Hausdorff Reductions to Sparse Sets and to Sets of High Information Content

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Abstract. We investigate the complexity of sets that have a rich internal structure and at the same time are reducible to sets of either low or very high information content. In particular, we show that every length-decreasing or word-decreasing self-reducible set that reduces to some sparse set via a non-monotone variant of the Hausdorff reducibility is low for Δ_2^p.
Measuring the information content of a set by the space-bounded Kolmogorov complexity of its characteristic sequence, we further investigate the (non-uniform) complexity of sets A in EXPSPACE/poly that reduce to some set having very high information content. Specifically, we show that if the reducibility used has a certain property, called "reliability," then A in fact is reducible to a sparse set (under the same reducibility). As a consequence of our results, the existence of hard sets (under "reliable" reducibilities) of very high information content is unlikely for various complexity classes as for example NP, PP, and PSPACE.

1 Introduction

An important subject of research in structural complexity theory is the study of reductions to sets of low information content (as for example sparse sets). Historically, this study originated in the Berman-Hartmanis conjecture that all NP-complete sets are polynomial-time isomorphic, a hypothesis that is empirically supported by all natural examples of NP-complete sets. An intuitive interpretation of the conjecture is that all NP-complete sets are only different encodings of the same information. Since polynomial-time isomorphic sets must have similar densities, and since there are NP-complete sets of exponential density, the Berman-Hartmanis conjecture implies that sparse sets cannot be NP-complete. Motivated by the Berman-Hartmanis conjecture, several results concerning reductions of NP-complete sets to sparse sets have been established. The basic results along this line of research were Mahaney's theorem that NP-complete sets are not many-one reducible to a sparse set unless P = NP [13], and the

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result of Karp, Lipton, and Sipser, that a sparse set cannot be Turing-hard for NP unless the polynomial hierarchy collapses to Σ_2^p [8]. The more recent result of Ogiwara and Watanabe [14], that the existence of sparse hard sets for NP under polynomial time bounded truth-table reductions implies P = NP, has been followed up by analogous results for more general reductions in [1, 15, 2].

Recently, Book and Lutz [7] have considered reductions to sets whose characteristic sequences are of very high space-bounded Kolmogorov complexity (they call the class of such sets HIGH). Book and Lutz obtained the surprising result that every set in ESPACE that is (polynomial-time) bounded truth-table reducible to a set in HIGH is actually bounded truth-table reducible to some sparse set. Thus, oracles of very high space-bounded Kolmogorov complexity are only as useful as sparse sets provided that the oracle access is restricted to a constant number of queries to retrieve the information from the oracle set. As a consequence, if SAT bounded truth-table reduces to a set in HIGH, then SAT bounded truth-table reduces to a sparse set, and thus, using the result of Ogiwara and Watanabe, it follows that P = NP.

Of course, one would expect similar results for a wider class of reducibility types, not only for the bounded truth-table case. The aim of the present paper is to identify the properties (of reducibilities) that are responsible for their inability to extract a substantial part of the information hidden in the sets in HIGH. We prove results analogous to the one of Book and Lutz for a wide range of reducibilities which we call "reliable reducibilities". Intuitively, A reduces to B via a reliable reducibility if not only A is determined by B (and the specific reduction used), but information about B can also be retrieved from the knowledge of A. More precisely, in the case of a reliable reduction, given A and the reduction procedure, it is possible to compute a *minimum* set B' of strings to which A reduces via the given reduction, that is, B' is a subset of every set to which A reduces.

For example, deterministic many-one reductions f are reliable, since for every $x \in A$ the membership of the image $f(x)$ in B is guaranteed, and A correctly reduces to the set $B' = \{f(x) \mid x \in A\}$ of all these images (a similar consideration applies for conjunctive reductions). In contrast, Turing reductions are not considered to be reliable since in general the information about B that can be derived from the knowledge of A and the oracle machine performing the reduction is rather vague. On the other hand, as we show in this paper, the Hausdorff reducibility [17] is reliable. In fact, it turns out that even an extension of the Hausdorff reducibility (called non-monotone Hausdorff reducibility) is reliable.

