Dynamic Programming and Reinforcement Learning

Infinite Time Markov Decision Processes (Week 2b)

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Outline

- Questions?

- Today: Infinite Horizon Problems

  Problem Examples

  Value Iteration

  Policy Iteration

Recap: Last Week

- Markov Policies in Finite Horizon MDPs
- Recursive Concept for Future Rewards
- The Value of “being in a certain state”
- Bellman Equation & Recursive Problem Decomposition
- Backward Induction Solution Approach

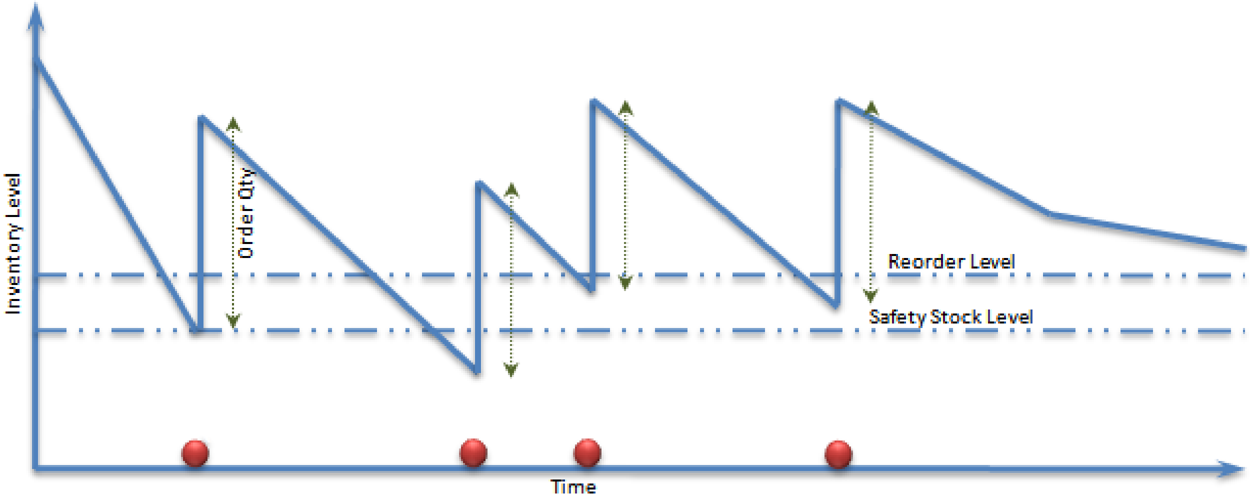
Solving MDP Problems

- Continuous Time Problems & Control Theory (not in focus)
- Discrete Time MDP Problems with **Finite Horizon** (last meeting)
  – *Time-dependent* Framework, Terminal Condition/Reward
  – Bellman Equation
  – Optimal numerical solutions via Backward induction
- Discrete Time MDP Problems with **Infinite Horizon** (today)
  – *Time-independent* Framework, Bellman Equation
  – Optimal numerical solutions via Value & Policy Iteration
  – Basis for **Reinforcement Learning**

Classification (Infinite Horizon Problems)

Example	Objective	State	Action	Dynamic	Payments
Inventory Mgmt.	min costs	#items	#order	order/holding	entry-sales
Advertising	max profits	image	#advertise	campaigns	effect-forget
Used Cars	min costs	age	replace(y/n)	buy/repair	aging/faults
Agriculture/Forestry					
Durable Products					
Chess/Go/Tetris					
Circular Economy					

Example Problem (Inventory Management)



Example Problem (Inventory Management)

- Problem context: Sell and order items over time
- Time Horizon: Infinite
- Demand: Stochastic (where price is fixed)
- Action: Replenish your inventory from time to time
- Rewards: Order cost vs. inventory holding costs
- Goal: Maximize expected discounted future profits

- How to find an optimal order policy?

Example MDP (Inventory Management)

• Framework:	$t = 0, 1, 2, \ldots, \infty$	Discrete time periods
• State:	$s \in S$	Number of items left
• Actions:	$a \in A$	Number of ordered items (replenish)
• Events:	$i \in I$, $P(i, a, s)$	Demand i (e.g., 0,1,2,3 with prob. 1/4 each)
• Rewards:	$r = r(i, a, s)$	Revenue – Order Cost – Holding Cost
	$:= p \cdot \min(i, s) - c \cdot a$	e.g., for given price p, variable order cost c,
	$-h \cdot s - 1_{\{a>0\}} \cdot f$	holding h, and fixed order costs f
• New State:	$s \to s' = \Gamma(i, a, s)$	Old – Sold + Replenish (end of period)
• Initial State:	$s_0 \in S$	Initial number of items in t=0

