Dynamic Programming and Reinforcement Learning

Finite Time Markov Decision Processes (Week 2a)

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Outline

- Questions?

- Today: Getting familiar with MDPs

  Finite Horizon Problems

  Dynamic Programming

  Bellman Equation & Value Function

Goals for Today

- Understand: States, Actions, Rewards, State Transitions
- Learn: Finite Horizon MDPs
- Learn: Concept of Expected Future Rewards
- Evaluate: Policies and their Performance
- Learn: Solve Problems using Dynamic Programming (Backward Induction)

What are Finite Horizon MDP Problems?

- We seek to control a dynamic system over a finite time
- We consider a finite *sequence* of decisions
- The system evolves according to a certain (time-dependent) dynamic
- The decisions are supposed to be chosen such that a certain objective is optimized (expected rewards)
- Find the right balance between short and long-term effects

Example Decision Problems with Finite Horizon

Examples with Finite Horizon

- Selling Airline Tickets
- Drinking at a Party
- Exam Preparation
- Eating Cake
- Selling Christmas Trees
- Accommodation Services
- Perishable Products, Fashion, etc.

Task: Describe & Classify

- Goal/Objective
- State of the System
- Actions
- Dynamic of the System
- Revenues/Costs
- Finite/Infinite Horizon
- Stochastic Components

Classification (Finite Horizon Problems)

Example	**Objective**	**State**	**Action**	**Dynamic**	**Payments**
Airline Tickets	**max revenue**	**#tickets**	**price**	**tickets sold**	**sales rewards**
Drinking at Party	max fun	‰	#beer	impact-rehab	fun/money
Exam Preparation	max mark/effort	#learned	#learn	learn-forget	effort, mark
Eating Cake	max utility	%cake	#eat	outflow	utility
Christmas Trees	. . .				
Accommodation	. . .				
Fashion Items	. . .				

General Problem Components (Finite Horizon)

• What do you want to optimize (e.g., expected rewards)		(Objective)
• Define the state of your system		(State)
• Define the set of possible actions (time+state dependent)		(Actions)
• Quantify event probabilities	(time+state+action dep.)	(Dynamics)
• Define rewards	(time+state+action+event dep.)	(Rewards)
• Define state transitions	(time+state+action+event dep.)	(Transitions)
• What happens at the end of the time horizon? (state dep.)		(Final Rewards)

Solving Finite Horizon MDPs via Dynamic Programming

- **Continuous** Time Problems & Control Theory (not in focus)
  - – Hamilton-Jacobi-Bellman equation
  - – Solve (partial) differential equations
  - – Allows for structural and analytical results
  - – See also: Pontryagin's maximum principle / Hamiltonian
  - – See also: Differential games & stochastic extensions

- **Discrete** Time Problems (our focus)
  - – Bellman equation, backward induction
  - – Optimal numerical solutions

Basic Notation (Discrete Time Models)

- Framework: $t = 0,1,2,...,T$ Discrete time periods
- State: $s_t \in S$ One- or multi-dimensional
- Actions: $a_t \in A$ One- or multi-dimensional
- Events: $i_t \in I$, $P_t(i,a,s)$ Probability of event i in $(t,t+1)$ under a in s
- Rewards: $r_t = r_t(i,a,s)$ Realized reward in $(t,t+1)$ for i, a in s
- New State: $s_t \rightarrow s_{t+1} = \Gamma_t(i,a,s)$ State transition (forr i, a in s)
- Initial State: $s_0 \in S$ State in t=0

Sequence of Events (Finite Horizon)

t=0 start in state s_0 at the beginning of period (0,1)

choose/play action a_0 for period (0,1)

observe realized reward r_0 of period (0,1)

t=1 observe realized new state s_1 after period (0,1) / the beginning of period (1,2)

choose/play action a_1 for period (1,2)

observe realized reward r_1 of period (1,2)

