Lowness and the Complexity of Sparse and Tally Descriptions

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Abstract

We investigate the complexity of obtaining sparse descriptions for sets in various reduction classes to sparse sets. Let A be a set in a certain reduction class $R_r(\text{SPARSE})$. Then we are interested in finding upper bounds for the complexity (relative to A) of sparse sets S such that $A \in R_r(S)$. By establishing such upper bounds we are able to derive the lowness of A.

In particular, we show that if a set A is in the class $R^p_{hd}(R^p_c(\text{SPARSE}))$ then A is in $R^p_{hd}(R^p_c(S))$ for a sparse set $S \in \text{NP}(A)$. As a consequence we can locate $R^p_{hd}(R^p_c(\text{SPARSE}))$ in the EL^{Θ}_3 level of the extended low hierarchy. Since $R^p_{hd}(R^p_c(\text{SPARSE})) \supseteq R^p_b(R^p_c(\text{SPARSE}))$ this solves the open problem of locating the closure of sparse sets under bounded truth-table reductions optimally in the extended low hierarchy. Furthermore, we show that for every $A \in R^p_d(\text{SPARSE})$ there exists a sparse set $S \in \text{NP}(A \oplus \text{SAT})/\text{F}\Theta^p_2(A)$ such that $A \in R^p_d(S)$. Based on this we show that $R^p_{1\text{-}tt}(R^p_d(\text{SPARSE}))$ is in EL^{Θ}_3.

Finally, we construct for every set $A \in R^p_c(\text{TALLY}) \cap R^p_d(\text{TALLY})$ (or equivalently, $A \in \text{IC}[\log, \text{poly}]$, as shown in [AHH$^+$92]) a tally set $T \in \text{P}(A \oplus \text{SAT})$ such that $A \in R^p_c(T) \cap R^p_d(T)$. This implies that the class IC[log, poly] of sets with low instance complexity is contained in EL^{Σ}_1.

1 Introduction

Sparse sets play a central role in structural complexity theory. The question of the existence of sparse hard sets for various complexity classes under different sorts of reducibilities is well studied (see for example [KL80, Mah82, OW91, AHH$^+$92, AKM92]). Besides that much work has been done on other issues concerning the complexity of sets reducible to sparse sets (see for example [Kad87, AH92, SL92,

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AHH+92]). A central motivation for most earlier work (and also for this paper) can be seen as seeking answers to the following two questions.

1. If a set A is reducible to a sparse set, does it follow that A is reducible to some sparse set that is "simple" relative to A?

2. If a set A is reducible to a sparse set, then how easy is it to access all the relevant information contained in the set A when it is used as oracle?

Question 1 originates in the study of the notions of equivalence and reducibility to sparse sets (see for example [TB91, AHOW, GW91]) and addresses the complexity (relative to A) of sparse descriptions for sets A which are reducible to sparse sets. For the case of Turing reductions, Gavaldà and Watanabe [GW91] established an $\mathrm{NP} \cap \text{co-NP}$ lower bound by constructing a set $B \in R^p_T(\mathrm{SPARSE})$ (in fact, B is even in $R^p_c(\mathrm{SPARSE})$) that is not Turing reducible to a sparse set in $\mathrm{NP}(B) \cap \text{co-NP}(B)$. For truth-table reduction classes to sparse sets, the first question is also investigated in [AHH+92] where various upper bounds for the relative complexity of sparse descriptions are presented.

The second question concerns lowness properties of sets reducible to sparse sets. Lowness is an important structural tool for systematically classifying sets in different complexity classes according to the power they provide when used as oracles. Building on ideas from recursive function theory, Schöning [Sch83] defined the low hierarchy inside NP. In order to classify sets outside of NP (e.g. sets reducible to sparse sets) Balcázar, Book, and Schöning [BBS86] defined the extended low hierarchy. The extended low hierarchy was subsequently refined by Allender and Hemachandra [AH92] and Long and Sheu [LS91] who showed the optimal location of various reduction and equivalence classes to sparse and tally sets in the (refined) extended low hierarchy. Very recently, Sheu and Long [SL92] proved that the extended low hierarchy is an infinite hierarchy.