Sequence of Events (Infinite Horizon)

t=0 start in state s_0 at the beginning of period (0,1)

choose/play action a_0 for period (0,1)

observe realized reward r_0 of period (0,1)

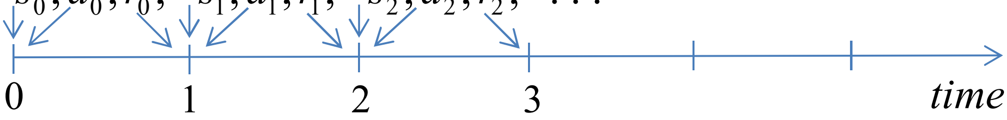
t=1 observe realized new state s_1 after period (0,1) / the beginning of period (1,2)

choose/play action a_1 for period (1,2)

observe realized reward r_1 of period (1,2)

. . .

=> $s_0, a_0, r_0, \quad s_1, a_1, r_1, \quad s_2, a_2, r_2, \quad \ldots$



Simulation of a Given Policy

- Assume a (stationary/time-indep.) order strategy: $\pi(s) = if\ s < 5\ then\ 12\ else\ 0$
- Parameters: $p = 10, c = 2, h = 0.5, f = 20$ Demand: $P(i) = 1/4,\ i = 0,1,2,3$

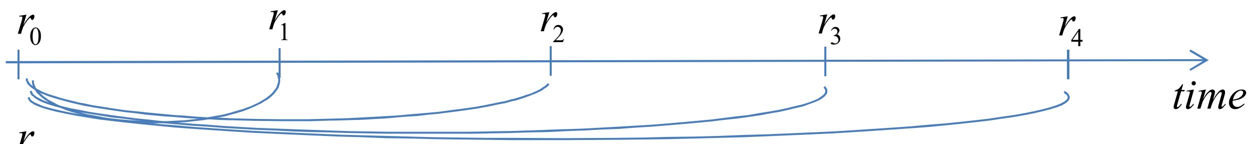
period	state	action	demand (event)	sales revenue	order cost	holding costs	reward	new state
0	5	0	1	10	0	-2.5	7.5	4
1	4	12	0	0	-24-20	-2	-46	16
2	16	0	2	20	0	-8	12	14
3	14	0	1	10	0	-7	-7	13
...								

- What is the tong-term performance of $\pi(s)$?

Discounting

- **Idea**: If delayed rewards are worth less, we use a penalty factor $\gamma < 1$ for each period to measure the value they have for us "now"

value of future rewards when being in $t = 0$?



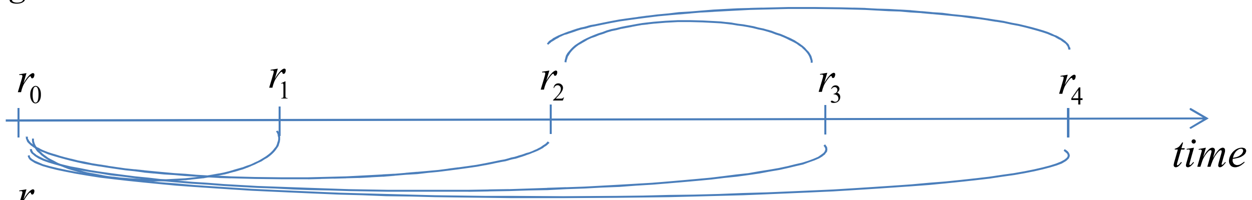
$$
\begin{aligned}
&r_0 \\
&+\gamma \cdot r_1 \\
&+\gamma^2 \cdot r_2 \\
&+\gamma^3 \cdot r_3 \\
&+\gamma^4 \cdot r_4
\end{aligned}
$$

Discounting

- **Idea**: If delayed rewards are worth less, we use a penalty factor $\gamma < 1$ for each period to measure the value they have for us "now"

value of future rewards when being in $t = 0$?

value of future rewards when being in $t = 2$?



$$r_0 + \gamma \cdot r_1 + \gamma^2 \cdot r_2 + \gamma^3 \cdot r_3 + \gamma^4 \cdot r_4$$

$$r_2 + \gamma \cdot r_3 + \gamma^2 \cdot r_4$$

Expected Discounted Future Rewards (Infinite Horizon)

- *Random* discounted reward stream: $r_0, \gamma \cdot r_1, \gamma^2 \cdot r_2, \gamma^3 \cdot r_3, \ldots$ with $\gamma \in [0,1)$