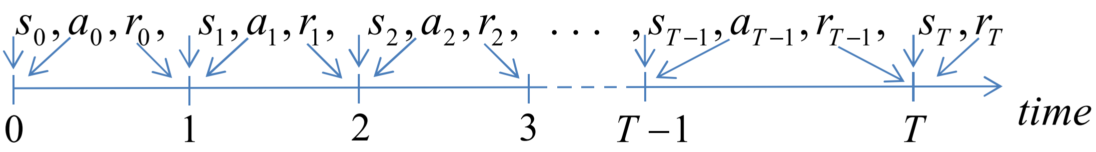
. . .

t=T observe realized new state s_T after period (T-1,T) / end of time horizon

no further action

observe realized terminal reward $r_T(s_T)$ (cf. state-dependent salvage value)

=>



Markov Property

- The distribution of rewards and state transitions do only depend on the current state and action – not the history to get there.

- This can formally be expressed as:

$$P\left(s_{t+1} \middle| s_0, \ldots, s_t, a_0, \ldots, a_t\right) = P\left(s_{t+1} \middle| s_t, a_t\right) \quad \forall t = 0, 1, \ldots, T-1$$

$$P\left(r_{t+1} \middle| s_0, \ldots, s_t, a_0, \ldots, a_t\right) = P\left(r_{t+1} \middle| s_t, a_t\right) \quad \forall t = 0, 1, \ldots, T-1$$

Policies

- A *non-anticipating* (Markov) *policy* $\pi_t(s_t)$ determines an action a_t to be chosen/played in state s_t at time *t*, for all $s_t \in S,\ \ t = 0,...,T-1$.

- A policy is usually **deterministic**, i.e., a unique action is chosen.

- A policy can also be *randomized* (mixed), i.e., an action is chosen according to a certain probability distribution within the set A.

- Note, already the number of potential det. policies is **large**, cf. $|A|^{T\cdot|S|}$!

Check: Could You Simulate Test Examples?

- Finite Horizon Problems

  – Eating cake (e.g., with deterministic utility)

  – **Selling Airline Tickets (stochastic demand)**

- Describe: Horizon, states, actions, dynamics, rewards, transitions
- Evaluate rewards and state realizations of a certain (Markov) policy (which contains "*what to do in which state at which point in time*")
- Simulate multiple runs (and policies) & evaluate **average total rewards**

Example Problem (Selling Airline Tickets)

Example Problem (Selling Airline Tickets)

- Problem context: Sell items (N tickets) over time
- Time Horizon: Finite (T)
- Action: Offer price (p)
- Demand: Stochastic (Price and Time-dependent)
- Rewards: Sales revenues (r) & final rewards (salvage value)
- Goal: Maximize expected total profits

- How to find an optimal pricing policy?

Example MDP (Selling Airline Tickets)

- Framework: $t = 0,1,2,...,T$ Time periods
- State: $s_t \in S := \{0,1,...,N\}$ Items left
- Actions: $a_t \in A := \{5,10,...,400\}$ Price
- Events: $i_t \in I := \{0,1\}$ with probabilities Demand

  $P_t(1,a,s) := (1-a/400)\cdot(1+t)/T$ $P_t(0,a,s) = 1 - P_t(1,a,s)$
- Rewards: $r_t = r(i,a,s) := a \cdot \min(i,s)$ Revenue
- New state: $s_t \rightarrow s_{t+1} = \Gamma(i_t, a_t, s_t) := \max(0, s_t - i_t)$ Old – sold
- Initial state: $s_0 \in S$ Initial items N
- Final reward: $r_T(s) := f \cdot s$ with $f = 10$ Weight for freight

Simulation of a Given Policy

- Assume (dynamic) pricing strategy $\pi_t(s) = 250$ (e.g., static price)
- Parameters: $T = 200, \;\; N = 50, \;\; f = 10$

time	state	action	sales	reward (revenue)	accum. revenue	new state
0	50	250	0	0	0	50
1	50	250	1	250	250	49
2	49	250	0	0	250	49
3	49	250	1	250	500	48
…						
200	5	/	/	50	11,300	/

- Expected performance of $\pi(s)$?
- What is the best possible performance? What is an optimal policy??