In order to investigate the first question, we generalize the definition of sparse set descriptions given in [GW91]. Let $\leq_r$ be a reducibility. A sparse set S is a sparse r-description for a set A if $A \leq_r S$. Similarly, a tally set T is a tally r-description for A if $A \leq_r T$. In particular a sparse set S is called a sparse description for A if $A \leq^p_T S$. We are interested in finding upper bounds for the relative complexity of sparse (tally) r-descriptions for sets in $R_r(\mathrm{SPARSE})$[1] (respectively $R_r(\mathrm{TALLY})$). We refer to a sparse r-description satisfying the established upper bound as a *simple* sparse r-description (with respect to that upper bound), and we refer to the corresponding upper bound result as a *simplicity* result.

In Section 3 we establish simplicity results for various reduction classes to sparse and tally sets. In Section 4 we apply the simplicity results to derive lowness properties. These results reveal a close interconnection between the lowness of sets which are reducible to sparse or tally sets and the complexity of computing small descriptions for these sets. More precisely, the general pattern to prove the lowness of a set A that reduces to a sparse set is the following: first we appropriately bound the complexity of a sparse description for A; based on such a simplicity result, we then apply a deterministic enumeration technique like that of Mahaney [Mah82] or a census technique similar to that of Hemachandra [Hem87] and Kadin [Kad87] to replace the sparse oracle S (and thus A). Intuitively, we use the simplicity result to extract enough information (a suitable initial segment of S or the census thereof) from the oracle set A in order to avoid further queries to A. Our approach clarifies that the

[1]For a reducibility $\leq_r$ and a class $\mathcal{C}$ of sets, $R_r(\mathcal{C}) = \{A \mid A \leq_r B \text{ for some } B \in \mathcal{C}\}$.

Name	Notation
many-one	$\leq_m^p$
one truth-table	$\leq_{1\text{-}tt}^p$
bounded truth-table	$\leq_b^p$
conjunctive	$\leq_c^p$
disjunctive	$\leq_d^p$
bounded Hausdorff	$\leq_{bhd}^p$
Hausdorff	$\leq_{hd}^p$
np many-one	$\leq_m^{np}$
co-np many-one	$\leq_m^{co\text{-}np}$

Table 1: Polynomial-time Reductions

appropriate simplicity result is the basic reason for the set to be low. Using this approach we are able to derive several new and optimal lowness results.

2 Preliminaries and notation

Let A be a set. χ_A denotes the characteristic function of A. $A^{=n}$ ($A^{\leq n}$) denotes the set of all strings in A of length n (up to length n, respectively). The cardinality of A is denoted by $|A|$. A set T is called a tally set if $T \subseteq 0^*$. The census function of a set A is $census_A(1^n) = |A^{\leq n}|$. A set S is called sparse if its census function is bounded above by a polynomial. We use TALLY and SPARSE to represent the classes of tally and sparse sets, respectively.

Let $\mathcal{C}$ be a class of sets. A set S is called P($\mathcal{C}$)-printable (see [HY84]) if the function $0^n \mapsto \langle S^{\leq n}\rangle$ can be computed inside FP($\mathcal{C}$).

The reducibilities discussed in this paper are the standard polynomial time bounded reducibilities defined by Ladner, Lynch, and Selman [LLS75] and the Hausdorff reduction introduced by Wagner [Wag87] (see Table 1 for an overview).

Definition 2.1 [Wag87] *A is Hausdorff-reducible to B ($A \leq_{hd}^p B$), if there exists a function f computable in polynomial time, such that for all x, f computes a tuple $f(x) = \langle y_1, y_2, \ldots, y_{2k}\rangle$ such that*

1. $\chi_B(y_1) \geq \chi_B(y_2) \geq \cdots \geq \chi_B(y_{2k})$, *and*
2. $\chi_A(x) = \bigvee_{i=1}^k [\chi_B(y_{2i-1}) \wedge \neg\chi_B(y_{2i})]$,

We call f a bounded Hausdorff reduction ($A \leq_{bhd}^p B$) if k is a constant.

Notation [AHOW] For any reducibility $\leq_r^\alpha$ where $\alpha \in \{p, np, co\text{-}np\}$ and $r \in \{m, c, d, 1\text{-}tt, b, hd, bhd\}$ and any class $\mathcal{C}$ of sets let $R_r^\alpha(\mathcal{C}) = \{A \mid A \leq_r^\alpha B$ for some $B \in \mathcal{C}\}$.