- Exp. disc. future rewards from time t on **(discounted on *current* time *t*)**:

$$V_t^{(\pi)}(s) = E\left(\sum_{k=t}^{\infty} \gamma^{k-t} \cdot r_k \,\middle|\, s_t = s, \pi \right)$$

- Exp. disc. future rewards from time t on (discounted on t=0):

$$\tilde{V}_t^{(\pi)}(s) = \gamma^t \cdot V_t^{(\pi)}(s)$$

Recursion for Future Rewards (Infinite Horizon)

- *Random* discounted reward stream: $r_0, \gamma \cdot r_1, \gamma^2 \cdot r_2, \gamma^3 \cdot r_3, \ldots$ with $\gamma \in [0,1)$
- **Recursion** for expected future rewards from time *t* on, $s \in S$:

$$V_t^{(\pi)}(s) = E\left(\sum_{k=t}^{\infty} \gamma^{k-t} \cdot r_k \middle| s_t = s, \pi \right) = E\left(\gamma^{t-t} \cdot r_t + \sum_{k=t+1}^{\infty} \gamma^{k-t} \cdot r_k \middle| s_t = s, \pi \right)$$

$$= E\left(r_t + \gamma \cdot \sum_{k=t+1}^{\infty} \gamma^{k-t-1} \cdot r_k \middle| s_t = s, \pi \right)$$

rewards now + from t+1 on?

Recursion for Future Rewards (Infinite Horizon)

- *Random* discounted reward stream: $r_0, \gamma \cdot r_1, \gamma^2 \cdot r_2, \gamma^3 \cdot r_3, \ldots$ with $\gamma \in [0,1)$
- **Recursion** for expected future rewards from time *t* on, $s \in S$:

$$V_t^{(\pi)}(s) = E\left(\sum_{k=t}^{\infty} \gamma^{k-t} \cdot r_k \,\middle|\, s_t = s, \pi \right) = E\left(\gamma^{t-t} \cdot r_t + \sum_{k=t+1}^{\infty} \gamma^{k-t} \cdot r_k \,\middle|\, s_t = s, \pi \right)$$

$$= E\left(r_t + \gamma \cdot \sum_{k=t+1}^{\infty} \gamma^{k-t-1} \cdot r_k \,\middle|\, s_t = s, \pi \right)$$

rewards now + from t+1 on?

$$= E\left(r_t + \gamma \cdot \underbrace{E\left(\sum_{k=t+1}^{\infty} \gamma^{k-(t+1)} \cdot r_k \,\middle|\, s_{t+1} = s' \right)}_{=V_{t+1}^{(\pi)}(s')} \,\middle|\, s_t = s, \pi \right)$$

yes :-)

Time Independence of Future Rewards (Infinite Horizon)

- Assume a given *policy* $\pi_t(s_t) = \pi(s_t)$, **which does not depend on time**
- Random discounted reward stream: $r_0, \gamma \cdot r_1, \gamma^2 \cdot r_2, \gamma^3 \cdot r_3, \ldots$ with $\gamma \in [0,1)$
- Expected future rewards from time *t* on (discounted on *t*), $s \in S$:

$$V_0^{(\pi)}(s) = E\left(\sum_{k=0}^{\infty} \gamma^k \cdot r_k \,\middle|\, s_0 = s, a_k = \pi(s_k) \right)$$

$$V_t^{(\pi)}(s) = V_0^{(\pi)}(s)? \quad \text{for all } t = 0,1,2,\ldots \quad ?$$

Time Independence of Future Rewards (Infinite Horizon)

- Assume a given *policy* $\pi_t(s_t) = \pi(s_t)$, **which does not depend on time**
- Random discounted reward stream: $r_0, \gamma \cdot r_1, \gamma^2 \cdot r_2, \gamma^3 \cdot r_3, \ldots$ with $\gamma \in [0,1)$
- Expected future rewards from time *t* on (discounted on *t*), $s \in S$:

$$V_0^{(\pi)}(s) = E\left(\sum_{k=0}^{\infty} \gamma^k \cdot r_k \,\middle|\, s_0 = s, a_k = \pi(s_k) \right)$$

$$V_t^{(\pi)}(s) = E\left(\sum_{k=t}^{\infty} \gamma^{k-t} \cdot r_k \,\middle|\, s_t = s, a_k = \pi(s_k) \right) = V_0^{(\pi)}(s) \quad \text{for all } t = 0,1,2,\ldots$$

- Same state, same actions, same expected reward stream, same discounting:

=> The value of "*being in a certain state*" is **time-independent** (cf. $V^{(\pi)}(s)$).