In spite of its reliability, the Hausdorff reducibility is much more powerful than the many-one reducibility, and in some cases it has been shown that the Hausdorff reducibility has the same power as the truth-table reducibility. For example, Wagner [17] proved that the closure of NP under the Hausdorff reducibility coincides with the truth-table closure of NP. Furthermore, for every set ring $\mathcal{C}$ it is known (cf. [18]) that the closure of $\mathcal{C}$ under the bounded Hausdorff reducibility is the same as its closure under the bounded truth-table reducibility (as an example we mention the equality $R^p_{bhd}(R^p_c(\text{SPARSE})) = R^p_{btt}(R^p_c(\text{SPARSE}))$ [3], where R_c^p(SPARSE) is the closure of sparse sets under the conjunctive reducibility).

In the first part of this paper we consider sets that have a rich internal structure (self-reducibility) and at the same time are Hausdorff reducible to sets of low information content (i.e. sparse sets). Exploiting properties of the Hausdorff reducibility we show that every length-decreasing or word-decreasing self-reducible set that non-monotone Hausdorff reduces to some sparse set is low for P^{NP}, i.e. it does not provide this class with any additional power when used as an oracle. This generalizes and improves results of Lozano and Torán [12] stating that any length-decreasing or word-decreasing self-reducible set that many-one reduces to a sparse set is low for Δ_2^p.

In the second part we show that if a set A in EXPSPACE/poly reduces via a reliable reducibility to some set in HIGH, then A reduces (via the same reducibility) to a sparse set. As a consequence of this result, it is unlikely that there exist hard sets (under the non-monotone Hausdorff and other reliable reducibilities) for complexity classes like UP, NP, PP, or PSPACE.

2 Preliminaries

Let $\Sigma = \{0, 1\}$ be the standard alphabet, and let $A \subseteq \Sigma^*$ be a set. The length of a string x is denoted by $|x|$. $A^{=n}$ ($A^{\leq n}$) denotes the set of all strings in A of length n (up to length n, respectively). χ_A denotes the characteristic function of A. $\chi^A_{\leq n}$ denotes the characteristic sequence of A for all strings up to length n, i.e., $|\chi^A_{\leq n}| = 2^{n+1} - 1$, and the i-th bit of $\chi^A_{\leq n}$ equals $\chi_A(s_i)$ where s_i is the i-th string in Σ^* in lexicographical order. The cardinality of A is denoted by $|A|$. The census function of a set A is $census_A(1^n) = |A^{\leq n}|$. A set S is called sparse if its census function is bounded above by a polynomial. A set T is called a tally set if $T \subseteq 0^*$. We use TALLY and SPARSE to represent the classes of tally and sparse sets, respectively. The empty string is denoted by ε. $\langle \cdot, \cdot \rangle$ denotes a standard polynomial time computable pairing function whose inverses are also computable in polynomial time. For any reducibility type α, let $R_\alpha(\mathcal{C}) = \{A \mid \exists B \in \mathcal{C} : A \leq_\alpha B\}$ denote the class of sets $\leq_\alpha$-reducible to some set in $\mathcal{C}$. The reducibilities discussed in this paper are the standard polynomial-time reducibilities defined by Ladner, Lynch, and Selman [10] and the following Hausdorff reducibility introduced by Wagner [16].

Definition 1. A is Hausdorff reducible to B (in symbols: $A \leq_{hd}^p B$), if there exists a polynomial-time computable function f mapping every string x to a sequence of queries, such that for all $x \in \Sigma^*$, if $f(x) = \langle y_1, \ldots, y_k \rangle$ then

- $y_{i+1} \in B$ implies $y_i \in B$ for all $i = 1, \ldots, k-1$, and
- $x \in A \iff \max\{j \mid 0 \leq j \leq k \text{ and for all } i = 1, \ldots, j : y_i \in B\}$ is odd.

We call f a bounded Hausdorff reduction ($A \leq_{bhd}^p B$) if the number $k(x)$ of queries produced by f on x is bounded by a constant for all x.