A class $\mathcal{K}$ of sets that includes $\emptyset$ and Σ^* and is closed under union and intersection is said to be a set ring. The following characterization of the boolean closure of set rings due to Hausdorff plays a key role in some of our results.

Theorem 2.2 [Hau14] *Let $\mathcal{K}$ be a set ring and let $BC(\mathcal{K})$ be the closure of $\mathcal{K}$ under union, intersection, and complement. Then every $A \in BC(\mathcal{K})$ can be represented as $A = \bigcup_{i=1}^{k}(A_{2i-1} - A_{2i})$, where $A_j \in \mathcal{K}$, $1 \le j \le 2k$, and $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{2k}$.*

The above representation for A is called a Hausdorff representation for A over $\mathcal{K}$. We state some useful properties relating boolean closures, bounded truth-table closures, and the bounded Hausdorff closures of language classes.

Lemma 2.3 [KSW87] *Let $\mathcal{K}$ be a class that contains P and is closed under many-one reductions. Then $BC(\mathcal{K}) = R_b^p(\mathcal{K})$.*

Lemma 2.4 *Let $\mathcal{K}$ be a class closed under many-one reductions and under join. Then A has a Hausdorff representation over $\mathcal{K}$ if and only if A is bounded Hausdorff reducible to some set in $\mathcal{K}$.*

The next lemma is obtained by combining Theorem 2.2 and the above two lemmas.

Lemma 2.5 *If $\mathcal{K}$ is a set ring which contains* P *and is closed under many-one reductions then $BC(\mathcal{K}) = R_{bhd}^p(\mathcal{K}) = R_b^p(\mathcal{K})$.*

Theorem 2.6 1. $R_{bhd}^p(R_c^p(\mathrm{SPARSE})) = R_b^p(R_c^p(\mathrm{SPARSE}))$

2. $R_{bhd}^p(R_m^{co\text{-}np}(\mathrm{SPARSE})) = R_b^p(R_m^{co\text{-}np}(\mathrm{SPARSE}))$

For a class $\mathcal{C}$ of sets and a class $\mathcal{F}$ of functions from Σ^* to Σ^* let $\mathcal{C}/\mathcal{F}$ [KL80] be the class of sets A such that there is a set $B \in \mathcal{C}$ and a function $h \in \mathcal{F}$ such that for all $x \in \Sigma^*$,

$$\chi_A(x) = \chi_B(x, h(0^{|x|}))$$

Although Karp and Lipton introduced the notion of advice functions in order to characterize nonuniform complexity classes by imposing a quantitative length restriction on the functions in $\mathcal{F}$, we will consider here complexity restricted advice function classes. We refer the reader to [KT90] for a general study of complexity restricted advice functions.

For a set A the class $\Theta(A)$ [LS91] contains all languages $L(M, A)$ accepted by a deterministic polymomial time bounded oracle machine M asking on inputs of length n at most $O(\log n)$ queries to A. A deterministic polynomial-time oracle machine M as in the definition above is called a Θ machine. The Θ levels of the (relativized) polynomial time hierarchy are defined as $\Theta_k^p(A) = \Theta(\Sigma_{k-1}^p(A)), k \ge 1$ [Wag90, LS91].

Similarly, the class $\mathrm{F}\Theta(A)$ contains all functions computable by some deterministic polymomial time bounded oracle transducer M asking on inputs of length n at most $O(\log n)$ queries to oracle A, and $\mathrm{F}\Theta_k^p(A) = \mathrm{F}\Theta(\Sigma_{k-1}^p(A)), k \ge 1$.

For further definitions used in this paper we refer the reader to standard books on structural complexity theory (for example, [BDG, Sch86]).

3 Upper bounds for sparse and tally descriptions

In this section we consider the following question: If a set A reduces to a sparse set S via a reduction of a certain type, does it follow that A reduces via a reduction of the same type to some sparse set S' that is "simple" relative to A?

We notice that a simplicity argument was already used in the proof of Mahaney's theorem [Mah82] that if NP has sparse hard sets then P = NP. Mahaney first showed that P = NP under the stronger assumption that NP has a sparse complete set. From that the theorem is derived by observing that if a set $A \in \mathrm{NP}$ many-one reduces in polynomial time to a sparse set then it actually many-one reduces to a sparse set in NP. This observation can be formalized as a simplicity result for sets in $R^p_m(\mathrm{SPARSE})$.