Problem Formulation (Infinite Horizon)

- Find a (**stationary**) *Markov policy* $\pi = \pi(s)$ that maximizes total expected (discounted) future rewards, $0 \leq \gamma < 1$:

$$\max_{\pi} E\left[\sum_{t=0}^{\infty} \underbrace{\gamma^t}_{\substack{discount\\factor}} \cdot \left(\sum_{i_t \in I} \underbrace{P(i_t, a_t, s_t)}_{\substack{probability\ for\ event\ i_t\\ under\ action\ a_t\ in\ state\ s_t}} \cdot \underbrace{r(i_t, a_t, s_t)}_{\substack{reward\ for\ event\ i_t\\ under\ action\ a_t\ in\ state\ s_t}} \right) \middle| \underbrace{s_0}_{initial\ state} \right],$$

where states evolve (time-independently) according to $s \rightarrow s' = \Gamma(i, a, s)$.

- How to solve such problems?
- Will the *value function* or the *optimal policy* be **time-dependent**?

Solution Approach (Dynamic Programming)

- What is the **best expected value** of having the chance to . . .

  "sell items (from any time t on, disc. on t) *starting in state s"?*

- Answer: That's easy $V(s)$!

Solution Approach (Dynamic Programming)

- What is the **best expected value** of having the chance to . . .

  “sell items (from any time t on, disc. on t) starting in state s”?

- Answer: That’s easy $V(s)$!

- We can assume V is independent of time and satisfies the Bellman equation!

- We don’t know the “Value Function V”, but V is determined by:

  Value (state today) = Best expected (profit today + Value (state tomorrow))

Solution Approach (Dynamic Programming)

- *Value (state today) = Best expected (profit today + Value (state tomorrow))*
- Idea: Consider potential events & transition dynamics within one period.

  What can happen during one time interval (under action a)?

state in t	event	reward	state in t+1	probability
	0	$r(0,a,s)$	$\Gamma(0,a,s)$	$P(0,a,s)$
s	1	$r(1,a,s)$	$\Gamma(1,a,s)$	$P(1,a,s)$
	2	$r(2,a,s)$	$\Gamma(2,a,s)$	$P(2,a,s)$

- What does that mean for the **value of state** s **(at any time** t**)**, i.e., $V(s)$?

Balancing Potential Short- and Long-Term Rewards

state in t	event	reward	state in t+1	probability
	0	$r(0,a,s)$	$\Gamma(0,a,s)$	$P(0,a,s)$
s	1	$r(1,a,s)$	$\Gamma(1,a,s)$	$P(1,a,s)$
	2	$r(2,a,s)$	$\Gamma(2,a,s)$	$P(2,a,s)$

$$V(s) = \underbrace{\max_{a \in A}}_{\substack{potential\\ actions}} \left\{ \underbrace{P(0,a,s)}_{probability\ not\ to\ sell} \cdot \left(\underbrace{r(0,a,s)}_{today's\ reward} + \underbrace{\gamma \cdot V\left(\Gamma(0,a,s)\right)}_{best\ disc.\ exp.future\ rewards} \right) \right.$$

$$\left. + \underbrace{P(1,a,s)}_{probability\ of\ demand=1} \cdot \left(\underbrace{r(1,a,s)}_{today's\ reward} + \underbrace{\gamma \cdot V\left(\Gamma(1,a,s)\right)}_{best\ disc.\ exp.future\ rewards} \right) + \ldots \right\}$$

Bellman Equation (Infinite Horizon)

- We obtain the Bellman Equation, which **determines** the Value Function:

$$V(s) = \underbrace{\max_{a \in A}}_{\substack{potential \\ actions}} \left\{ \sum_{i \in I} \underbrace{P(i,a,s)}_{probability} \cdot \left(\underbrace{r(i,a,s)}_{today's\ reward} + \underbrace{\gamma \cdot V\left(\Gamma(i,a,s)\right)}_{best\ disc.\ exp.\ future\ rewards\ of\ new\ state} \right) \right\}$$

- Does it reveal optimal policies?

Value Function & Optimal Policy

- We obtain the Bellman Equation, which **determines** the Value Function:

$$V(s) = \underbrace{\max_{a \in A}}_{\substack{\textit{potential}\\ \textit{actions}}} \left\{ \sum_{i \in I} \underbrace{P(i,a,s)}_{\textit{probability}} \cdot \left(\underbrace{r(i,a,s)}_{\textit{today's reward}} + \underbrace{\gamma \cdot V\left(\Gamma(i,a,s)\right)}_{\textit{best disc. exp. future rewards of new state}} \right) \right\}$$

- Does it reveal optimal policies?

- Yes, $a^*(s) = \arg\max_{a \in A} \{\ldots\}$ is the *optimal policy.*

- But, how can we compute the Value Function? By *backward induction*?