In this context, the i-th query y_i computed by $f(x)$, $1 \le i \le k(x)$, is also denoted by $f(x,i)$. By removing in the definition of the Hausdorff reducibility the monotony condition we obtain the following generalization of the Hausdorff reducibility which, as we will see in Proposition 3, is equivalent in power to the composition of the Hausdorff and the conjunctive reducibilities.

Definition 2. A is non-monotone Hausdorff reducible to B ($A \le_{nhd}^{p} B$), if there exists a polynomial-time computable function f mapping every string x to a sequence of queries, such that for all $x \in \Sigma^*$, if $f(x) = \langle y_1, \ldots, y_k \rangle$ then

$$x \in A \iff \max\{j \mid 0 \le j \le k \text{ and for all } i = 1, \ldots, j : y_i \in B\} \text{ is odd.}$$

The following proposition gives examples for equalities of reduction classes defined by (bounded) truth-table reducibilities and Hausdorff reducibilities.

Proposition 3.

1. *For every class $\mathcal{C}$ of languages,* $R_{nhd}^{p}(\mathcal{C}) = R_{hd}^{p}(R_{c}^{p}(\mathcal{C}))$.
2. [17] $R_{hd}^{p}(\mathrm{NP}) = R_{nhd}^{p}(\mathrm{NP}) = R_{tt}^{p}(\mathrm{NP})$ $(= \Theta_2^p)$.
3. [3] $R_{bhd}^{p}(R_{c}^{p}(\mathrm{SPARSE})) = R_{btt}^{p}(R_{c}^{p}(\mathrm{SPARSE}))$.

Let M be a Turing machine, z be a string and let d, s be natural numbers. We say that $z \in KS_M[d,s]$, if M on some input of length at most d outputs z using space at most s. In other words, $KS_M[d,s]$ is the set of strings whose *s-space-bounded Kolmogorov complexity* relative to M is bounded by d. Similarly, for a string y, $KS_M[d,s|y]$ is the set of all strings z for which there is a string x of length at most d such that M on input $\langle x,y \rangle$ outputs z using space at most s. Well known simulation-techniques (see [11]) show that there is a Universal Turing machine U such that for every machine M there is a constant c such that for all d, s: $KS_M[d,s] \subseteq KS_U[d+c, cs+c]$. Henceforth, we fix such a Universal Turing machine and omit the subscript. Note that there is a constant c such that for every set A and for all n, the characteristic sequence $\chi_{\le n}^{A}$ of A restricted to the set of strings up to length n is in $KS[2^{n+1}+c, 2^{cn}]$.
Next we give a definition for the class HIGH containing only sets of very high information content. (We say that a property holds "for almost every n" if it holds for all but finitely many n.)

Definition 4. A set B is in HIGH, if for every constant $c > 0$ there exists a polynomial q such that for almost every n, $\chi_{\le n}^{B} \notin KS[2^{n+1} - q(n), 2^{cn}]$.

Observe that the definition above for the class HIGH is an extension of the original one given by Book and Lutz [7] who required for q the fixed polynomial $2n$. It is easy to see that for every set A in the class EXPSPACE/poly there is a polynomial p such that for every n, $\chi_{\le n}^{A} \in KS[p(n), 2^{p(n)}]$, where EXPSPACE $=$ DSPACE$(2^{n^{O(1)}})$.
Self-reducibility is an important structural property. The types of self-reducibility we will use are the length-decreasing (as defined by Ko [9]) and the word-decreasing self-reducibility (see [5]).

Definition 5. [9] A partial order $\prec$ on Σ^* is said to be polynomially well-founded and length-related (abbr. *polynomially related*) if there is a polynomial p such that

1. for all $x, y \in \Sigma^*$: if $x \prec y$, then $|x| \leq p(|y|)$,
2. $x \prec y$ is decidable in time polynomial in $|x| + |y|$, and
3. for all $x_1, x_2, \ldots, x_k \in \Sigma^*$: if $x_1 \prec x_2 \prec \cdots \prec x_k$, then $k \leq p(|x_k|)$.

