to disjunctive reductions or conjunctive reductions.

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1 Overview

This paper is concerned with three basic questions about sparse sets, and a related question about sets of low instance complexity:

Question 1 With respect to what types of reductions might NP have hard or complete sparse sets?[1]

Question 2 If a set A reduces to a sparse set, does it follow that A is reducible to some sparse set that is "simple" relative to A?

Question 3 With respect to what types of reductions might NP have hard or complete sets of low instance complexity, and, relatedly, what is the structure of the class of sets with low instance complexity?

With respect to the first and third questions, intuitively one would expect that even with respect to flexible reductions NP is unlikely to have complete sets whose information content is low. With respect to the second question, one might intuitively feel that the structure imposed on a set by the fact that it reduces to a sparse set makes it plausible that we can indeed find a simple sparse set that can masquerade as the original sparse set. These two intuitions are in many ways certified by the current literature, and by the results of this paper.

The rest of this section summarizes the results of this paper and compares them with previous work.

With regard to Question 1:

- We show that if any NP-complete set conjunctively reduces to a sparse set,[2] then P = NP. This result, which has been obtained independently by Ranjan and Rohatgi [RR], extends Mahaney's Theorem [Mah82] and is incomparable with the strongest previously known result, which is due to Ogiwara and Watanabe: if any NP-complete set bounded truth-table reduces to a sparse set then P = NP [OW91].

  We obtain the above result as a corollary to a more general result we establish regarding the impossibility (if P $\neq$ NP) of NP-complete sets reducing to sparse sets, even via quite flexible reductions.

- One might ask whether in the above-mentioned result of Ogiwara and Watanabe the bounded truth-table case is optimal, or whether it can be

[1] For the reductions we will discuss, the question of sparse hard sets is equivalent to asking what types of reductions might reduce many-one complete sets for NP to some sparse set; we will often use this formulation.

[2] A conjunctive reduction from A to B means that there is a Turing machine with oracle B that accepts A, and the Turing machine's acceptance mechanism is that it accepts if and only if every string it queries is a member of B [LLS75].

extended by making the bound on the number of queries bigger than constant, for example, to any function that is $\omega(\log n)$. We show that there are relativized worlds, having tally NP-complete sets with respect to such reductions, in which the boolean hierarchy is infinite. This provides relativized upper bounds to the work of Ko, Orponen, Schöning, and Watanabe [KOSW86,Orp90,OKSW], of Ogiwara and Watanabe, and of the current paper. Our result strengthens the results of Homer and Longpré [HL91], who independently of the work of this paper showed that there are relativized worlds in which there are sparse sets that are NP-complete with respect to such bounds and yet (relativized) $\mathrm{P} \neq \mathrm{NP}$.

With regard to Question 2:

- In the context of recent comparisons between equivalence and reducibility to sparse sets [AHOW,GW91], it is interesting to know for various reduction classes to sparse sets how easy a sparse set can be relative to the set that is reduced to it. We show that any set A that disjunctively reduces (respectively, disjunctive bounded truth-table reduces, 2-truth-table reduces) to a sparse set in fact disjunctively reduces (respectively, disjunctive bounded truth-table reduces, 2-truth-table reduces) to a sparse set that is in $\mathrm{P}^{\mathrm{NP}^A}$ (respectively, $\mathrm{P}^{\mathrm{NP}^A[\log]}$ (sic), $\mathrm{P}^{\mathrm{NP}^A[\log]}$ (sic)).[3]

With regard to Question 3:

- We completely characterize the sets of low instance complexity (that is, the class $\mathrm{IC}[\log, \mathrm{poly}]$ [KOSW86,Orp90,OKSW]) in terms of reductions to tally sets: $\mathrm{IC}[\log, \mathrm{poly}]$ is exactly the class of sets that both disjunctively and conjunctively reduce to tally sets.
- We show that Orponen's result [Orp90] on 1-truth-table-complete sets for NP applies to disjunctive and conjunctive reductions: If $\mathrm{P} \neq \mathrm{NP}$ and A is $\leq_c^p$-hard or $\leq_d^p$-hard for NP, then $A \notin \mathrm{IC}[\log, \mathrm{poly}]$.

In this extended abstract all proofs are omitted. Complete versions of this paper are available as technical reports of the University of Ulm (Ulmer Informatik-Bericht Nr. 91-08) and of the University of Rochester (in preparation).