Value Iteration for Infinite Horizon MDPs

Value Iteration for Infinite Horizon MDPs

- We want to determine the values $V^*(s)$ that solve the equation system

$$V^*(s) = \max_{a \in A} \left\{ \sum_{i \in I} P(i,a,s) \cdot \left(r(i,a,s) + \gamma \cdot V^*\left(\Gamma(i,a,s)\right) \right) \right\}$$

- **Value Iteration**: Use the "Finite horizon" *backward induction* approach:

$$V_t(s) = \max_{a \in A} \left\{ \sum_{i \in I} P(i,a,s) \cdot \left(r(i,a,s) + \gamma \cdot V_{t+1}\left(\Gamma(i,a,s)\right) \right) \right\}, \quad V_T(s) = r_T(s) = 0$$

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$$r_0,\ \gamma \cdot r_1,\ \gamma^2 \cdot r_2,\ \gamma^3 \cdot r_3,\ \ldots,\ \gamma^{999} \cdot r_{999}\ \overbrace{\gamma^{1000} \cdot r_{1000}, \gamma^{1001} \cdot r_{1001}, \gamma^{1002} \cdot r_{1002}, \ldots}^{\gamma^T \cdot \sum_{k=0}^{\infty} \gamma^k \cdot r_{T+k}}$$

$0 \qquad T = 1000$

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$$r_0,\ \gamma\cdot r_1,\ \gamma^2\cdot r_2,\ \gamma^3\cdot r_3,\ \ldots,\ \gamma^{999}\cdot r_{999}\quad \overbrace{\underbrace{\gamma^{1000}\cdot r_{1000},\gamma^{1001}\cdot r_{1001},\gamma^{1002}\cdot r_{1002},\ldots}_{\le\gamma^T\cdot\sum_{k=0}^{\infty}\gamma^k\cdot r_{\max}\ =\ ???}}^{\gamma^T\cdot\sum_{k=0}^{\infty}\gamma^k\cdot r_{T+k}}$$

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Value Iteration for Infinite Horizon MDPs

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$$r_0,\ \gamma \cdot r_1,\ \gamma^2 \cdot r_2,\ \gamma^3 \cdot r_3,\ \ldots,\ \gamma^{999} \cdot r_{999}\quad \overbrace{\underbrace{\gamma^{1000} \cdot r_{1000}, \gamma^{1001} \cdot r_{1001}, \gamma^{1002} \cdot r_{1002}, \ldots}_{\leq \gamma^T \cdot \sum_{k=0}^{\infty} \gamma^k \cdot r_{\max} = \gamma^T \cdot r_{\max} \cdot \frac{1}{1-\gamma}}}^{\gamma^T \cdot \sum_{k=0}^{\infty} \gamma^k \cdot r_{T+k}}$$

$0 \qquad\qquad T = 1000$

Value Iteration for Infinite Horizon MDPs

- We want to determine the values $V^*(s)$ that solve the equation system

$$V^*(s) = \max_{a \in A} \left\{ \sum_{i \in I} P(i,a,s) \cdot \left(r(i,a,s) + \gamma \cdot V^*\left(\Gamma(i,a,s)\right) \right) \right\}$$

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$$V_t(s) = \max_{a \in A} \left\{ \sum_{i \in I} P(i,a,s) \cdot \left(r(i,a,s) + \gamma \cdot V_{t+1}\left(\Gamma(i,a,s)\right) \right) \right\}, \; V_T(s) = r_T(s) = 0$$

- For "large" T and for any initial $V_T(s)$ the values $V_0(s)$ converge

  to the exact values $V^*(s)$ with $\left| V_0(s) - V^*(s) \right| \le \frac{\gamma^T}{1-\gamma} \cdot r_{\max} \xrightarrow{T \to \infty} 0$

- The optimal policy $a^*(s)$, $s \in S$, is determined by

  the *arg max* of the **last iteration step**, i.e., $a_0(p)$

Value Iteration Tabular Schema (Inventory Example)

$time \,/\, periods \,/\, iterations \;\; (choose\ a\ suitable\ T)$

starting values

all states $s \in S$

	0	1		$T-2$	$T-1$	T
$s=4$						$V_T(4) = r_T(4) = 0$
$s=3$						$V_T(3) = r_T(3) = 0$
$s=2$						$V_T(2) = r_T(2) = 0$
$s=1$						$V_T(1) = r_T(1) = 0$
$s=0$						$V_T(0) = r_T(0) = 0$

Value Iteration Tabular Schema (Inventory Example)