Problem Formulation (Finite Horizon)

- Find a *Markov policy* $\pi = \pi_t(s)$ that maximizes

  the **total expected future rewards**, i.e.,

$$\max_{\pi} E\left[\sum_{t=0}^{T}\left(\sum_{i_t \in I_t} \underbrace{P_t(i_t, a_t, s_t)}_{\substack{\text{probability for event } i_t \\ \text{under action } a_t \text{ in state } s_t}} \cdot \underbrace{r_t(i_t, a_t, s_t)}_{\substack{\text{reward for event } i_t \\ \text{under action } a_t \text{ in state } s_t}}\right) \middle| \underbrace{s_0}_{\text{initial state}}\right],$$

  where states evolve according to $s_t \rightarrow s_{t+1} = \Gamma_t(i_t, a_t, s_t)$

- How to solve such problems?   Answer: Dynamic Programming

Expected Future Rewards (Finite Horizon)

- Assume a given *policy* $\pi = \pi_t(s_t)$
- *Random* reward stream: $r_0, r_1, r_2, r_3, r_4, \ldots, r_{T-1}, r_T$ (finite horizon)
- Expected future rewards . . .

. . . from time t=0 on: $$V_0^{(\pi)}(s) = E\left(\sum_{t=0,\ldots,T} r_t \,\middle|\, s_0 = s, a_t = \pi_t(s_t) \right)$$

. . . from time t=*3* on:

Expected Future Rewards (Finite Horizon)

- Assume a given *policy* $\pi = \pi_t(s_t)$
- *Random* reward stream: $r_0, r_1, r_2, r_3, r_4, \ldots, r_{T-1}, r_T$ (finite horizon)
- Expected future rewards . . .

. . . from time *t*=0 on: $$V_0^{(\pi)}(s) = E\left(\sum_{t=0,\ldots,T} r_t \,\middle|\, s_0 = s, a_t = \pi_t(s_t) \right)$$

. . . from time *t=3* on: $$V_3^{(\pi)}(s) = E\left(\sum_{t=3,\ldots,T} r_t \,\middle|\, s_3 = s, a_t = \pi_t(s_t) \right)$$

- $V_t^{(\pi)}(s)$ describes "*the **value** of being in a certain state s at time t*"

for a given policy π , $s \in S,\ t = 0,\ldots,T$.

Recursion for Expected Future Rewards (Finite T)

- *Random* reward stream: $r_0, r_1, r_2, r_3, r_4, \ldots, r_{T-1}, r_T$ (finite horizon)
- Recursion for expected future rewards from time *t* on, $s \in S$:

$$V_t^{(\pi)}(s) = E\left(\sum_{k=t,\ldots,T} r_k \,\middle|\, s_t = s, \pi \right) = E\left(r_t + \sum_{k=t+1,\ldots,T} r_k \,\middle|\, s_t = s, \pi \right)$$

$$= E\left(r_t + \underbrace{E\left(\sum_{k=t+1,\ldots,T} r_k \,\middle|\, s_{t+1} = s', \pi \right)}_{?} \,\middle|\, s_t = s, \pi \right)$$

Recursion for Expected Future Rewards (Finite T)

- *Random* reward stream: $r_0, r_1, r_2, r_3, r_4, \ldots, r_{T-1}, r_T$ (finite horizon)
- Recursion for expected future rewards from time t on, $s \in S$:

$$V_t^{(\pi)}(s) = E\left(\sum_{k=t,\ldots,T} r_k \middle| s_t = s, \pi \right) = E\left(r_t + \sum_{k=t+1,\ldots,T} r_k \middle| s_t = s, \pi \right)$$

$$= E\left(r_t + \underbrace{E\left(\sum_{k=t+1,\ldots,T} r_k \middle| s_{t+1} = s', \pi \right)}_{?} \middle| s_t = s, \pi \right)$$

$$= E\left(r_t + V_{t+1}^{(\pi)}(s_{t+1}) \middle| s_t = s, \pi \right)$$ sum of rewards now + from t+1 on

Solution Approach (Dynamic Programming)

- What is the **best expected value** of having the chance to . . .

  "sell items from time t on being in state s"?

- Answer: That's easy $V_t(s)$! ?????

Solution Approach (Dynamic Programming)

- What is the **best expected value** of having the chance to . . .