2 Notation

A set T is said to be a *tally* set if $T \subseteq 0^*$. A set S is said to be *sparse* if it has at most polynomially many elements at each length: S is sparse if and only if for some

[3] The [log] means that there is an $\mathcal{O}(\log n)$ bound on the number of calls made to the NP^A oracle (see [Wag90]).

Name	Notation
k disjunctive (truth-table or Turing)	$\leq^p_{k\text{-}d}$
bounded (truth-table or Turing)	$\leq^p_b$
$f(n)$ truth-table	$\leq^p_{f(n)\text{-}tt}$
conjunctive (truth-table or Turing)	$\leq^p_c$
disjunctive (truth-table or Turing)	$\leq^p_d$
bounded conjunctive (truth-table or Turing)	$\leq^p_{bc}$
bounded disjunctive (truth-table or Turing)	$\leq^p_{bd}$

Table 1: Reductions

polynomial p it holds that $(\forall n)[||S \cap \Sigma^n|| \leq p(n)]$. We use TALLY and SPARSE to represent, respectively, the classes of tally and sparse sets. Tally and sparse sets have come to play a large role in modern complexity theory (see, e.g., the surveys [Mah86, Mah89,HOW]).

The reductions discussed in this paper are polynomial-bounded reductions defined by Ladner, Lynch, and Selman [LLS75] (see Table 1).

We will use the following notation to describe downward closures of classes under various reductions; the interrelations among $\mathrm{R}^p_r(\mathrm{SPARSE})$ classes have been systematically studied by Book and Ko [BK88] and Ko [Ko89].

Notation 2.1 [BK88,AHOW] For any reducibility $\leq^p_r$ and any class of sets $\mathcal{C}$, let $\mathrm{R}^p_r(\mathcal{C}) = \{A \mid (\exists B \in \mathcal{C})[A \leq^p_r B]\}$.

3 Sets Reducing to Sparse Sets

The study of sparse complete sets was started by the conjecture of L. Berman and J. Hartmanis [BH77] that there are no sparse NP-complete sets; they were motivated to make this conjecture since, if it fails, then not all NP-complete sets are polynomial-time isomorphic (and at that time they conjectured that all NP-complete sets were polynomial-time isomorphic, though recent work has dimmed hopes on that issue [JY85,KMR89]).

The first result along the lines of their sparseness conjecture was P. Berman's proof that $\mathrm{P} = \mathrm{NP}$ if some subset of 0^* is NP-complete [Ber78]. This result was quickly followed by Fortune's proof that if there is a sparse coNP-complete set, then $\mathrm{P} = \mathrm{NP}$ [For79]. Finally, Mahaney showed that $\mathrm{P} = \mathrm{NP}$ if any NP-complete set many-one reduces to a sparse set [Mah82].

Although Mahaney obtained the complete collapse of the polynomial hierarchy in the case of many-one reducibility, possible collapses in the case of more flexible reducibilities have remained an active research area. For the case of Turing reductions, it is known that the existence of sparse Turing-complete sets for NP would collapse the polynomial hierarchy to $\mathrm{P}^{\mathrm{NP}[\log]}$ [Kad89], and the existence of sparse Turing-hard sets for NP would collapse the polynomial hierarchy to Σ_2^p [KL80]; both of these results are known to be essentially optimal with respect to relativizable proof techniques [Kad89,Hel86].

As just noted, for the cases of many-one and Turing reductions, the consequences of sparse NP-complete sets are well-understood. However, with respect to the reductions whose strength lies between Turing and many-one reductions, the question of extending Mahaney's many-one result has proved considerably more challenging. For the case of bounded truth-table reductions, Ukkonen [Ukk83] generalized Berman's result [Ber78] by showing that if there is a tally bounded truth-table hard set for NP, then P = NP. Yesha [Yes83] generalized Fortune's result [For79] by showing that if there is a sparse bounded positive truth-table hard set for coNP, then P = NP. Yesha also (partially) generalized Mahaney's Theorem [Mah82] by showing that if there is a sparse bounded positive truth-table complete set for NP, then P = NP. These results regarding bounded truth-table reductions have been recently subsumed by Ogiwara and Watanabe ([OW91], see also [Wat88] and [JY90, footnote 9]), who showed that if there is a sparse bounded truth-table hard set for NP, then P = NP.

For the case of conjunctive reductions, Ukkonen [Ukk83] and Yap [Yap83] generalized Fortune's result [For79] by showing that if there is a sparse conjunctive hard set for coNP, then P = NP. Yap [Yap83] also (partially) generalized Mahaney's Theorem [Mah82] by showing that if there is a sparse set that is both conjunctive and disjunctive complete for NP, then P = NP. However, in the decade since Mahaney's Theorem, it has remained an open question whether his result can be extended to the case of conjunctive reductions.