Theorem 3.1 [LS91] *If $A \in R^p_m(\mathrm{SPARSE})$ then there exists a sparse set $S \in R^{np}_m(A)$ such that $A \leq^p_m S$.*

It is easy to see that the same upper bound holds for $R^p_c(\mathrm{SPARSE})$, i.e., every set $A \in R^p_c(\mathrm{SPARSE})$ has a sparse c-description in $R^{np}_m(A)$. On the other hand, for a set A in $R^p_b(\mathrm{SPARSE})$ the only known upper bound for the complexity of sparse b-descriptions is $\Delta^p_3(A)$ (this can be seen by a minor modification of the proof that every set A in P/poly has an advice function computable in $\Delta^p_3(A)$ [Sch86]).

Surprisingly, it turns out (see the following theorem) that each set A in the reduction class $R^p_{hd}(R^p_c(\mathrm{SPARSE}))$, which is larger than $R^p_b(\mathrm{SPARSE})$, has a sparse $hd(c)$-description in $\mathrm{NP}(A)$. From this it follows (see Section 4) that $R^p_{hd}(R^p_c(\mathrm{SPARSE}))$, and therefore $R^p_b(\mathrm{SPARSE})$ are in EL^{Θ}_3.

Theorem 3.2 *For every set $A \in R^p_{hd}(R^p_c(\mathrm{SPARSE}))$ there is a sparse set $\hat{S} \in \mathrm{NP}(A)$ such that $A \in R^p_{hd}(R^p_c(\hat{S}))$.*

Corollary 3.3 *For every set $A \in R^p_b(R^p_c(\mathrm{SPARSE}))$ there is a sparse set $S' \in \mathrm{NP}(A)$ such that $A \in R^p_b(R^p_c(S'))$.*

Next we consider sets in $R^p_d(\mathrm{SPARSE})$ and show that every set A in $R^p_d(\mathrm{SPARSE})$ has a sparse d-description in $\mathrm{NP}(A \oplus \mathrm{SAT})/\mathrm{F}\Theta^p_2(A)$. Furthermore we show that there exists a set in $\mathrm{NP}(A)/\mathrm{F}\Theta^p_2(A)$ such that every element $\langle 0^n, W\rangle$ in this set encodes a finite set W to which A is disjunctively reducible with respect to strings of length n, i.e., $A^{=n} \leq^p_d W$ via the given disjunctive reduction function (cf. the notion of $\mathrm{CIR}(A)$ [KoSc85]). Our proof technique is a refinement and an extension of the one developed in [AHH+92] where it is shown that for every set A in $R^p_d(\mathrm{SPARSE})$ there is a sparse set $S' \in \mathrm{P}(\mathrm{NP}(A))$ such that $A \in R^p_d(S')$.

Theorem 3.4 *Let A be a set in $R^p_d(\mathrm{SPARSE})$ witnessed by the sparse set S and the reduction function f. Then*

1. *there exists a set $C \subseteq \{\langle 0^n, W\rangle \mid A \leq^p_d W$ via f for inputs of length $n\}$ and a polynomial p such that $C \in \mathrm{NP}(A)/\mathrm{F}\Theta^p_2(A)$ and for every n there is at least one pair of the form $\langle 0^n, W\rangle$ in $C^{\leq p(n)}$,*

2. *there exists a sparse set $\hat{S} \in \mathrm{NP}(A \oplus \mathrm{SAT})/\mathrm{F}\Theta^p_2(A)$ such that $A \leq^p_d \hat{S}$ via f.*

The following theorem shows in general that a simplicity result for a reduction class $R^p_r(\mathrm{SPARSE})$ can be translated into a simplicity result for the reduction class $R^p_{1\text{-}tt}(R^p_r(\mathrm{SPARSE}))$.