Definition 6. [9, 5] A set A is length-decreasing self-reducible, if there is some polynomial-time oracle machine M and a polynomially related ordering $\prec$ such that $A = L(M, A)$ and, on any input x, M queries the oracle only about words $q \prec x$. [3] A set A is word-decreasing self-reducible, if there is some polynomial-time oracle machine M such that $A = L(M, A)$ and, on any input x, M queries the oracle only about words lexicographically smaller than x.

A set A is called $\mathrm{FP}^{\mathrm{NP}}$-*printable*, if there exists a polynomial-time algorithm that uses an oracle from NP and computes on input 1^n (an encoding of) the finite set $A^{\leq n}$. For further definitions used in this paper we refer the reader to some standard book on structural complexity theory (as for example [6]).

3 On Hausdorff reductions of self-reducible sets to sparse sets

In this section we consider consequences of the assumption that a self-reducible set A $\leq^p_{nhd}$-reduces to a sparse set S. Exploiting the self-reducibility structure of the set A we are able to construct an $\mathrm{FP}^{\mathrm{NP}}$-printable subset of S to which A reduces. We only sketch the proof for the case of length-decreasing self-reducible sets, the word-decreasing case can be proved similarly.

Theorem 7. *If a length-decreasing self-reducible set A $\leq^p_{nhd}$-reduces to a sparse set S, then there exists an $\mathrm{FP}^{\mathrm{NP}}$-printable subset $S' \subseteq S$ such that $A \leq^p_{nhd} S'$.*

Proof Sketch. Let A be a length-decreasing self-reducible set and let S be a sparse set such that $A \leq^p_{nhd} S$ via some reduction function f. Let k be a polynomial such that f on input x produces $k(|x|)$ many queries $f(x, i)$, $1 \leq i \leq k(|x|)$, all of length $r(|x|)$ for some strictly increasing polynomial r. Let M be an oracle Turing machine and let $\prec$ be a polynomially related ordering witnessing the self-reducibility of A.
Let $j(x, B) = \max\{0 \leq j \leq k(|x|) \mid \text{for all } i = 1, \ldots, j : f(x, i) \in B\}$, i.e., $x \in A \iff j(x, S)$ is odd. Let M' be an oracle Turing machine that works as follows: M' simulates M; if M asks an oracle query then M uses its oracle (say O) to compute $j(q, O)$, and continues the simulation with answer "yes" if $j(q, O)$ is odd, and with answer "no" otherwise. It is clear that $A = L(M', S)$.
Next we show how to construct for $i = 0, 1, 2, \ldots$ a subset S_i of $S^{=r(i)}$ such that the reduction from A to S_i via f is correct for all instances in Σ^i (i.e.

[3] Observe that by part 3 of the preceding definition, $x \not\prec x$.

$x \in \Sigma^i \Rightarrow [x \in A \iff j(x, S_i)$ is odd$])$. Using the modified self-reduction machine M', the sets S_i can be constructed by a polynomial-time oracle machine using an NP-oracle. Define the *distance* (cf. [12]) between two strings y and x as the maximum length of an increasing chain from y to x w.r.t. $\prec$ (if such a chain exists): $dist(y, x) = \max\{k \mid \exists x_0, x_1, \dots, x_k : y = x_0 \prec x_1 \prec \cdots \prec x_k = x\}$. Since $\prec$ is polynomially related, there exists a polynomial p such that $dist(y, x) \leq p(|x|)$ for every pair (y, x) on wich $dist$ is defined. Clearly, $dist(x, x) = 0$. By the "inductive structure" of self-reducible sets it follows for all $x \in \Sigma^*$ and $i \in \mathbb{N}$: if for all y with $dist(y, x) > i$ it holds that $y \in L(M', Q) \iff j(y, Q)$ is odd, then for all y with $dist(y, x) \geq i$ it holds that $y \in L(M', Q) \iff y \in A$. The following function uses these properties of M', j, and A to construct on input 0^i a set Q, which is a finite subset of S to which all instances of A having length at most i are correctly reduced by f. Define $E = \{\langle y, 0^m, 0^i, Q\rangle \mid \exists x \in \Sigma^i : dist(y, x) \geq m \wedge y \in L(M', Q) \not\Leftrightarrow j(y, Q)$ is odd $\}$. Note that E is in NP.