We show that if there is a sparse set that is conjunctive hard for NP, then P = NP (Corollary 3.4). This result is incomparable to the result of [OW91] on bounded truth-table reductions to sparse sets. In fact, we prove that $\mathrm{NP} \subseteq \mathrm{R}^p_{bc}(\mathrm{R}^p_{1\text{-}tt}(\mathrm{R}^p_c(\mathrm{SPARSE})))$ implies P = NP. (See Figure 1; note that $\mathrm{R}^p_b(\mathrm{R}^p_c(\mathrm{SPARSE})) = \mathrm{R}^p_{bd}(\mathrm{R}^p_{bc}(\mathrm{R}^p_{1\text{-}tt}(\mathrm{R}^p_c(\mathrm{SPARSE}))))$.

Definition 3.1 [OW91] For a set A in NP, let $W \in \mathrm{P}$ and q be a polynomial such that $A = \{\, x \mid (\exists\, w \in \Sigma^{q(|x|)})\, [\langle x, w\rangle \in W]\,\}$. For $x \in A$, let $w_{max}(x) = \max\{\, w \in \Sigma^{q(|x|)} \mid \langle x, w\rangle \in W\,\}$. Then $Left(A) = \{\, \langle x, w\rangle \mid x \in A,\ w \in \Sigma^{q(|x|)}$ and $w \leq$

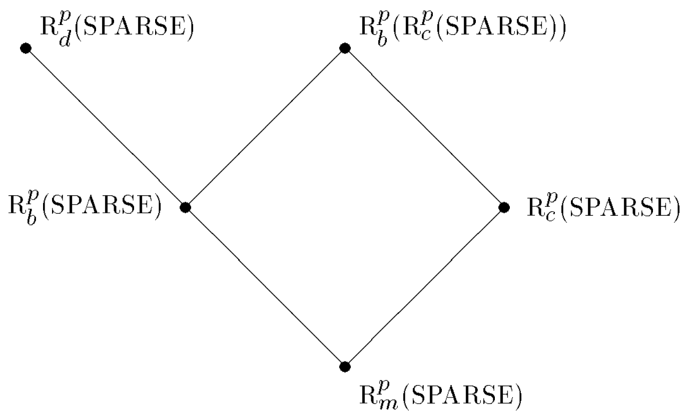

Figure 1: Inclusion structure of some reduction classes to sparse sets (see [Ko89, AHOW]). All inclusions indicated are proper [BK88,Ko89,Gav92].

$w_{max}(x)\}$ is the *left set of* A.[4]

Theorem 3.2 If $A \in$ NP and $Left(A) \in R_{bc}^p(R_{1\text{-}tt}^p(R_c^p(\text{SPARSE})))$, then $A \in$ P.

Corollary 3.3 If $A \in$ NP and $Left(A) \in R_c^p(\text{SPARSE})$, then $A \in$ P.

Of course, from Corollary 3.3, it immediately follows that Mahaney's Theorem generalizes to the case of conjunctive reductions. Corollary 3.4 has been obtained independently by Ranjan and Rohatgi [RR].

Corollary 3.4 If any NP-complete set conjunctively reduces to a sparse set, then P = NP.

Indeed, $Left(A) \leq_m^p A$ for any NP-complete set A, and thus the above theorems apply to the case when some NP-complete (or NP-hard) set reduces (via the reductions named above) to some sparse set.

Recently, Buhrman, Longpré, and Spaan [BLS92] have proved the surprising result that SPARSE $\subseteq R_c^p(\text{TALLY})$, which implies that $R_c^p(\text{SPARSE}) = R_c^p(\text{TALLY})$ and $R_b^p(R_c^p(\text{SPARSE})) = R_b^p(R_c^p(\text{TALLY}))$. Using this result, it can be shown [AKM] that if NP $\subseteq R_b^p(R_c^p(\text{SPARSE}))$ then P = NP, which unifies Theorem 3.2 with the result of Ogiwara and Watanabe [OW91].

We now discuss applications of the above results to the classes UP, PP, and $C_=$P.

Since for every set $A \in$ UP it holds that $Left(A)$ is in UP, it also follows that if UP is contained in $R_{bc}^p(R_{1\text{-}tt}^p(R_c^p(\text{SPARSE})))$ then UP = P. This strengthens the

[4]The left set tacitly depends on the particular witness relation chosen.

results of Watanabe [Wat91], who showed that if $\mathrm{P} \neq \mathrm{UP}$ then there exists a set in UP that does not many-one polynomial-time reduce to any sparse set.