$time\,/\,periods\,/\,iterations\ \ (choose\ a\ suitable\ T)$

$starting\ values$

$all\ states\ s \in S$

	0	1		$T-2$	$T-1$	T
$s=4$					$V_{T-1}(4),$ $\left(a^*_{T-1}(4)\right)$	$V_T(4)=r_T(4)=0$
$s=3$					$V_{T-1}(3),$ $\left(a^*_{T-1}(3)\right)$	$V_T(3)=r_T(3)=0$
$s=2$					$P(0,a,2)$ $P(1,a,2)$	$V_T(2)=r_T(2)=0$
$s=1$					$P(2,a,2)$ $P(3,a,2)$	$V_T(1)=r_T(1)=0$
$s=0$						$V_T(0)=r_T(0)=0$

Value Iteration Tabular Schema (Inventory Example)

result / policy

time / periods / iterations

try different starting values

all states $s \in S$

	0	1		$T-2$	$T-1$	T
$s=4$	$V_0(4)$, $a_0^*(4)$	$V_1(4)$, $a_1^*(4)$	$\cdots$	$V_{T-2}(4)$, $a_{T-2}^*(4)$	$V_{T-1}(4)$, $a_{T-1}^*(4)$	$V_T(4)=r_T(4)=0$
$s=3$	$V_0(3)$, $a_0^*(3)$	$V_1(3)$, $a_1^*(3)$	$\cdots$	$V_{T-2}(3)$, $a_{T-2}^*(3)$	$V_{T-1}(3)$, $a_{T-1}^*(3)$	$V_T(3)=r_T(3)=0$
$s=2$	$V_0(2)$, $a_0^*(2)$	$V_1(2)$, $a_1^*(2)$	$\cdots$	$V_{T-2}(2)$, $a_{T-2}^*(2)$	$V_{T-1}(2)$, $a_{T-1}^*(2)$	$V_T(2)=r_T(2)=0$
$s=1$	$V_0(1)$, $a_0^*(1)$	$V_1(1)$, $a_1^*(1)$	$\cdots$	$V_{T-2}(1)$, $a_{T-2}^*(1)$	$V_{T-1}(1)$, $a_{T-1}^*(1)$	$V_T(1)=r_T(1)=0$
$s=0$	$V_0(0)$, $a_0^*(0)$	$V_1(0)$, $a_1^*(0)$	$\cdots$	$V_{T-2}(0)$, $a_{T-2}^*(0)$	$V_{T-1}(0)$, $a_{T-1}^*(0)$	$V_T(0)=r_T(0)=0$

Policy Iteration for Infinite Horizon MDPs

Policy Iteration for Infinite Horizon MDPs

- We want to determine the values $V^*(s)$ that solve the system above

- **Policy iteration**: Subsequently *evaluate & improve* a policy:

  (1)

  (2)

  (3)

Policy Iteration for Infinite Horizon MDPs

- We want to determine the values $V^*(s)$ that solve the system above
- **Policy iteration**: Subsequently *evaluate & improve* a policy:

(1) Choose any starting policy $\pi(s)$, $s \in S$

(2) **Evaluate** the policy $\pi(s)$ by solving the linear system $\forall s \in S$

(3)

Policy Iteration for Infinite Horizon MDPs

- We want to determine the values $V^*(s)$ that solve the system above

- **Policy iteration**: Subsequently *evaluate* & *improve* a policy:

(1) Choose any starting policy $\pi(s)$, $s \in S$

(2) **Evaluate** the policy $\pi(s)$ by solving the linear system $\forall s \in S$

$$V^{(\pi)}(s) = \sum_{i \in I} P(i,\pi(s),s) \cdot \left(r(i,\pi(s),s) + \gamma \cdot V^{(\pi)}\left(\Gamma(i,\pi(s),s)\right)\right)$$

e.g., using a simplified Value Iteration (*without max* !) or via LP

(3)

Policy Iteration for Infinite Horizon MDPs

- We want to determine the values $V^*(s)$ that solve the system above
- **Policy iteration**: Subsequently *evaluate* & *improve* a policy:

(1) Choose any starting policy $\pi(s)$, $s \in S$

(2) **Evaluate** the policy $\pi(s)$ by solving the linear system $\forall s \in S$

$$V^{(\pi)}(s) = \sum_{i \in I} P(i, \pi(s), s) \cdot \left(r(i, \pi(s), s) + \gamma \cdot V^{(\pi)} \left(\Gamma(i, \pi(s), s) \right) \right)$$

e.g., using a simplified Value Iteration (*without max* !) or via LP

(3) **Update** $\pi(s) \leftarrow \arg\max_{a \in A} \left\{ \sum_{i \in I} P(i, a, s) \cdot \left(r(i, a, s) + \gamma \cdot V^{(\pi)} \left(\Gamma(i, a, s) \right) \right) \right\}$

Repeat Step (2) & (3) $\forall s$ until no improvement (=> optimal solution!)