CONSTRUCT(0^i)
  $Q := \emptyset$
  while $\exists j \leq p(i), y \in \Sigma^{\leq p(i)} : \langle y, 0^j, 0^i, Q\rangle \in E$ **do**
    $k := \max\{m \mid \exists z \in \Sigma^{\leq p(i)} : \langle z, 0^m, 0^i, Q\rangle \in E\}$
    $x := \min\{y \mid \langle y, 0^k, 0^i, Q\rangle \in E\}$ w.r.t. lexicographical order
    $Q := Q \cup \{f(x, j(x, Q) + 1)\}$
  end
  return $Q \cap \Sigma^{r(i)}$

Define $S' = \bigcup_{i \geq 0}$ CONSTRUCT(0^i). It follows that S' is sparse, that $A \leq^p_{nhd} S'$ via f, and finally that S' is $\mathrm{FP}^{\mathrm{NP}}$-printable. ∎

Theorem 8. *If a word-decreasing self-reducible set A $\leq^p_{nhd}$-reduces to a sparse set S, then there exists an $\mathrm{FP}^{\mathrm{NP}}$-printable subset $S' \subseteq S$ such that $A \leq^p_{nhd} S'$.*

Corollary 9. *If a length-decreasing or word-decreasing self-reducible set A $\leq^p_{nhd}$-reduces to a sparse set, then A is low for Δ^p_2, i.e. $\Delta^p_2(A) = \Delta^p_2$.*

Since the PSPACE-complete set QBF and the NP-complete set SAT are word-decreasing self-reducible we can state the following theorem.

Theorem 10.
1. *If* SAT *is $\leq^p_{nhd}$-reducible to a sparse set, then* $\mathrm{PH} = \Delta^p_2$.
2. *If QBF is $\leq^p_{nhd}$-reducible to a sparse set, then* $\mathrm{PSPACE} = \Delta^p_2$.

4 Reliable reducibilities

In this section, we introduce the concept of a reliable reducibility and prove the reliability of various reducibilities as for example the (non-monotone) Hausdorff reducibility, the composition of the bounded Hausdorff and the conjunctive reducibilities (henceforth called *bhd-c* reducibility), and the *co-np* as well as the *co-rp* many-one reducibilities.

Definition 11. A reducibility $\leq_\alpha$ is reliable if for all sets A, B such that $A \leq_\alpha B$ there exist a polynomial p and a sequence $B_n \subseteq B$, $n \geq 0$, of sets fulfilling the following properties:

- all instances for A up to length n are correctly reduced (via the given reduction) to B_n,
- for every $n \geq 0$, $\chi^{B_n}_{\leq n} \in KS\left[p(n), 2^{p(n)} \mid \chi^A_{\leq n}\right]$.

Theorem 12. *The composition of the Hausdorff reducibility and the conjunctive reducibility is reliable.*

Proof Sketch. Let $A \leq^p_{hd} B$ via a polynomial time computable function f, and let B conjunctively reduce to C via a polynomial time computable function g. Let $k(x)$ be the number of queries in the list $f(x)$. The following algorithm computes on input $\chi^A_{\leq n}$ the characteristic string $\chi^{C_n}_{\leq n}$ of a finite set $C_n \subseteq C$ such that A reduces to C_n via f and g for all instances up to length n. In fact, C_n is the smallest set with this property.

> **input** $\chi^A_{\leq n}$
> $C_n := \emptyset$; $x := \varepsilon$;
> **repeat**
> $l := \max(\{0\} \cup \{j \mid 1 \leq j \leq k(|x|) \text{ and } g(f(x,j)) \subseteq C_n\})$
>   **if** $\bigcup\{g(f(x,j)) \mid 1 \leq j \leq l\} \not\subseteq C_n$ **then**
> $C_n := C_n \cup \bigcup\{g(f(x,j)) \mid 1 \leq j \leq l\}$
> $x := \varepsilon$
>   **elsif** $x \in A \iff l$ is even **then**
> $C_n := C_n \cup g(f(x, l+1))$
> $x := \varepsilon$
>   **else** $x := succ(x)$ (* in lexicographical order *)
>   **end**
> **until** $x = 0^{n+1}$
> **output** $\chi^{C_n}_{\leq n}$