  "sell items from time t on being in state s"?

- Answer: That's easy $V_t(s)$! ?????

- We have renamed the problem. Awesome. But - that's a solution approach!

- We don't know the "Value Function *V*", but *V* has to satisfy the relation:

  Value (state today) = Best expected (profit today + Value (state tomorrow))

Solution Approach (Dynamic Programming)

- *Value (state today) = Best expected (profit today + Value (state tomorrow))*
- Idea: Consider potential events & transitions within one period.

  What can happen during one time interval (under action a)?

state in t	event	reward	state in t+1	probability
	0	$r(0,a,s)$	$\Gamma(0,a,s)$	$P_t(0,a,s)$
s	1	$r(1,a,s)$	$\Gamma(1,a,s)$	$P_t(1,a,s)$
	2	$r(2,a,s)$	$\Gamma(2,a,s)$	$P_t(2,a,s)$

- What does that mean for the **value** of state s at time t, i.e., $V_t(s)$?

Balancing Potential Short- and Long-Term Rewards

state in t	event	reward	state in t+1	probability
	0	$r(0,a,s)$	$\Gamma(0,a,s)$	$P_t(0,a,s)$
s	1	$r(1,a,s)$	$\Gamma(1,a,s)$	$P_t(1,a,s)$
	2	$r(2,a,s)$	$\Gamma(2,a,s)$	$P_t(2,a,s)$

$$V_t(s) = \underbrace{\max_{a\in A}}_{\substack{potential\\actions}} \left\{ \underbrace{P_t(0,a,s)}_{probability\ not\ to\ sell\ in\ (t,t+1)} \cdot \left(\underbrace{r(0,a,s)}_{periods's\ reward} + \underbrace{\gamma \cdot V_{t+1}\left(\Gamma(0,a,s)\right)}_{best\ disc.\ exp.future\ rewards} \right) \right.$$

$$\left. + \underbrace{P_t(1,a,s)}_{probability\ of\ 1\ sale\ in\ (t,t+1)} \cdot \left(\underbrace{r(1,a,s)}_{period's\ reward} + \underbrace{\gamma \cdot V_{t+1}\left(\Gamma(1,a,s)\right)}_{best\ disc.\ exp.future\ rewards} \right) + \ldots \right\}$$

Bellman Equation (Finite Horizon)

- We obtain the *Bellman Equation*, which **determines** the Value Function:

$$V_t(s) = \underbrace{\max_{a \in A}}_{\substack{potential\\actions}} \left\{ \sum_{i \in I} \underbrace{P_t(i,a,s)}_{probability} \cdot \left(\underbrace{r(i,a,s)}_{today's\ reward} + \underbrace{\gamma \cdot V_{t+1}\big(\Gamma(i,a,s)\big)}_{best\ disc.\ exp.future\ rewards\ of\ new\ state} \right) \right\}$$

- Ok, but why is that interesting?

Value Function & Optimal Policy

- We obtain the *Bellman Equation*, which **determines** the Value Function:

$$V_t(s) = \underbrace{\max_{a\in A}}_{\substack{potential\\ actions}} \left\{ \sum_{i\in I} \underbrace{P_t(i,a,s)}_{probability} \cdot \left(\underbrace{r(i,a,s)}_{today's\ reward} + \underbrace{\gamma \cdot V_{t+1}\big(\Gamma(i,a,s)\big)}_{best\ disc.\ exp. future\ rewards\ of\ new\ state} \right) \right\}$$

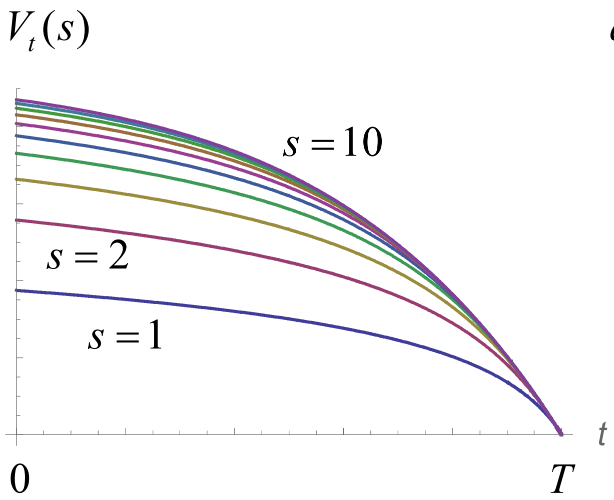
- Ok, but why is that interesting?