Policy Iteration Schema (Inventory Example)

Choose a starting policy $\pi^{(0)}(s)$, $s \in S$, cf. Step (1).

(i) Compute the value function $V^{(\pi^{(0)})}(s)$ of policy $\pi^{(0)}(s)$, cf. Step (2).
Use $V^{(\pi^{(0)})}(s)$ to improve policy $\pi^{(0)}(s)$ to a new policy $\pi^{(1)}(s)$, cf. Step (3).

(ii) Compute the value function $V^{(\pi^{(1)})}(s)$ of policy $\pi^{(1)}(s)$, cf. Step (2).
Use $V^{(\pi^{(1)})}(s)$ to improve policy $\pi^{(1)}(s)$ to a new policy $\pi^{(2)}(s)$, cf. Step (3).

. . .

Stop if after an iteration k we have $\pi^{(k)}(s) = \pi^{(k-1)}(s)$ for all $s \in S$.

Policy Iteration Step (2) via "Value Iteration"

- Assume the current policy $\pi(s)$, $s \in S$

- We want to determine the values $V^{(\pi)}(s)$ that solve the system, cf. Step (2)

$$V^{(\pi)}(s) = \sum_{i \in I} P(i,\pi(s),s) \cdot \left(r(i,\pi(s),s) + \gamma \cdot V^{(\pi)}\left(\Gamma(i,\pi(s),s)\right)\right), \ \forall s \in S$$

- **Iterative solution** (value iteration **with fixed *a***) starting with $V_T^{(\pi)}(s) = 0$:

$$V_t^{(\pi)}(s) = \max_{\substack{a \in A: \\ a = \pi(s)}} \left\{ \sum_{i \in I} P(i,a,s) \cdot \left(r(i,a,s) + \gamma \cdot V_{t+1}^{(\pi)}\left(\Gamma(i,a,s)\right)\right)\right\}$$

Policy Iteration Step (2) via "Value Iteration"

- Assume the current policy $\pi(s)$, $s \in S$

- We want to determine the values $V^{(\pi)}(s)$ that solve the system, cf. Step (2),

  $$V^{(\pi)}(s) = \sum_{i \in I} P(i, \pi(s), s) \cdot \left(r(i, \pi(s), s) + \gamma \cdot V^{(\pi)}\left(\Gamma(i, \pi(s), s)\right)\right), \ \forall s \in S$$

- **Iterative solution** (value iteration **with fixed *a***) starting with $V_T^{(\pi)}(s) = 0$:

  $$V_t^{(\pi)}(s) = \sum_{i \in I} P(i, \pi(s), s) \cdot \left(r(i, \pi(s), s) + \gamma \cdot V_{t+1}^{(\pi)}\left(\Gamma(i, \pi(s), s)\right)\right)$$

- For "large" *T* and for any initial $V_T^{(\pi)}(s)$ the values $V_0^{(\pi)}(s)$ converge to the exact values $V^{(\pi)}(s)$ with $\left|V_0^{(\pi)}(s) - V^{(\pi)}(s)\right| \xrightarrow{T \to \infty} 0$

Policy Iteration Step (2) via Linear Programming

- Assume a step k's current policy $\pi(s)$, $s \in S$

- We want to determine the values $V^{(\pi)}(s)$ that solve the system, cf. Step (2),

$$V^{(\pi)}(s) = \sum_{i \in I} \underbrace{P(i,\pi(s),s)}_{probability} \cdot \left(\underbrace{r(i,\pi(s),s)}_{period's\ reward} + \underbrace{\gamma \cdot V^{(\pi)}\left(\underbrace{\Gamma(i,\pi(s),s)}_{s'} \right)}_{disc.\ exp.\ future\ profits\ of\ new\ state} \right), \forall s \in S$$

- We have to solve a system of $|S|$ linear equations! *(not in focus)*

- Standard LP solvers can be applied, e.g., Gurobi, Cplex, etc.

Summary (Solving Discrete Time Infinite Horizon MDPs)

Policy Iteration

(+) provides optimal solutions for infinite horizon MDPs
(+) fast convergence (stop if no improvement is possible)
(–) full information required (cf. events & transitions)
(–) only for smaller state spaces

Value Iteration

(+) provides near-optimal solutions for infinite horizon MDPs
(+) numerically simple
(+) fast convergence (if the discount factor is not close to 1)
(–) full information required (cf. events & transitions)
(–) for medium size state spaces, does not scale (curse of dimensionality)

Recall - Questions?