Theorem 3.5 *Let $\mathcal{C}$ be a relativizable complexity class closed under join such that $\mathcal{C}(A \oplus B) = \mathcal{C}(A) = \mathcal{C}(\overline{A})$ for all sets $A \subseteq \Sigma^*$ and $B \in \mathrm{P}$. If every set L in $R^p_r(\mathrm{SPARSE})$ has a sparse r-description in $\mathcal{C}(L)$ then every set A in $R^p_{1\text{-}tt}(R^p_r(\mathrm{SPARSE}))$ has a sparse 1-$tt(r)$-description in $\mathcal{C}(A)$.*

Corollary 3.6 *For every set A in $R^p_{1\text{-}tt}(R^p_d(\mathrm{SPARSE}))$ there is a sparse set S' in $\mathrm{NP}(A \oplus \mathrm{SAT})/\mathrm{F}\Theta^p_2(A)$ such that $A \in R^p_{1\text{-}tt}(R^p_d(S'))$.*

Next we consider nondeterministic reduction classes to sparse sets. The notion of $\leq^{co\text{-}np}_m$ reducibility can be seen as a generalization of the deterministic polynomial-time conjunctive reducibility.

Definition 3.7 *A set A is co-np many-one reducible to a set B (denoted $A \leq^{co\text{-}np}_m B$) if there exists a polynomial-time nondeterministic Turing transducer M such that for every $x \in \Sigma^*$, $x \in A$ if and only if all outputs of M on input x are in B.*

Note that $A \leq^{co\text{-}np}_m B$ if and only if $\overline{A} \leq^{np}_m \overline{B}$ where $\leq^{np}_m$ is the more familiar polynomial-time nondeterministic many-one reducibility [LLS75]. Clearly, for every set B, $R^p_c(B)$ is contained in $R^{co\text{-}np}_m(B)$, and $R^{co\text{-}np}_m(B)$ is closed downward under $\leq^p_c$ and $\leq^{co\text{-}np}_m$ reductions.

Theorem 3.8 *1. For every set $A \in R^{co\text{-}np}_m(\mathrm{SPARSE})$ there exists a sparse set $S' \in R^{np}_m(A)$ such that $A \in R^{co\text{-}np}_m(S')$.*

2. For every set $A \in R^p_{hd}(R^{co\text{-}np}_m(\mathrm{SPARSE}))$ there is a sparse set $S' \in \mathrm{NP}(\mathrm{SAT} \oplus A)$ such that $A \in R^p_{hd}(R^{co\text{-}np}_m(S'))$.

3. For every set $A \in R^p_b(R^{co\text{-}np}_m(\mathrm{SPARSE}))$ there is a sparse set $S' \in \mathrm{NP}(\mathrm{SAT} \oplus A)$ such that $A \in R^p_b(R^{co\text{-}np}_m(S'))$.

At the end of this section we consider reduction classes to tally sets. The class $R^p_c(\mathrm{TALLY}) \cap R^p_d(\mathrm{TALLY})$ is of particular interest since it coincides with the class IC[log,poly] of sets containing only strings of low instance complexity [AHH+92]. IC[log,poly] and the notion of instance complexity were introduced by Ko, Orponen, Schöning, and Watanabe (see [OKSW]).

Theorem 3.9 *1. For every set $A \in R^p_c(\mathrm{TALLY}) \cap R^p_d(\mathrm{TALLY})$ there is a tally set $T' \in \mathrm{P}(\mathrm{SAT} \oplus A)$ such that $A \in R^p_c(T') \cap R^p_d(T')$.*

2. For every set $A \in R^{np}_m(\mathrm{TALLY}) \cap R^{co\text{-}np}_m(\mathrm{TALLY})$ there is a tally set $T' \in \mathrm{P}(\mathrm{SAT} \oplus A)$ such that $A \in R^{np}_m(T') \cap R^{co\text{-}np}_m(T')$.

The class IC[log, poly] is known to be closed downward under polynomial time bounded truth-table reductions [OKSW]. It is interesting to note that IC[log, poly] and $R^{np}_m(\mathrm{TALLY}) \cap R^{co\text{-}np}_m(\mathrm{TALLY})$ are also closed downward under polynomial time Hausdorff reductions.