- Answer: Because $a_t^*(s) = \arg\max_{a\in A} \{...\}$ is the *optimal policy.*

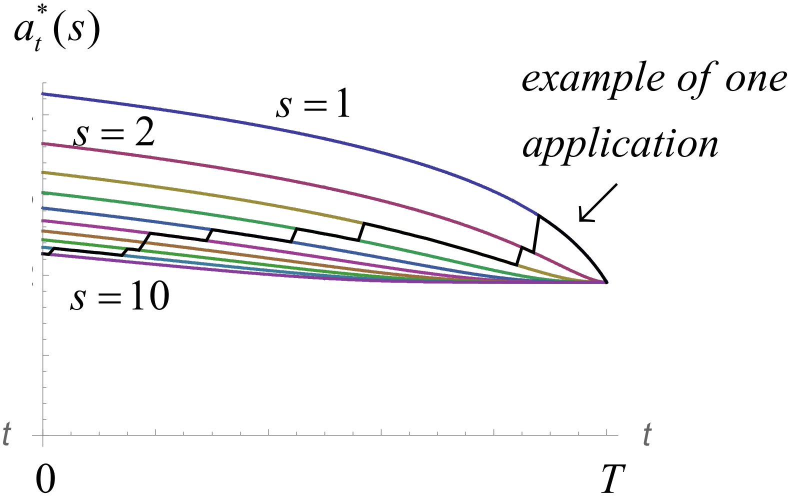
- Ok! Now, we just need to compute the Value Function!

Value Function & Optimal Policy (Illustration)

Value Function



Pricing Policy



Backward Induction for Discrete Finite Horizon MDPs

- Starting with the **terminal condition** $V_T(s) := r_T(s)$ at the horizon T

  we can compute the value function *recursively* $\forall s \in S_t, t = 0,1,...,T-1$:

$$V_t(s) = \underbrace{\max_{a \in A_t(s)}}_{\substack{potential\\actions}} \left\{ \sum_{i \in I_t} \underbrace{P_t(i,a,s)}_{probability} \cdot \left(\underbrace{r_t(i,a,s)}_{today's\ reward} + \underbrace{\gamma \cdot V_{t+1}\left(\Gamma_t(i,a,s)\right)}_{best\ disc.\ exp.future\ rewards\ of\ new\ state} \right) \right\}$$

- The optimal strategy $a_t^*(s)$, $t = 0,1,...,T-1$, $s \in S$,

  is determined by the *arg max* of the value function $V_t(s)$

- The approach is general applicable & *optimal* for finite horizon problems

  The numerical complexity increases with T, $|S|$, $|A|$, and $|I|$

Backward Induction Tabular Schema (Airline Example)

time / periods

	0				$T-1$	T
$s=4$						$V_T(4)=r_T(4)=0$
$s=3$						$V_T(3)=r_T(3)=0$
$s=2$						$V_T(2)=r_T(2)=0$
$s=1$						$V_T(1)=r_T(1)=0$
$s=0$						$V_T(0)=r_T(0)=0$

$(f=0)$

all states $s \in S$

Backward Induction Tabular Schema (Airline Example)

time / periods

all states $s \in S$

	0				$T-1$	T
$s=4$					$P_{T-1}(0,a,4)$ $P_{T-1}(1,a,4)$	$V_T(4)=r_T(4)=0$
$s=3$		$P_{T-1}(0,a,4)\cdot\left(r_{T-1}(0,a,4)+\gamma\cdot V_T(4)\right)$ $+P_{T-1}(1,a,4)\cdot\left(r_{T-1}(1,a,4)+\gamma\cdot V_T(3)\right)$				$V_T(3)=r_T(3)=0$
$s=2$						$V_T(2)=r_T(2)=0$
$s=1$						$V_T(1)=r_T(1)=0$
$s=0$						$V_T(0)=r_T(0)=0$