- States, Action, Rewards, Transitions
- Discounted Future Rewards
- Value Function
- Bellman Equation
- Backward Induction
- Value Iteration
- Policy Iteration

Could You Solve Different Test Problems?

- Finite Horizon (*use Backward Induction*)
  - – Eating cake (deterministic utility)
  - – Selling Airline Tickets (stochastic demand)

- Infinite Horizon (*use Value & Policy Iteration*)
  - – Car replacement problem (deterministic costs)
  - – Inventory management (stochastic demand)

Overview

Week	Dates	Topic	
1	April 21	Introduction	
2	April 25/28	Finite + Infinite Time MDPs	
3	**May 2/5**	**Approximate Dynamic Programming (ADP) + Exercise**	
4	May 9/12	Q-Learning (QL)	
5	May 16/19	Deep Q-Networks (DQN)	
6	May 23	DQN Extensions	(Thu May 26 “Himmelfahrt”)
7	May 30/June 2	Policy Gradient Algorithms	
8	June 9	Project Assignments	(Mon June 6 “Pfingstmontag”)
9	June 13/16	Work on Projects: Input/Support	
10	June 20/23	Work on Projects: Input/Support	
11	June 27/30	Work on Projects: Input/Support	
12	July 4/7	Work on Projects: Input/Support	
13	July 11/14	Work on Projects: Input/Support	
14	July 18/21	Final Presentations	
	Sep 15	Finish Documentation	

Exercise **Value Iteration & Policy Iteration**

Consider the inventory management example. Maximize expected discounted rewards. Find an optimal ordering policy. Items ordered at the beginning of a period are delivered at the beginning of the same period; holding cost only occur for items left at the beginning of the period. If demand exceeds the inventory (at the beginning of the period) then sales are lost, cf. $r(i,a,s) = p \cdot \min(i,s) - c \cdot a - h \cdot s - 1_{\{a>0\}} \cdot f$, where $p = 10, c = 2, h = 0.5, f = 20$. Demand probabilities are $P(i,a,s) = P(i) := 1/4$, $i = 0,1,...,3$. Consider the state space $s \in S{=}\{0,1,...,50\}$. Feasible ordering decisions are $a \in A{:=}\{0,1,...,50\}$. The discount factor is $\gamma = 0.95$. The initial state is 10 items.

(a) Solve the problem using value iteration.

(b) Solve the problem using policy iteration.

(c) Simulate the optimal policy $\pi(s)$ over runs of 100 periods and accumulate total rewards discounted on *t*=0. Compare the mean with the value function for the initial state (Bonus).

Problem Embedding (Finite as Infinite Horizon MDP)

- Finite Horizon (time-dependent) with discount $\gamma \leq 1$
  - $r_t = r_t(i,a,s)$ with probabilities $P_t(i,a,s)$, $s \in S$, $t = 0,1,2,...,T$
  - $s_t \rightarrow s_{t+1} = \Gamma_t(i,a,s)$ with $i \in I_t(a,s)$, $a \in A_t(s)$

- Infinite Horizon Value It. Embedding (time-independent) with same $\gamma \leq 1$
  - $\tilde{s}_0 := (0, s_0)$, $\tilde{s} := (t,s)$, $s \in S$, $t = 0,1,2,...,T$
  - $\tilde{r} = \tilde{r}(i,a,\tilde{s}) = \tilde{r}(i,a,t,s) := if\ t < T\ then\ r_t(i,a,s)\ else\ r_T(s)$
  - $\tilde{P}(i,a,\tilde{s}) := P_t(i,a,s)$, $a \in \tilde{A}(\tilde{s}) := A_t(s)$, $i \in \tilde{I}(a,\tilde{s}) := I_t(a,s)$
  - $\tilde{s} \rightarrow \tilde{s}' = \tilde{\Gamma}(i,a,\tilde{s}) = \tilde{\Gamma}(i,a,t,s) := \left(t+1,\ if\ t < T\ then\ \Gamma_t(i,a,s)\ else\ s\right)$
  - $V_k(t,s) = \max_{a \in \tilde{A}} \left\{ \sum_{i \in \tilde{I}} \tilde{P}(i,a,t,s) \cdot \left(\tilde{r}(i,a,t,s) + \gamma \cdot 1_{\{t<T\}} \cdot V_{k+1}\left(\tilde{\Gamma}(i,a,t,s)\right)\right)\right\}$, i.e. $V_k(t,s) = 0\ \forall t > T$