Theorem 3.10 *1. $R^p_{hd}(\mathrm{IC}[\log, \mathrm{poly}]) = \mathrm{IC}[\log, \mathrm{poly}]$*

2. $R^p_{hd}(R^{np}_m(\mathrm{TALLY}) \cap R^{co\text{-}np}_m(\mathrm{TALLY})) = R^{np}_m(\mathrm{TALLY}) \cap R^{co\text{-}np}_m(\mathrm{TALLY})$

4 Lowness

The low and high hierarchies inside NP were introduced by Schöning [Sch83]. The notion of lowness has turned out to be an important structural tool for classifying problems and subclasses of NP not known to be NP-complete or in P. This idea was extended by Balcázar, Book, and Schöning [BBS86] who defined the extended low hierarchy in order to classify decision problems and classes not contained in NP. Allender and Hemachandra [AH92] and Long and Sheu [LS91] refined the extended low hierarchy and proved the optimality of the location of various classes therein.

Definition 4.1 [BBS86, AH92, LS91] *The Σ, Δ, and Θ levels of the extended low hierarchy (denoted EL_k^{Σ}, EL_k^{Δ}, and EL_k^{Θ}, respectively) are defined as below.*

1. $EL_k^{\Sigma} = \{A \mid \Sigma_k^p(A) \subseteq \Sigma_{k-1}^p(\mathrm{SAT} \oplus A)\}$, $k \geq 1$
2. $EL_k^{\Delta} = \{A \mid \Delta_k^p(A) \subseteq \Delta_{k-1}^p(\mathrm{SAT} \oplus A)\}$, $k \geq 2$
3. $EL_k^{\Theta} = \{A \mid \Theta_k^p(A) \subseteq \Theta_{k-1}^p(\mathrm{SAT} \oplus A)\}$, $k \geq 2$

Various classes of sets reducible to sparse and tally sets have been shown to be included in the extended low hierarchy (see for example [BBS86, AH92, LS91]). Using the simplicity results of the previous section we are able to derive lowness results for the reduction classes to sparse sets considered here. Some of our extended lowness proofs contain a census argument similar to that used by Hemachandra [Hem87] and Kadin [Kad87].

Theorem 4.2
1. $R_{hd}^p(R_c^p(\mathrm{SPARSE})) \subseteq EL_3^{\Theta}$
2. $R_d^p(\mathrm{SPARSE}) \subseteq EL_3^{\Theta}$.
3. $R_{1\text{-}tt}^p(R_d^p(\mathrm{SPARSE})) \subseteq EL_3^{\Theta}$.

These results are optimal since SPARSE $\not\subseteq EL_2^{\Sigma}$ as proved in [AH92]. Item 4 of the following corollary answers an open question in [AH92] and also extends the recent result by Long and Sheu [LS91] that $R_{1\text{-}tt}^p(\mathrm{SPARSE}) \subseteq EL_3^{\Theta}$.

Corollary 4.3
1. $R_c^p(\mathrm{SPARSE}) \subseteq EL_3^{\Theta}$
2. $R_c^p(\text{co-SPARSE}) \subseteq EL_3^{\Theta}$
3. $R_d^p(\text{co-SPARSE}) \subseteq EL_3^{\Theta}$
4. $R_b^p(\mathrm{SPARSE}) \subseteq EL_3^{\Theta}$
5. $E_T^p(R_{hd}^p(R_c^p(\mathrm{SPARSE}))) \subseteq EL_3^{\Theta}$
6. $E_T^p(R_d^p(\mathrm{SPARSE})) \subseteq EL_3^{\Theta}$

The following theorem states the generalized Θ_2^p-lowness of $\mathrm{NP}(\mathrm{NP} \cap \mathrm{SPARSE}) \cap R_m^{co\text{-}np}(\mathrm{SPARSE})$ and is an improvement of Kadin's result that every sparse NP set is low for Θ_2^p.

Theorem 4.4 *If $A \in \mathrm{NP}(\mathrm{NP} \cap \mathrm{SPARSE})$ and $A \leq_m^{co\text{-}np} S$ for a sparse set S then $\Theta_2^p(A) \subseteq \Theta_2^p$, i.e. A is low for Θ_2^p.*

Using this theorem we can improve the result of Lozano and Torán [LT91] that every disjunctive self-reducible set that is many-one reducible to some sparse set is low for Θ_2^p. Since co-NP/log $\subseteq R_m^{co\text{-}np}(\mathrm{TALLY})$ [AKM92] it subsumes all previously known Θ_2^p-lowness results (e.g. for $\mathrm{NP} \cap R_m^p(\mathrm{SPARSE})$ and for $\mathrm{NP} \cap$ co-NP/log, cf. [LS91]) regarding NP sets reducible to sparse or tally sets.