Claim 13.
1. *C_n is a subset of C for all $n \geq 0$.*
2. *For all $n \geq 0$, A reduces to C_n via f and g for all instances $x \in \Sigma^{\leq n}$.*

Observe that $|C_n| = 2^{O(n)}$, and therefore the algorithm can be performed in time $2^{O(n)}$. As a consequence, it is easy to see that there exists a constant c such that for every n, $\chi^{C_n}_{\leq n} \in KS\left[c, 2^{cn} \mid \chi^A_{\leq n}\right]$. ∎

Since the many-one, conjunctive, (bounded) Hausdorff, and *bhd-c* reducibilities are all special cases of the composed Hausdorff and conjunctive reducibilities, we can state the following corollary.

Corollary 14. *The many-one, conjunctive, (bounded) Hausdorff, and bhd-c reducibilities as well as the non-monotone Hausdorff reducibility are reliable.*

As shown in [4], also the *co-np* many-one reducibility, the *co-rp* many-one reducibility, and the reducibility obtained by composing the Hausdorff, the *co-rp* many-one, and the conjunctive reducibilities are reliable.

5 On reliable reductions to HIGH sets

In this section we prove that no set A in EXPSPACE/poly reduces via a reliable reduction to a set B in HIGH unless A is reducible via a reduction of the same type to a sparse set. In order to give an intuitive explanation how the proof works, consider the case that A reduces to B via a many-one reduction function f. Since the set A is of relatively low space-bounded Kolmogorov complexity, membership in B can be decided for all the instances in the range of f using only a relatively small amount of resources (assuming that f is honest). Therefore, the range of f cannot be too large since otherwise a substantial part of the characteristic sequence of B would contain only little information, contradicting the fact that B is in HIGH.

For the proof we need two lemmas that are of independent interest. Intuitively spoken, the following lemma shows that the Kolmogorov complexity of a string cannot be very high, if "many" 1's of the string are easily computable. For $b \in \{0,1\}$ let $\#_b(y)$ be the number of bits equal to b in the string y. Further, let $\preceq$ be the partial ordering on Σ^* defined by $a_1 \dots a_k \preceq b_1 \dots b_l$, if $k \le l$ and $a_i \le b_i$, for $i = 1, \dots, k$.

Lemma 15. *There exists a constant c^* such that for all $c, d \in \mathbb{N}$ and for all $x, y \in \Sigma^*$, if $x \preceq y$ and $x \in KS[d, c]$, then $y \in KS[2\log|d| + d + |y| - \#_1(x) + c^*, \max(c, c^* \cdot |y|)]$.*

The second lemma that we need states that if in the reduction of a set A to some set B the number of positively answered queries on instances of length n is polynomially bounded, then the given reduction can be modified (by padding the queries) to yield a reduction from A to a sparse set.

Lemma 16. *Assume that a set A reduces via a given reduction to some set. If there is a polynomial r such that for every $n \ge 0$ there is a set S_n of cardinality $|S_n| \le r(n)$ such that all length n instances for A are correctly reduced to S_n, then A reduces via the same type of reduction (that is, only the queries need to be padded) to a sparse set $\hat{S}$.*

Now we are ready to prove the main result of this section. It states that in order to decide a set in EXPSPACE/poly, all the information that a reliable reduction is able to extract out of a set containing a lot of information can be provided just as well by a sparse set (which is of very low information content).