Backward Induction Tabular Schema (Airline Example)

time / periods

all states $s \in S$

	0				$T-1$	T
$s=4$					$P_{T-1}(0,a^*,4)$ $P_{T-1}(1,a^*,4)$	$V_T(4)=r_T(4)=0$
$s=3$						$V_T(3)=r_T(3)=0$
$s=2$						$V_T(2)=r_T(2)=0$
$s=1$						$V_T(1)=r_T(1)=0$
$s=0$						$V_T(0)=r_T(0)=0$

Backward Induction Tabular Schema (Airline Example)

$time\,/\,periods$

all states $s \in S$

	0				$T-1$	T
$s=4$					$V_{T-1}(4)$, $a^*_{T-1}(4)$	$V_T(4)=r_T(4)=0$
$s=3$						$V_T(3)=r_T(3)=0$
$s=2$						$V_T(2)=r_T(2)=0$
$s=1$						$V_T(1)=r_T(1)=0$
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Backward Induction Tabular Schema (Airline Example)

$time\,/\,periods$

all states $s \in S$

	0				$T-1$	T
$s=4$					$V_{T-1}(4)$, $a^*_{T-1}(4)$	$V_T(4)=r_T(4)=0$
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$s=2$						$V_T(2)=r_T(2)=0$
$s=1$						$V_T(1)=r_T(1)=0$
$s=0$						$V_T(0)=r_T(0)=0$

Backward Induction Tabular Schema (Airline Example)

$time\,/\,periods$

all $states$ $s \in S$	0				$T-1$	T
$s=4$					$V_{T-1}(4),$ $a^*_{T-1}(4)$	$V_T(4)=r_T(4)=0$
$s=3$					$V_{T-1}(3),$ $a^*_{T-1}(3)$	$V_T(3)=r_T(3)=0$
$s=2$						$V_T(2)=r_T(2)=0$
$s=1$						$V_T(1)=r_T(1)=0$
$s=0$						$V_T(0)=r_T(0)=0$

Backward Induction Tabular Schema (Airline Example)

time / periods

all states $s \in S$

	0			$T-2$	$T-1$	T
$s=4$					$V_{T-1}(4)$, $a^*_{T-1}(4)$	$V_T(4)=r_T(4)=0$
$s=3$					$V_{T-1}(3)$, $a^*_{T-1}(3)$	$V_T(3)=r_T(3)=0$
$s=2$					$V_{T-1}(2)$, $a^*_{T-1}(2)$	$V_T(2)=r_T(2)=0$
$s=1$					$V_{T-1}(1)$, $a^*_{T-1}(1)$	$V_T(1)=r_T(1)=0$
$s=0$					$V_{T-1}(0)$, $a^*_{T-1}(0)$	$V_T(0)=r_T(0)=0$

Backward Induction Tabular Schema (Airline Example)

time / periods

all states $s \in S$

	0	1		$T-2$	$T-1$	T
$s=4$			$\cdots$	$V_{T-2}(4)$, $a^*_{T-2}(4)$	$V_{T-1}(4)$, $a^*_{T-1}(4)$	$V_T(4)=r_T(4)=0$
$s=3$			$\cdots$	$V_{T-2}(3)$, $a^*_{T-2}(3)$	$V_{T-1}(3)$, $a^*_{T-1}(3)$	$V_T(3)=r_T(3)=0$
$s=2$			$\cdots$	$V_{T-2}(2)$, $a^*_{T-2}(2)$	$V_{T-1}(2)$, $a^*_{T-1}(2)$	$V_T(2)=r_T(2)=0$
$s=1$			$\cdots$	$V_{T-2}(1)$, $a^*_{T-2}(1)$	$V_{T-1}(1)$, $a^*_{T-1}(1)$	$V_T(1)=r_T(1)=0$
$s=0$			$\cdots$	$V_{T-2}(0)$, $a^*_{T-2}(0)$	$V_{T-1}(0)$, $a^*_{T-1}(0)$	$V_T(0)=r_T(0)=0$