Corollary 4.5
1. *If $A \in \mathrm{NP} \cap R_c^p(\mathrm{SPARSE})$ then $\Theta_2^p(A) \subseteq \Theta_2^p$.*
2. *If $A \in \mathrm{NP} \cap R_m^{co\text{-}np}(\mathrm{SPARSE})$, then $\Theta_2^p(A) \subseteq \Theta_2^p$.*

Although we have simplicity results concerning nondeterministic reduction classes to sparse sets (e.g. Theorem 3.8) extended lowness is not a meaningful measure of lowness for such reduction classes since they contain either NP or co-NP. Nevertheless, the next theorem is a kind of lowness result. We show that the full power of a Θ_3^p computation relative to an oracle in $R_{hd}^p(R_m^{co\text{-}np}(\mathrm{SPARSE}))$ is not needed in order to access the information contained in the oracle.

Theorem 4.6 *If $A \in R_{hd}^p(R_m^{co\text{-}np}(\mathrm{SPARSE}))$ then $\Theta_3^p(A) \subseteq \Theta_2^p(\Sigma_2^p \oplus A)$.*

Finally we locate IC[log,poly] in the first level of the extended low hierarchy. Since $EL_1^\Sigma = EL_2^\Theta = EL_2^\Delta$ the class $\mathrm{IC}[\log, \mathrm{poly}]$ which equals $R_c^p(\mathrm{SPARSE}) \cap R_d^p(\text{co-SPARSE})$ is located two levels below $R_c^p(\mathrm{SPARSE})$ in the extended low hierarchy.

Theorem 4.7 *If A has a sparse description that is $\mathrm{P}(\mathrm{SAT} \oplus A)$ printable then $A \in EL_1^\Sigma$.*

Since the class APT (almost polynomial time [MP79]) is easily seen to be contained in $\mathrm{IC}[\log, \mathrm{poly}]$ the following corollary subsumes all previously known EL_1^Σ-lowness results (e.g. for APT [LS91], for $E_T^p(\mathrm{TALLY})$ [BB86] and for $R_m^p(\mathrm{TALLY})$ [AH92]) regarding sets reducible to sparse or tally sets.

Corollary 4.8

1. $\mathrm{IC}[\log, \mathrm{poly}] \subseteq EL_1^\Sigma$
2. $E_T^p(\mathrm{IC}[\log, \mathrm{poly}]) \subseteq EL_1^\Sigma$
3. $R_m^{np}(\mathrm{TALLY}) \cap R_m^{co\text{-}np}(\mathrm{TALLY}) \subseteq EL_1^\Sigma$

We summarize some of our results on simplicity and lowness in the following table.

For A in the reduction class	simplicity	lowness
$R_{hd}^p(R_c^p(\mathrm{SPARSE}))$	$\mathrm{NP}(A)$	EL_3^Θ
$R_{1\text{-}tt}^p(R_d^p(\mathrm{SPARSE}))$	$\mathrm{NP}(\mathrm{SAT} \oplus A)/\mathrm{F}\Theta_2^p(A)$	EL_3^Θ
$R_c^p(\mathrm{TALLY}) \cap R_d^p(\mathrm{TALLY})$	$\mathrm{P}(\mathrm{SAT} \oplus A)$	EL_1^Σ
$R_m^{np}(\mathrm{TALLY}) \cap R_m^{co\text{-}np}(\mathrm{TALLY})$	$\mathrm{P}(\mathrm{SAT} \oplus A)$	EL_1^Σ
$R_m^{co\text{-}np}(\mathrm{SPARSE}) \cap \mathrm{NP}^{\mathrm{NP} \cap \mathrm{SPARSE}}$	$R_m^{np}(A)$	$\Theta_2^p(A) \subseteq \Theta_2^p$
$R_{hd}^p(R_m^{co\text{-}np}(\mathrm{SPARSE}))$	$\mathrm{NP}(\mathrm{SAT} \oplus A)$	$\Theta_3^p(A) \subseteq \Theta_2^p(\Sigma_2^p \oplus A)$

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