Consider the set $\{\langle x, m\rangle \mid$ there are at least m satisfying assignments for $x\}$, which has properties similar to left sets and is complete for PP. Under the assumption that this set is in the reduction class of Theorem 3.2 we can compute in polynomial time a set of numbers that includes #SAT(x), the number of satisfying assignments of formula x. Now we can use the result of Cai and Hemachandra [CH91] and Toda (see [ABG90]) that P = PP if there is an FP function that computes on input x a set of numbers that includes #SAT(x).

Theorem 3.5 If PP is contained in $\mathrm{R}^p_{bc}(\mathrm{R}^p_{1\text{-}tt}(\mathrm{R}^p_c(\mathrm{SPARSE})))$ then P = PP.

By the fact that there exist complete sets in $\mathrm{C_{=}P}$ that are one word decreasing self-reducible [OL], we can show that if $\mathrm{C_{=}P}$ is contained in $\mathrm{R}^p_{bc}(\mathrm{R}^p_{1\text{-}tt}(\mathrm{R}^p_c(\mathrm{SPARSE})))$ then $\mathrm{C_{=}P} = \mathrm{P}$. In [AKM] it is shown that for $K \in \{\mathrm{UP}, \mathrm{PP}, \mathrm{C_{=}P}\}$, the assumption that K is contained in $\mathrm{R}^p_b(\mathrm{R}^p_c(\mathrm{SPARSE}))$ implies $\mathrm{P} = K$.

By applying a different proof technique we obtain results similar to Corollary 3.4 for the classes $\mathrm{Mod}_k\mathrm{P}$, $k \geq 2$ [CH90,BGH90].

Definition 3.6 A set L is *perfectly one word decreasing self-reducible* if there exist a function $f \in \mathrm{FP}$ and polynomials p and q such that

1. $L \subseteq \{\langle x, y, i\rangle \mid |y| = p(|x|) \text{ and } 1 \leq i \leq q(|x|)\}$,
2. $(\forall\, x, i : 1 \leq i \leq q(|x|))[f(\langle x, 0^{p(|x|)}, i\rangle) = \chi_L(\langle x, 0^{p(|x|)}, i\rangle)]$, and
3. $(\forall\, x)(\forall\, y \in \Sigma^{p(|x|)} - \{0^{p(|x|)}\})(\exists\, k)(\forall\, i : 1 \leq i \leq q(|x|))[f(\langle x, y, i\rangle) = \langle x, pred(y), (i+k) \bmod q(|x|)\rangle$ and $(\langle x, y, i\rangle \in L$ iff $f(\langle x, y, i\rangle) \in L)]$.

Theorem 3.7 If a perfectly one word decreasing self-reducible set L is conjunctively reducible in polynomial time to a sparse set then L is in P.

Since there exist complete sets in the classes $\mathrm{Mod}_k\mathrm{P}, k \geq 2$, that are perfectly one word decreasing self-reducible (in fact, these are essentially the strictly one word decreasing self-reducible complete sets for $\mathrm{Mod}_k\mathrm{P}$ that are defined in [OL]), we have the following corollary.

Corollary 3.8 If $k > 1$ and some $\mathrm{Mod}_k\mathrm{P}$-complete set polynomial-time conjunctively reduces to a sparse set, then $\mathrm{Mod}_k\mathrm{P} = \mathrm{P}$.

For nearly near-testable sets [HH91], the class of sets that have "implicit" polynomial-time membership tests, a similar result holds.

Theorem 3.9 If a nearly near testable set is conjunctively reducible in polynomial time to a sparse set then it is in P.

Finally, we show that the "P = NP if for some k there is a sparse k-tt hard set for NP" result of Ogiwara and Watanabe [OW91] cannot be improved to $\omega(\log n)$-bounded truth-table reductions by any relativizable proof technique, even with respect to the weakened conclusion that the boolean hierarchy [CGH+88] collapses. This extends work of Longpré and Homer (done independently of this work) that shows that for the same truth-table bounds, P = NP results are precluded in some relativized worlds [HL91] (see also [IM89,CGH+89,Kad89] for earlier results on oracles that limit the extent of the collapse one gets from assumptions of NP having sparse hard sets[5]).

Theorem 3.10 For each non-increasing, polynomial-time computable function $f(n) = \omega(\log n)$, there are sets A and T such that T is a tally set that is $\leq^{p,A}_{f(n)\text{-}tt}$-complete for NP^A, and yet the boolean hierarchy relative to A does not collapse.