Backward Induction Tabular Schema (Airline Example)

time / periods

all states $s \in S$	0	1		$T-2$	$T-1$	T
$s=4$	$V_0(4)$, $a_0^*(4)$	$V_1(4)$, $a_1^*(4)$	$\cdots$	$V_{T-2}(4)$, $a_{T-2}^*(4)$	$V_{T-1}(4)$, $a_{T-1}^*(4)$	$V_T(4)=r_T(4)=0$
$s=3$	$V_0(3)$, $a_0^*(3)$	$V_1(3)$, $a_1^*(3)$	$\cdots$	$V_{T-2}(3)$, $a_{T-2}^*(3)$	$V_{T-1}(3)$, $a_{T-1}^*(3)$	$V_T(3)=r_T(3)=0$
$s=2$	$V_0(2)$, $a_0^*(2)$	$V_1(2)$, $a_1^*(2)$	$\cdots$	$V_{T-2}(2)$, $a_{T-2}^*(2)$	$V_{T-1}(2)$, $a_{T-1}^*(2)$	$V_T(2)=r_T(2)=0$
$s=1$	$V_0(1)$, $a_0^*(1)$	$V_1(1)$, $a_1^*(1)$	$\cdots$	$V_{T-2}(1)$, $a_{T-2}^*(1)$	$V_{T-1}(1)$, $a_{T-1}^*(1)$	$V_T(1)=r_T(1)=0$
$s=0$	$V_0(0)$, $a_0^*(0)$	$V_1(0)$, $a_1^*(0)$	$\cdots$	$V_{T-2}(0)$, $a_{T-2}^*(0)$	$V_{T-1}(0)$, $a_{T-1}^*(0)$	$V_T(0)=r_T(0)=0$

Summary (Solving Discrete Time Finite Horizon MDPs)

Backward Induction

(+) provides optimal solutions for finite horizon MDPs

(+) allows for time-dependent frameworks

(+) general applicable

(+) numerically simple, no solver needed

(–) full information required (cf. event & transition probabilities)

(–) only for medium size state spaces,
does not scale (curse of dimensionality)

Recall - Questions?

- Markov Policies
- Recursive Concept for Future Rewards
- The Value of "being in a certain state"
- Bellman Equation & Recursive Problem Decomposition
- Backward Induction Solution Approach

Overview

Week	Dates	Topic	
1	April 21	Introduction	
2	April 25/28	Finite + **Infinite Time** MDPs	
3	May 2/5	Dynamic Programming (DP) Exercise	
4	May 9/12	Approximate Dynamic Programming (ADP) + Q-Learning (QL)	
5	May 16/19	Deep Q-Networks (DQN)	
6	May 23	DQN Extensions	(Thu May 26 “Himmelfahrt”)
7	May 30/June 2	Policy Gradient Algorithms	
8	June 9	Project Assignments	(Mon June 6 “Pfingstmontag”)
9	June 13/16	Work on Projects: Input/Support	
10	June 20/23	Work on Projects: Input/Support	
11	June 27/30	Work on Projects: Input/Support	
12	July 4/7	Work on Projects: Input/Support	
13	July 11/14	Work on Projects: Input/Support	
14	July 18/21	Final Presentations	
	Sep 15	Finish Documentation	

Exercise (Bonus)   Dynamic Programming / Backward Induction

We want to sell event tickets using Dynamic Programming. We seek to find a pricing policy that optimizes expected profits. We have *N*=50 tickets and *T*=200 periods of time. Tickets cannot be sold after *T*. We do not use a discount factor and there is no salvage value for unsold items. We consider the following demand probabilities, i.e., $P_t(1,a):=(1-a/400)\cdot(1+t)/T$ and $P_t(0,a):=1-P_t(1,a)$, $a\in A:=\{5,10,...,400\}$, $t=0,1,...,T-1$.

(a) Formulate a general model to sell tickets under given *N*, *T*, and demand probabilities P_t.

(b) Solve the given example and output the solution in an appropriate way.

(c) Simulate 1000 runs of applying the optimal policy over *T* periods. Show the distribution of realized total profits of these 1000 runs. Compare the mean with the value function.