Theorem 17. *If a set $A \in$ EXPSPACE/poly reduces via a reliable reduction to a set $B \in$ HIGH, then A reduces via the same type of reduction (that is, only the queries need to be padded) to a sparse set.*

Proof Sketch. Let $A \in$ EXPSPACE/poly, $B \in$ HIGH and assume that A reduces to B via a reliable reduction. Intuitively, using the reliability of the reduction and the fact that $A \in$ EXPSPACE/poly, it is possible to compute within the resource bounds provided by the complexity class EXPSPACE/poly a sequence

of sets $B_n \subseteq B$ such that for all instances up to length n, A reduces correctly to B_n. Since $B \in$ HIGH, the number of 1's in the characteristic sequences of the sets B_n cannot be large, and therefore, as we will see, A reduces to a sparse set. In the sequel we give the formal proof.
Since $A \in$ EXPSPACE/poly, there exists a polynomial s such that for every n, $\chi^A_{\leq n} \in KS[s(n), 2^{s(n)}]$. Also, since A reduces to B via a reliable reduction, there are a polynomial s' and a sequence $B_n \subseteq B$, $n \geq 0$, of sets such that for every $n \geq 0$, all instances for A up to length n are correctly reduced (via the given reduction) to B_n, and $\chi^{B_n}_{\leq n} \in KS\left[s'(n), 2^{s'(n)} \,\middle|\, \chi^A_{\leq n}\right]$. As an immediate consequence of the reliability of the reduction from A to B and the fact that $A \in$ EXPSPACE/poly we have (1) $\chi^{B_n}_{\leq n} \in KS[p(n), 2^{p(n)}]$ for some polynomial p and every $n \geq 0$. Next we show that there is a polynomial q such that (2) $census_{B_n}(1^n) \leq q(n)$ for all n. Assume otherwise, then for every polynomial q there are infinitely many n such that $census_{B_n}(1^n) > q(n)$. Since $B_n \subseteq B$ it follows by (1) above and by Lemma 15 (letting $x = \chi^{B_n}_{\leq n}$ and $y = \chi^B_{\leq p(n)}$) that there is a constant c^* such that for every polynomial q there are infinitely many n for which $\chi^B_{\leq p(n)} \in KS[2\log|p(n)| + p(n) + 2^{p(n)} - q(n) + c^*, c^* \cdot 2^{p(n)}]$, contradicting the fact that $B \in$ HIGH.
To complete the proof of the theorem let r be a (non-decreasing) polynomial bounding the length of the queries of the given reliable reduction from A to B. Then for all length n instances, A reduces correctly to the set $B^{\leq r(n)}_{r(n)}$. By (2) above it follows that $census_{B_{r(n)}}(1^{r(n)}) \leq q(r(n))$ for all n, and thus the theorem follows by Lemma 16. ■

As a consequence of the preceding theorem and the reliability results of Section 4 we obtain the following theorem.

Theorem 18. *Let A be in* EXPSPACE/poly *and let α be one of the following reducibility types:*

> *non-monotone Hausdorff reducibility, bhd-c reducibility, co-np many-one reducibility, or the composition of the bhd, co-rp many-one, and conjunctive reducibilities.*

Then A is in R_α(HIGH) if and only if A is in R_α(SPARSE).

By Theorem 17 we know that the existence of hard sets in HIGH for any complexity class $\mathcal{C} \subseteq$ EXPSPACE/poly with respect to a reliable reducibility implies the existence of a sparse hard set for $\mathcal{C}$ with respect to that reducibility. Thus the existence of hard sets in HIGH leads to the same consequences as the existence of sparse hard sets. Using Theorems 10 and 18, and results of [2] we get

Corollary 19.

1. *Let $\mathcal{C}$ be a class from $\{\mathrm{UP}, \mathrm{NP}, \mathrm{C_{=}P}, \mathrm{PP}\}$. If $\mathcal{C} \subseteq R^p_{bhd}(R^p_c(\mathrm{HIGH}))$, then $\mathcal{C} = \mathrm{P}$.*
2. *For $\mathcal{C} \in \{\mathrm{NP}, \mathrm{PSPACE}\}$, if $\mathcal{C} \subseteq R^p_{nhd}(\mathrm{HIGH})$, then $\mathcal{C}$ is low for Δ^p_2.*
3. *If NP is contained in $R^p_{bhd}(R^{co\text{-}rp}_m(R^p_c(\mathrm{HIGH})))$, then $\mathrm{NP} = \mathrm{RP}$.*

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