4 Reductions to Simple Sparse Sets

In this section, we explore the second question mentioned in the introduction:

> If a set A reduces to a sparse set, does it follow that A is reducible to some sparse set that is "simple" relative to A?

Earlier work along these lines has been done both for the case (which is also the case of this paper) of reductions less flexible than Turing reductions [AHOW] and for the case of Turing reductions [GW91]. (Note that Theorems 4.2 and 4.3 are incomparable.)

Theorem 4.1 If $A \leq^p_{bd} S$ for some sparse set S, then there is a sparse set $\widehat{S}$ such that $A \leq^p_{bd} \widehat{S}$ and $\widehat{S} \in \mathrm{P}^{\mathrm{NP}^A[\log]}$.

Theorem 4.2 If $A \leq^p_{2\text{-}tt} S$ for some sparse set S, then there is a sparse set $\widehat{S}$ such that $A \leq^p_{2\text{-}tt} \widehat{S}$ and $\widehat{S} \in \mathrm{P}^{\mathrm{NP}^A[\log]}$.

Theorem 4.3 For $k = 2$ and $k = 3$: If $A \leq^p_{k\text{-}d} S$ for some sparse set S, then there is a sparse set $\widehat{S}$ such that $A \leq_{k\text{-}d} \widehat{S}$ and $\widehat{S} \in \mathrm{P}^{\mathrm{NP} \oplus A}$.

Also, we have the following results for unbounded disjunctive and unbounded conjunctive reductions.

Theorem 4.4 If $A \leq^p_d S$ for some sparse set S, then there is a sparse set $\widehat{S}$ such that $A \leq^p_d \widehat{S}$ and $\widehat{S} \in \mathrm{P}^{\mathrm{NP}^A}$.

[5]Kadin's [Kad89] Turing-hardness relativized claim in fact applies equally well to the truth-table case.

Theorem 4.5 If $A \leq_c^p S$ for some sparse set S, then there is a sparse set $\widehat{S}$ such that $A \leq_c^p \widehat{S}$ and $\widehat{S} \in \mathrm{NP}^A$.

5 Low Instance Complexity and Polynomial-Time Reductions

Instance complexity, as defined by Ko, Orponen, Schöning, and Watanabe [KOSW86,Orp90,OKSW], is a notion of the complexity of specific instances of a problem—a topic that standard complexity theory is ill-suited to study (as any single instance is trivial). In this paper, we are primarily concerned with sets of "low" instance complexity: IC[log,poly] (introduced in [KOSW86]).

We show that the sets of low instance complexity are intimately related to the study of reductions to sets of low information content. In particular, we show that a set A is of low instance complexity if and only if it both conjunctively and disjunctively reduces to tally sets.

Theorem 5.1 A is in IC[log,poly] if and only if there exist tally sets T_0 and T_1 such that $A \leq_c^p T_0$ and $A \leq_d^p T_1$.

Using the above characterization of IC[log,poly] it is easy to see that IC[log,poly] is closed under bounded truth-table reductions (a different proof that explicitly constructs programs appears in [OKSW]):

Theorem 5.2 [OKSW] IC[log,poly] is closed downward under $\leq_b^p$-reductions.

Since IC[log,poly] $\subseteq \mathrm{R}_c^p(\mathrm{SPARSE})$, the following corollary is an easy consequence of the above theorem and Corollary 3.4.

Corollary 5.3 [OKSW] If $\mathrm{P} \neq \mathrm{NP}$ and A is $\leq_b^p$-hard for NP, then $A \notin \mathrm{IC}[\log, \mathrm{poly}]$.

Using Theorem 5.1 and the fact that NP has neither $\leq_c^p$-hard nor $\leq_d^p$-hard tally sets unless $\mathrm{P} = \mathrm{NP}$, it follows that NP has neither $\leq_c^p$-hard nor $\leq_d^p$-hard sets in IC[log,poly] unless $\mathrm{P} = \mathrm{NP}$.

Corollary 5.4 If $\mathrm{P} \neq \mathrm{NP}$ and A is $\leq_d^p$-hard or $\leq_c^p$-hard for NP, then $A \notin \mathrm{IC}[\log, \mathrm{poly}]$.

Finally, using Theorem 3.10 and the fact that every tally set is in $\mathrm{IC}[\log, \mathrm{poly}]$, we conclude that for truth-tables of size $\omega(\log n)$, no analog of Corollary 5.3 can be proved by any relativizable proof technique.

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